A Nonlinear Autonomous Boundary Value Problem for a Non-Degenerate Differential-Algebraic System

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Abstract

We have found the constructive necessary and sufficient conditions for solvability and the scheme for constructing solutions of a nonlinear autonomous boundary value problem for a nondegenerate differential-algebraic system. The nonlinear boundary value problem for the autonomous system significantly differs from similar autonomous boundary value problems by its dependence on an arbitrary continuous vector function. We have also constructed a convergent iterative scheme for finding approximate solutions of the nonlinear autonomous boundary value problem for the nondegenerate differential-algebraic system in critical and noncritical cases. Proposed in this paper scheme for studying the nonlinear autonomous boundary value problem for the nondegenerate system of differential-algebraic equations can be transferred to degenerate systems of differential-algebraic equations in the same way.

Keywords [1]

Differential-algebraic equations, nonlinear autonomous boundary value problems.

1. Statement of the problem

Let A and B are $(m \times n)$ dimensional matrices and $Z(z, \varepsilon)$ is a n dimensional vector function. We will call a weakly nonlinear autonomous periodic differential-algebraic boundary value problem the problem of finding solution [1]

$$z(t,\varepsilon): z(\cdot,\varepsilon) \in C^1[a, b(\varepsilon)], \;\; z(t,\cdot) \in C[0, \varepsilon_0], \;\; b(0) := b^*$$

of a differential-algebraic system

$$Az' = Bz + \varepsilon Z(z, \varepsilon), \tag{1}$$

which satisfy the boundary condition

$$\ell z(\cdot, \varepsilon) = \alpha. \tag{2}$$

Here, $\ell z(\cdot, \varepsilon)$ is a linear bounded vector functional:

$$\ell z(\cdot, \varepsilon): C[a, b(\varepsilon)] \to R^q.$$

The solution of the problem (1), (2) is found in a small neighbourhood of the solution $z_0(t) \in C^1[a, b^*]$ of the Noether $(q \neq n)$ differential-algebraic generating boundary value problem

$$A z'_0 = B z_0, \, \ell z_0(\cdot) = \alpha \in R^q. \tag{3}$$

The vector function $Z(z, \varepsilon)$ we assume to be continuously differentiable with respect to the unknown $z(t, \varepsilon)$ in a small neighbourhood of the solution of the generating problem and continuously differentiable with respect to a small parameter ε in a small positive neighbourhood of zero. The

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matrix A we assume, in general, to be rectangular: $m \neq n$, or square, but degenerate [2, 3, 4]. Under the condition

$$P_{A^*} = 0 \tag{4}$$

the generating system (3) is reduced to the traditional system of ordinary differential equations [5]

$$z'_0 = A^+ B z_0 + P_{A_{\rho_0}} \nu_0(t); \tag{5}$$

here

$$\text{rank}\, A := m < n.$$

In addition, A^+ is a pseudo-inverse (by Moore-Penrose) matrix, P_{A^*} is a matrix-orthoprojector:

$$P_{A^*}: R^m \to N(A^*),$$

$P_{A_{\rho_0}}$ is a $(n \times \rho_0)$ matrix formed from ρ_0 linearly independent columns of $(n \times n)$ matrix-orthoprojector

$$P_A: R^n \to N(A),$$

$\nu_0(t) \in R^{\rho_0}$ is an arbitrary continuous vector function. Under the condition (4) the system (1) we will call nondegenerate. In the critical case

$$P_{Q^*} \neq 0, \quad Q := \ell X_0(\cdot)$$

for a fixed vector function $\nu_0(t) \in C[a, b^*]$ under the condition

$$P_{Q_d^*}\{\alpha - \ell K[P_{A_{\rho_0}} \nu_0(s)](\cdot)\} = 0 \tag{6}$$

the generating problem (3) has r parametric family of solutions [5]

$$z_0(t, c_r) = X_r(t)\, c_r + G[P_{A_{\rho_0}} \nu_0(s)](t), \; c_r \in R^r.$$

Here $X_0(t)$ is a normal $(X_0(a) = I_n)$ fundamental matrix of the homogeneous part of the differential system (5), and $G[P_{A_{\rho_0}} \nu_0(s)](t)$ is the generalized Green's operator [3, 5] of the generating periodic differential-algebraic boundary value problem (3), $K[P_{A_{\rho_0}} \nu_0(s)](t)$ is the generalised Green's operator [3, 5] of the Cauchy problem $z(a) = 0$ for the differential-algebraic system (3). The matrix $P_{Q_d^*}$ is formed from d linearly independent rows of the matrix-orthoprojector P_{Q^*}, and the matrix P_{Q_r} is formed from r linearly independent columns of the matrix-orthoprojector P_Q. Under the condition (4) the system (1) leads to the traditional system of ordinary differential equations

$$z' = A^+ B\, z + P_{A_{\rho_0}} \nu_0(t) + \varepsilon\, A^+ Z(z, \varepsilon); \tag{7}$$

The periodic boundary value problem for an autonomous system (6) significantly differs [1, 6] from similar autonomous boundary value problems by its dependence on an arbitrary vector function $\nu_0(t) \in C[a, b^*]$. In addition, only in exceptional cases, the autonomous boundary value problem (1), (2) is solvable on a segment of fixed length.

Example 1. *Let us find a solution to the autonomous nonlinear differential-algebraic boundary value problem for equation*

$$Az' = Bz + \varepsilon\, Z(z, \varepsilon), \quad t \in [0, T], \quad \ell z(\cdot, \varepsilon) = 0; \tag{8}$$

here,

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \;\; B := \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \;\; \Omega := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$Z(z(t, \varepsilon), \varepsilon) := \Omega\, z(t, \varepsilon)(1 - z^*(t, \varepsilon)\, z(t, \varepsilon)), \quad \ell z(\cdot, \varepsilon) := z(0, \varepsilon) - z(T, \varepsilon).$$

The condition (4) is satisfied, so the system (8) is nondegenerate. In this case, the matrix A is rectangular, and

$$\rho_0 = 1 \neq 0, \;\; P_{A_{\rho_0}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

therefore, the homogeneous part of the system (8) has a solution that depends on an arbitrary continuous function; we put $\nu_0(t) := 0$:

$$z_0(t, c) = X_0(t)c + K[P_{A_{\rho_0}} \nu_0(s)](t), \quad c \in R^3,$$

where

$$X_0(t) = \begin{pmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{pmatrix}, \quad K[P_{A_{\rho_0}} \nu_0(s)](t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The generating solution

$$z_0(t, c_r) = z(t, 0), \quad c_r := \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix}, \quad c_2 \in R^1$$

determines the partial solution

$$z(t, \varepsilon) = z_0(t, c_r) = c_r$$

of nonlinear autonomous differential-algebraic boundary value problem (8) on the segment $[0, T]$ of fixed length.

2. Necessary condition of solvability

An autonomous boundary value problem for the system (7) is significantly different from similar nonautonomous boundary value problems; in that the length of the interval $[a, b(\varepsilon)]$, on which we determined the solution of a nonlinear boundary value problem for the system (7), in general, is unknown. We will use the technique [1, 6], which consists in representing the unknown function

$$b(\varepsilon) = b^* + \varepsilon\,(b^* - a)\,\beta(\varepsilon)$$

through a new unknown

$$\beta(\varepsilon) \in C[0, \varepsilon_0], \quad \beta(0) := \beta^*.$$

Function $\beta(\varepsilon)$ is to be determined in the process of finding a solution of the boundary value problem for the system (7). The technique consists in replacing the of the independent variable

$$t = a + (\tau - a)(1 + \varepsilon\beta(\varepsilon))$$

and finding the solution of the nonlinear boundary value problem (2), (7) and the function $\beta(\varepsilon)$, as a function of a small parameter. In the critical case, under the condition (6) for a fixed function $\nu_0(\tau)$ the condition of solvability of the nonlinear boundary value problem (2), (7) takes the form

$$P_{Q_d^*}\{\big(1 + \varepsilon\beta(\varepsilon)\big)\,\alpha - \ell K[\beta(\varepsilon)(A^+ B z(s, \varepsilon) + P_{A_{\rho_0}} \nu_0(s) + \\ +(1 + \varepsilon\,\beta(\varepsilon))\,A^+ Z(z(s, \varepsilon), \varepsilon)](\cdot)\} = 0. \qquad (9)$$

Using the continuity of the nonlinear vector function $Z(z(t, \varepsilon), \varepsilon)$, on ε in a small positive neighborhood of zero, we find the limit for $\varepsilon \to 0$ in equality (9) and obtain the necessary condition

$$F(\check{c}_0) := P_{Q_d^*}\{\alpha - \ell K[\beta^*(A^+ B\, z_0(s, c_r^*) + P_{A_{\rho_0}} \nu_0(s) + A^+ Z(z_0(s, c_r^*), 0)](\cdot)\} = 0 \qquad (10)$$

of existence of a solution to the boundary value problem (1), (2) in the critical case; here

$$\check{c}_0 := \begin{pmatrix} c_r^* \\ \beta^* \end{pmatrix} \in R^{r+1}.$$

Thus, the following lemma is proved.

Lemma. *Assume that the autonomous differential-algebraic boundary value problem* (1), (2) *for a fixed constant* $\nu_0 \in R^{\rho_0}$ *under the conditions* (4) *and* (6) *represents the critical case* $P_{Q^*} \neq 0$ *and has the solution*

$$z(t, \varepsilon) = \mathrm{col}(z^{(1)}(t, \varepsilon), \ \ldots, \ z^{(n)}(t, \varepsilon)), \ z^{(i)}(\cdot, \varepsilon) \in C^1[a, b(\varepsilon)], \\ z^{(i)}(t, \cdot) \in C[0, \varepsilon_0], \ i = 1, \ 2, \ \ldots, \ n,$$

which for $\varepsilon = 0$ *turns into the generating* $z(t, 0) = z_0(t, c_r^*)$. *Then the vector* $\check{c}_0$ *satisfies equation (10).*

The first r components of $c_r^* \in R^r$ of the root of the equation (10) determine the amplitude of the generating solution $z_0(t, c_r^*)$, in small neighborhood of which the desired solution of the initial problem (1), (2) can exist. In addition, from the equation (10) the value β^*, which determines the first approximation to the unknown function

$$b_1(\varepsilon) = b^* + \varepsilon(b^* - a)\beta^*,$$

can be found. If the equation (10) has no real roots, then the initial differential algebraic problem (1), (2) has no desired solutions.

The equation (10) will be further called the equation for the generating constants of the autonomous nonlinear differential-algebraic boundary value problem (1), (2). The statement of the lemma generalizes the corresponding results of [1, 6, 7, 9] on the case of an autonomous nonlinear differential-algebraic boundary value problem (1), (2), namely, on the case $A \neq I_n$. Similarly to [1, 6, 9] we demonstrate that the periodic problem for the system (7) is solvable provided that the roots of the equation for the generating constants (10) are simple.

3. Sufficient condition of solvability

Assume that the equation for the generating constants (10) has real roots. Fixing one of the solutions $\check{c}_0 \in R^{r+1}$ of the equation (10), we come to the problem of finding solutions of the problem (1), (2)

$$z(\tau,\varepsilon) = z_0(\tau, c_r^*) + x(\tau,\varepsilon)$$

in the neighbourhood of the generating solution

$$z_0(\tau, c_r^*) = X_r(\tau)c_r^* + G[P_{A_{\rho_0}} \nu_0(s)](\tau).$$

The deviation of $x(t,\varepsilon)$ from the generating solution is determined by the boundary value problem

$$x' = A^+Bx + \varepsilon\{\beta A^+Bz + \beta P_{A_{\rho_0}} \nu_0(s) + (1+\varepsilon\beta)A^+Z(z,\varepsilon)\}, \;\; \ell x(\cdot,\varepsilon) = 0. \qquad (11)$$

Using the continuous differentiability of the function $Z(z,\varepsilon)$ with respect to both the first and second arguments in the neighbourhood of the generating solution $z_0(\tau, c_r^*)$ and the point $\varepsilon = 0$, we get the expansion of this function

$$Z(z_0(\tau, c_r^*) + x(\tau,\varepsilon), \varepsilon) = Z(z_0(\tau, c_r^*),0) +$$
$$+A_1(\tau)x(\tau,\varepsilon) + \varepsilon A_2(\tau) + R(z_0(\tau, c_r^*) + x(\tau,\varepsilon), \varepsilon),$$

where

$$\mathrm{A}_1(\tau) = \left.\frac{\partial Z(z,\varepsilon)}{\partial \mathrm{z}}\right|_{\substack{z=z_0(\tau,c_r^*),\\ \varepsilon=0,}}, \quad \mathrm{A}_2(\tau) = \left.\frac{\partial Z(z,\varepsilon)}{\partial \varepsilon}\right|_{\substack{z=z_0(\tau,c_r^*),\\ \varepsilon=0,}}$$

The residual $R(z(\tau,\varepsilon),\varepsilon)$ of the expansion of the function $Z(z(\tau,\varepsilon),\varepsilon)$ of higher order of smallness on x and ε in the neighbourhood of the points $x = 0$ and $\varepsilon = 0$, than the first three terms of the expansion, so

$$\left.R(z,\varepsilon)\right|_{\substack{z=z_0(\tau,c_r^*),\\ \varepsilon=0,}} \equiv 0, \quad \left.\frac{\partial R(z,\varepsilon)}{\partial z}\right|_{\substack{z=z_0(\tau,c_r^*),\\ \varepsilon=0,}} \equiv 0, \quad \left.\frac{\partial R(z,\varepsilon)}{\partial \varepsilon}\right|_{\substack{z=z_0(\tau,c_r^*),\\ \varepsilon=0,}} \equiv 0$$

Let us denote by $(d \times (r+1))$ the dimensional matrix

$$B_0 = -P_{Q_d^*}\ell K\{A^+[\beta^* B + A_1(s)]X_r(s);\; P_{A_{\rho_0}} \nu_0(s) + A^+Bz_0(s, c_r^*)\}(\cdot).$$

In the critical case, under the condition (6) for a fixed function $\nu_0(\tau)$ and the solution $\check{c}_0$ of equation (10) the solution of the nonlinear boundary value problem (11) has the form

$$x(\tau,\varepsilon) = X_r(\tau)c_r(\varepsilon) + x^{(1)}(\tau,\varepsilon);$$

here,

$$x^{(1)}(\tau,\varepsilon) := \varepsilon\, G[\beta A^+Bz + \beta P_{A_{\rho_0}} \nu_0(s) + (1+\varepsilon\beta)\, A^+Z(z,\varepsilon)](\tau).$$

and

$$\beta(\varepsilon) := \beta^* + \gamma(\varepsilon), \quad c(\varepsilon) := \mathrm{col}(c_r(\varepsilon), \beta(\varepsilon)) \in C[0,\varepsilon_0],$$

while the condition of solvability of the nonlinear boundary value problem (11) leads to the equation

$$B_0 c(\varepsilon) = -P_{Q_d^*}\ell K\{\beta^* A + Bx^{(1)}(s,\varepsilon) + A^+[\varepsilon A_2(s) + R(z(s,\varepsilon),\varepsilon)]\}(\cdot),$$

which solvable under the condition [1, 10, 11]

$$P_{B_0^*}P_{Q_d^*}\ell K\{\beta^* A + Bx^{(1)}(s,\varepsilon) + A^+[\varepsilon A_2(s) + R(z(s,\varepsilon),\varepsilon)]\}(\cdot) = 0.$$

In particular, the condition of solvability of the nonlinear boundary value problem (11) is satisfied in the case of

$$P_{B_0^*}P_{Q_d^*} = 0. \qquad (12)$$

Thus, under the condition (12) we get an operator system that is equivalent to the problem of finding solutions of the boundary value problem (1), (2)

$$z(\tau,\varepsilon) = z_0(\tau, c_r^*) + x(\tau,\varepsilon), \;\; x(\tau,\varepsilon) = X_r(\tau)c_r(\varepsilon) + x^{(1)}(\tau,\varepsilon),$$
$$x^{(1)}(\tau,\varepsilon) := \varepsilon G[\beta A^+ Bz + \beta P_{A_{\rho_0}} \nu_0(s) + (1+\varepsilon\beta)A^+ Z(z,\varepsilon)](\tau),$$
$$c(\varepsilon) = -B_0^+ P_{Q_d^*} \ell K\{\beta^* A + Bx^{(1)}(s,\varepsilon) + A^+[\varepsilon A_2(s) + R(z(s,\varepsilon),\varepsilon)]\}(\cdot). \qquad (13)$$

The operator system (13) belongs to the class of systems for which the method of simple iterations is applicable [1]. Thus, the following theorem is proved.

Theorem. *Assume that the autonomous differential-algebraic boundary value problem (1), (2) for a fixed constant* $\nu_0 \in R^{\rho_0}$ *under the conditions* (4) *and* (6) *represents the critical case* $P_{Q^*} \neq 0$. *Suppose also that the equation for the generating constants* (10) *has real roots. Under the conditions* (4), (6) *and* (12) *for the fixed function* $\nu_0(\tau)$ *and for the solution* $\check{c}_0$ *of equation* (10) *the operator system* (13) *is equivalent to the problem of finding solutions of the boundary value problem* (1), (2) *and has at least one solution. To find the solution to the operator system* (13) *the method of simple iteration is applicable.*

For finding the solution of the autonomous boundary value problem (1), (2) in the neighbourhood of the generating solution, the Newton-Kantorovich method can also be used [8].

Example 2. *Let's find a solution to the autonomous nonlinear differential*-algebraic boundary *value problem for the equation*

$$Az' = Bz + \varepsilon\, Z(z,\varepsilon), \quad t \in [0,T], \;\; \ell z(\cdot,\varepsilon) = 0; \qquad (14)$$

here

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \;\; B := \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \;\; \Omega := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \;\; z(t,\varepsilon) := \begin{pmatrix} u(t,\varepsilon) \\ v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix},$$

and

$$Z(z(t,\varepsilon),\varepsilon) := \begin{pmatrix} u(t,\varepsilon)v(t,\varepsilon) \\ u(t,\varepsilon)v(t,\varepsilon) \end{pmatrix}, \;\; \ell z(\cdot,\varepsilon) := z(0,\varepsilon) - z(T,\varepsilon).$$

Since the condition (4) is satisfied, the system (14) is nondegenerate. In this case, the matrix A is rectangular, and

$$\rho_0 = 1 \neq 0, \;\; P_{A_{\rho_0}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

so the homogeneous part of the system (14) has a solution that depends on an arbitrary continuous function; we put $\nu_0(t) := \cos 3t$:

$$z_0(t,c) = X_0(t)\, c + K[P_{A_{\rho_0}} \nu_0(s)](t), \;\; c \in R^3,$$

where

$$X_0(t) = \begin{pmatrix} \cos t & \sin t & \sin t \\ 0 & 1 & 0 \\ -\sin t & -1+\cos t & \cos t \end{pmatrix}, \;\; K[P_{A_{\rho_0}} \nu_0(s)](t) = \frac{1}{24}\begin{pmatrix} 3\cos t - 3\cos 3t \\ 8\sin 3t \\ -3\sin t + \sin 3t \end{pmatrix}.$$

In this case, the condition (6) is satisfied, so the linear part of the problem (14) has a solution that depends on the continuous function $\nu_0(t) := \cos 3t$:

$$z_0(t,c_r) = X_r(t)c + G[P_{A_{\rho_0}} \nu_0(s)](t), \;\; X_r(t) := X_0(t);$$

here

$$G[P_{A_{\rho_0}} \nu_0(s)](t) = K[P_{A_{\rho_0}} \nu_0(s)](t), \;\; c_r \in R^3.$$

The equation for the generating constants (10) has a real root

$$\check{c}_0 := \begin{pmatrix} c_r^* \\ \beta^* \end{pmatrix} \in R^4, \; c_r^* = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in R^3, \; \beta^* = 0,$$

which corresponds to the full rank matrix

$$B_0 = \frac{\pi}{8}\begin{pmatrix} 0 & 1 & 0 & 16 \\ 0 & 17 & 0 & -16 \end{pmatrix},$$

thus, the condition of solvability (12) of the nonlinear boundary value problem (14) is satisfied.

Proposed in this paper scheme for studying a nonlinear autonomous boundary value problem for a nondegenerate system of differential-algebraic equations [12,13,14,15,16,17,18] can be, analogously [5], transferred to degenerate systems of differential-algebraic equations in the same way. For finding the solution of an autonomous boundary value problem for a nondegenerate system of differential-algebraic equations (1), (2) the main requirement is the solvability requirement (12), which is equivalent to the condition of simplicity of the roots of the equation for the generating constants (10) [1, 6]. The scheme for studying a nonlinear autonomous boundary value problem for a nondegenerate system of differential-algebraic equations proposed in this paper can be similarly transferred to systems of differential-algebraic equations with a variable rank of derivative matrix.

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