Approximation Properties of Generalized Hölder Classes in Context of Signal Processing

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Abstract

Means of clandestine information acquisition most often intercepted or received information is transmitted over a radio channel. Therefore, obtaining information about radio signals located near objects in which confidential information circulates is a very important scientific task in many aspects of information technologies. Therefore, methods of processing radio signals require constant improvement. When processing signals, especially continuous ones, the class to which the studied signals belong plays an important role. This is one of the key aspects in solving a number of tasks, one of which is to restore the continuous appearance of the signal after its digital processing. Considering the class of the signal helps to explore more deeply both the properties of the signal itself and the methods of its further processing, digital or analog, especially in the case of a complex nature, what is very useful in context of some tasks of system analysis. This allows to choose the optimal way to solve this problem depending on the class to which the signal belongs. The quality of approximation of signals described by functions from generalized Hölder classes and Poisson-type operators is investigated in the paper. In particular, the measure of deviation, which analytically describes the quality of the specified approximation, as well as the direct influence of the parameter describing the order of the fractional derivative function describing the signal, is established and investigated.

Keywords [1]

Signal processing, generalized Hölder classes, Poisson-type operator, approximation, information technologies.

1. Introduction

The space where we operate by means of a signal plays the important role in the construction of mathematical models of one or another real process [1], including the process of signal transformation and restoration. It may have a big interest in different tasks of computer modeling and related fields. Since in many cases the investigated signals are continuous and can be described using periodic functions, we consider the space of continuous 2π -periodic functions $C_{2\pi}$, being quite relevant in a number of problems of digital signal theory and related fields of science and technology [2, 3].

According to [4, 5], the norm in this space is given by the equality

$$\|g\|_C = \max_t |g(t)| \tag{1}$$

Among various classes of signals, the class of signals, described (characterized) by the following inequality, is quite popular

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$$\left|g(t+x)-2g(t)+g(t-x)\right|\le 2|x|^{\alpha},\ 0\le\alpha\le 2, -\pi\le t,x\le\pi, \tag{2}$$

and $g(\cdot)$ is the function, with the help of which the natural investigated signal is described. The class of all such functions $g(\cdot)$ from space $C_{2\pi}$ we will call the generalized Hölder class [6] and will denote by H_*^{α}. Since the signal in the most cases is an oscillatory process, the effectiveness of the constructed mathematical model of this oscillatory process will directly depend on how harmonic (polyharmonic) the function describing one of the types of such functions, which are solutions of polyharmonic equations of elliptic type [7-10] with corresponding boundary conditions [11, 12].

2. Estimation of deviation measure of signals from generalized Hölder classes from their generalized Abel-Poisson operators

Let's have polygarmonic functions in the form of an operator

$$P_{s,q}(n;g;t)=\pi^{-1}\int_{-\pi}^{\pi} g(t+x)K_{s,q}(n;x)dx, \tag{3}$$

where $q\ge 1, 0\le s\le\frac{1}{2}$,

$$K_{s,q}(n;x)=\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\frac{k}{n}}\left(1+kse^{-\frac{q+1}{n}}\left(e^{\frac{1}{n}}-1\right)^{q}\left(e^{\frac{1}{n}}+1\right)\right)\cos kx, \tag{4}$$

is the kernel of this operator. In the case $s=0$ from the relations (3) and (4) we have

$$P_{o,q}(n;g;t)=A(n;g;t)=\pi^{-1}\int_{-\pi}^{\pi} g(t+x)\left(\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\frac{k}{n}}\cos kx\right)dx, \tag{5}$$

the harmonic Abel-Poisson function [13, 14] or Poisson [15, 16]. If in (3) and (4) we put $q=1$, and $s=\frac{1}{2}$, then we get a biharmonic function [7]

$$P_{\frac{1}{2},1}(n;g;t)=B(n;g;t)=\pi^{-1}\int_{-\pi}^{\pi} g(t+x)\left(\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{\infty}e^{-\frac{k}{n}}\left(2+k\left(1-e^{-\frac{2}{n}}\right)\right)\cos kx\right)dx\,. \tag{6}$$

As we noted above, the most effective mathematical models of the process of signal transmission and restoration are those considered in the space of generalized Hölder classes H_*^{α}. Therefore, if the function $g(\cdot)$ from the class H_*^{α} $(0\le\alpha\le 2)$ describes the studied signal using the polyharmonic operator $P_{s,q}(n,g,t)$, then we will consider the quantity

$$\sup_{g\in H_*^{\alpha}}\max_{-\pi\le t\le\pi}\left|P_{s,q}(n;g;t)-g(t)\right|=\Delta\left(H_*^{\alpha};n\right). \tag{7}$$

The value in the left side of the equality (7) characterizes the deviation error for the thematic model $P_{s,q}(n;\varphi;t)$ of the signal from the real one, described using the function $g(\cdot)$ of the generalized Hölder class H_*^{α}. To study the value $\Delta\left(H_*^{\alpha};n\right)$, first of all, we consider the so-called integral representation:

$$P_{s,q}(n;g;t)-\varphi(t)=\pi^{-1}\int_{-\pi}^{\pi}\left(g(t+x)-g(x)\right)K_{s,q}(n,x)dx=$$

$$=(2\pi)^{-1}\int_{-\pi}^{\pi}\left(g(t+x)-2g(t)+g(t-x)\right)K_{s,q}(n,x)dx. \tag{8}$$

Further, combining the relations (7), (8), (2), (1), we obtain that

$$\Delta\left(H_*^{\alpha};n\right)=(2\pi)^{-1}\sup_{g\in H_*^{\alpha}}\max_{-\pi\le t\le\pi}\left|\int_{-\pi}^{\pi}\left(g(t+x)--2g(t)+g(t-x)\right)K_{s,q}(n;x)dx\ \right|\le$$

$$\leq (2\pi)^{-1} \sup_{g\in H_*^\alpha} \max_{-\pi\leq t\leq \pi-\pi} \int_{-\pi}^{\pi} |g(t+x)-2g(t)+g(t-x)||K_{s,q}(n,x)|dx. \quad (9)$$

Since, according to [17], the kernel $K_{s,q}(n;x)$ is always positive for all values of the parameters $q\geq 1; 0\leq s\leq \frac{1}{2}$ and $-\pi\leq x\leq \pi$, specified in (3), it follows from (9) that

$$\Delta(H_*^\alpha;n)\leq (2\pi)^{-1} \sup_{g\in H_*^\alpha} \max_{-\pi\leq t\leq \pi-\pi} \int_{-\pi}^{\pi} |g(t+x)-2g(t)+g(t-x)||K_{s,q}(n,x)|dx \leq$$

$$\leq \pi^{-1}\int_{\pi}^{\pi}|x|^\alpha \left(\frac{1}{2}+\sum_{k=1}^{\infty}\left(e^{-\frac{k}{n}}++kse^{-\frac{k}{n}}\left(1-e^{-\frac{1}{n}}\right)^q\left(1+e^{-\frac{1}{n}}\right)\right)\cos kx\right)dx. \quad (10)$$

Having made some mathematical transformations in the right-hand side of the equality (4), we get:

$$\frac{1}{2}+\sum_{k=1}^{\infty}\left(e^{-\frac{k}{n}}+kse^{-\frac{k}{n}}\left(1-e^{-\frac{1}{n}}\right)^q\left(1+e^{-\frac{1}{n}}\right)\right)\cos kx=$$

$$=\left(1-e^{-\frac{1}{n}}\right)^q\left(2\cos\frac{\pi-x}{2}\right)^{-1}\sum_{k=0}^{\infty}\frac{\pi-(2k+1)}{2}\left(\left(1-e^{-\frac{1}{n}}\right)^{1-q}+s\left(1+e^{-\frac{1}{n}}\right)\left(k\left(1-e^{-\frac{1}{n}}\right)-e^{-\frac{1}{n}}\right)\right). \quad (11)$$

Therefore, if both parts of the equality (11) are multiplied by $(2\pi)^{-1}\leq|x|^\alpha\ (0\leq\alpha\leq 2)$ and integrated term by term for all $x\in[-\pi;\pi]$, then we have that

$$\pi^{-1}\int_{-\pi}^{\pi}|x|^\alpha\left(\frac{1}{2}+\sum_{k=1}^{\infty}\left(e^{-\frac{k}{n}}+kse^{-\frac{k}{n}}\left(1-e^{-\frac{1}{n}}\right)\right)\right)\cos kx dx=$$

$$=\pi^{-1}\left(1-e^{-\frac{1}{n}}\right)^q\sum_{k=0}^{\infty}\int_0^{\pi}x^\alpha\left(\cos\frac{\pi-x}{2}\right)^{-1}\cos\frac{\pi-(2k+1)x}{2}dx\times$$

$$\times\left(\left(1-e^{-\frac{1}{n}}\right)^{1-q}+s\left(1+e^{-\frac{1}{n}}\right)\left(k\left(1-e^{-\frac{1}{n}}\right)-e^{-\frac{1}{n}}\right)\right)e^{-\frac{k}{n}} \quad (12)$$

To evaluate the integral in the right-hand side of (12), we will use the equality

$$\int_0^{\pi}x^\alpha\left(\cos\frac{\pi-x}{2}\right)^{-1}\cos\frac{\pi-(2k+1)x}{2}dx=$$

$$=\int_0^{\pi}\left(x^{\alpha-1}-\left(x^\alpha\cos\frac{\pi-x}{2}\right)^{-1}\right)\cos\frac{\pi-(2k+1)x}{2}dx+\int_0^{\pi}x^{\alpha-1}\cos\frac{\pi-(2k+1)x}{2}dx. \quad (13)$$

Since,

$$\int_0^{\pi}\left(x^{\alpha-1}-x^\alpha\left(\cos\frac{\pi-x}{2}\right)^{-1}\right)\cos\frac{\pi-(2k+1)x}{2}dx==O\left(\left(\frac{2}{2k+1}\right)^2\right), \quad (14)$$

$$\int_0^{\pi}x^{\alpha-1}\cos\frac{\pi-(2k+1)x}{2}dx==2^\alpha(\alpha-1)\frac{\Gamma(\alpha-1)}{(2k+1)^\alpha}\cos\frac{\pi(\alpha-1)}{2}+O\left(\left(\frac{2}{2k+1}\right)^2\right), \quad (15)$$

then by combining the relations (14), (15), (13), (11) and (10) we obtain that

$$\Delta(H_*^\alpha;n)=\Gamma(\alpha)\cos\frac{\pi-\alpha}{2}\left(1-e^{-\frac{1}{n}}\right)^q\times$$

$$\times\sum_{k=0}^{\infty}\left(\frac{2}{2k+1}\right)^\alpha e^{-\frac{k}{n}}\left(\left(1-e^{-\frac{1}{n}}\right)^{1-q}++s\left(1+e^{-\frac{1}{n}}\right)\left(k\left(1-e^{-\frac{1}{n}}\right)-e^{-\frac{1}{n}}\right)\right)+$$

$$+O\left(1-e^{-\frac{1}{n}}\right)\sum_{k=0}^{\infty}\left(\frac{2}{2k+1}\right)^2\left(1+s\left(k\left(1-e^{-\frac{1}{n}}\right)-e^{-\frac{1}{n}}\right)\left(1+e^{-\frac{1}{n}}\right)\left(1-e^{-\frac{1}{n}}\right)^{q-1}\right). \quad (16)$$

Let's analyze the right-hand side of the equality (16) as the deviation error of the mathematical model $P_{s,q}(n;\varphi;t)$ of the signal from the real one, described by means of the function $\varphi(\cdot)$ from the generalized Hölder class. To begin with, let's simplify the estimate (16). We can show that (16)

$$\Delta\left(H_*^{\alpha};n\right)=2^{\alpha}\Gamma(\alpha)\cos\frac{\pi-\alpha}{2}\left(1-e^{-\frac{1}{n}}\right)^{q}\left(\left(1-e^{-\frac{1}{n}}\right)^{1-q}-s\left(1+e^{-\frac{1}{n}}\right)e^{-\frac{1}{n}}\right)+$$

$$+O\left(1-e^{-\frac{1}{n}}\right)\left(1-se^{-\frac{1}{n}}\left(1+e^{-\frac{1}{n}}\right)\left(1-e^{-\frac{1}{n}}\right)^{q-1}\right). \tag{17}$$

Now we can visually represent the main term of the deviation error (16) depending on α and different values of the parameters s and q.

Above we considered two cases of parameter values s and q: $s=0, q\geq 1$ and $s=\frac{1}{2}, q=1$. For each of these two cases, we build the graph of the principal error term (17).

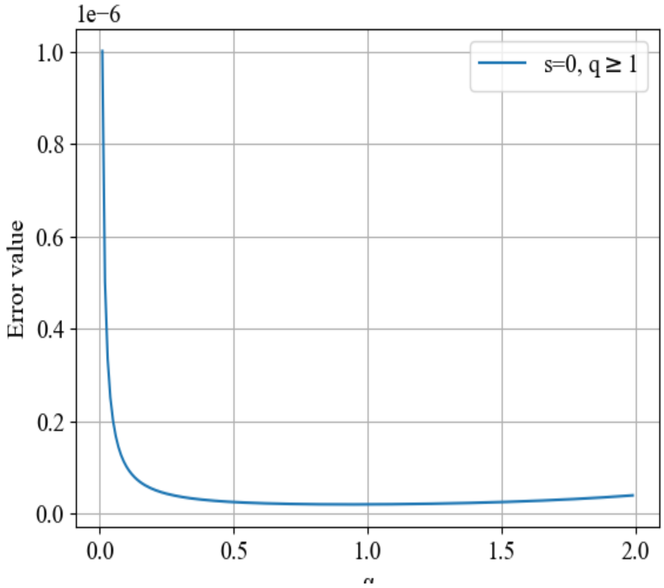

Figure 1: The behavior of the principal error term (17) in the case when $s=0, q\geq 1$

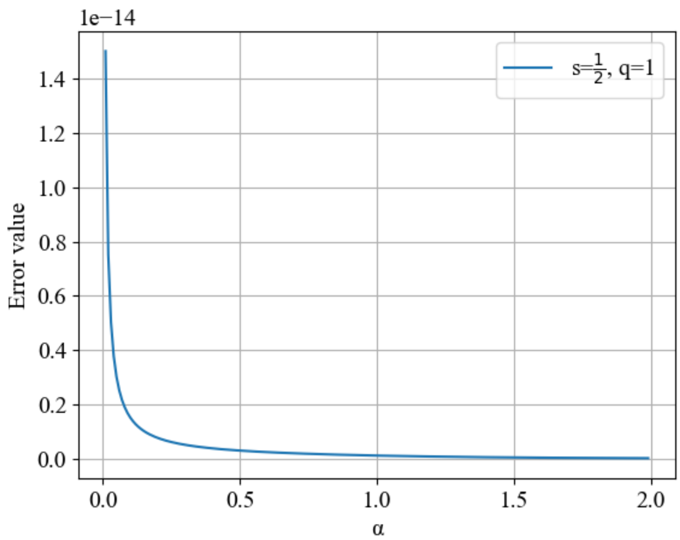

Figure 2: The behavior of the principal error term (17) in the case when $s=1/2, q=1$

As we can see from the above graphs, whatever parameter values s and q, we take, the smoothness of the function that describes the signal is a key aspect affecting the description of the signal using the Poisson-type operator in the case when the signal belongs to the generalized Lipschitz class.

3. Conclusions

In this work, the approximation of signals belonging to the generalized Lipschitz class by Poisson-type operators, as well as the application of this process in information technologies [18-21], have been investigated. The error measure, which is obtained with this approximation, has been derived analytically. With its help, we showed that the quality of the approximation of the investigated signal depends the most significantly on the parameter $\alpha = 2$ describing the smoothness order of the function or, what is the same, the order of its fractional derivative. According to the presented result, the quality of signal approximation by Poisson-type operators is better, the higher the value of the parameter is, confirmed by the graphs presented in the work. The application of the obtained results can be applied in various areas of information technologies, in particular, in network technologies, for example, in computer modelling [22-24], security [25], cybersecurity [26, 27], engineering [28-30], etc. We also have found that the presented results indicate the more optimal approximation of the signal by Poisson-type operators than by the methods presented in foreign sources, for example, in. This clearly shows that Poisson-type operators are effective mathematical tools for solving a number of information technology problems.

4. References

[1] Oleg Barabash, Oleksandr Laptiev, Valentyn Sobchuk, Ivanna Salanda, Yulia Melnychuk, Valerii Lishchyna. Comprehensive Methods of Evaluation of Distance Learning System Functioning. International Journal of Computer Network and Information Security (IJCNIS), Vol. 13, No. 3, June. 2021, pp. 62 – 71. doi: 10.5815/ijcnis.2021.03.06.

[2] O. Barabash, A. Musienko, V. Sobchuk, N. Lukova-Chuiko, O. Svynchuk. “Distribution of Values of Cantor Type Fractal Functions with Specified Restrictions“. Chapter in Book “Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics”. Editors V.A. Sadovnichiy, M.Z. Zgurovsky. Publisher Name: Springer, Cham, Switzerland AG 2021. P. 433–455. doi: 10.1007/978-3-030-50302-4_21.

[3] V.Mukhin, V.Zavgorodnii, O.Barabash, R. Mykolaichuk, Y. Kornaga, A. Zavgo-rodnya, V. Statkevych, “Method of Restoring Parameters of Information Objects in a Unified Information Space Based on Computer Networks,” International Journal of Computer Network and Information Secu-rity (IJCNIS), 2020, vol.12 (2), pp.11–21. doi: 10.5815/ijcnis.2020.02.02.

[4] A. Radosevic, D. Fertonani, T.M. Duman, J.G. Proakis, M. Stojanovic Bounds on the information rate for sparse channels with long memory and i.u.d. inputs (2011) IEEE Transactions on Communications, 59 (12), pp. 3343 – 3352. doi: 10.3390/axioms11040161.

[5] M. C. Ndinechi, N. Onwuchekwa, G. A. Chukwudebe. Algorithm for applying interpolation in digital signal processing systems. International Journal of Natural and Applied Sciences. 2009. P. 114–119. doi: 10.15330/cmp.14.1.223-229.

[6] D. Manolakis, J. Proakis Digital signal processing: Principles, Algorithms and Applications. 3rd ed., New Jersey : Prentice-Hall, International, 1996. 1032 p. doi: 10.1007/s10559-023-00598-8.

[7] L. Jing, M. Wang, Y. Lu, M. Stojanovic, R. Murch Differential orthogonal frequency division multiplexing communication in water pipeline channels (2020) Journal of the Acoustical Society of America, 148 (2), pp. EL130 - EL134. doi: 10.1007/s11253-019-01705-9.

[8] N. Ghaddar, Y.-H. Kim, L.B. Milstein, L. Ma, B.K. Yi Joint Channel Estimation and Coding over Channels with Memory Using Polar Codes (2021) IEEE Transactions on Communications, 69 (10), pp. 6575 – 6589. doi: 10.1007/s10958-022-06190-3.

[9] R. G. Lyons Understanding digital signal processing. 2nd ed. New Jersey : Prentice-Hall, International, 2004. 684 p. doi: 10.15330/cmp.13.3.805-817.

[10] R. Hegazy, J. Kadifa, L. Milstein, P. Cosman Subcarrier Mapping for Underwater Video Transmission over OFDM (2021) IEEE Journal of Oceanic Engineering, 46 (4), pp. 1408 – 1423. doi: 10.1007/s10958-023-06603-x.
[11] S. Brkic, P. Ivanis, B. Vasic Guaranteed error correction of faulty bit-flipping decoders under data-dependent gate failures (2016) IEEE International Symposium on Information Theory - Proceedings, 2016-August, art. pp. 1561 – 1565. doi: 10.1007/s10559-022-00457-y.
[12] R.D. Hegazy, Q. Song, A. Vosoughi, L.B. Milstein, P.C. Cosman Joint Source-Channel Rate-Distortion Optimization for Wireless Video Transmission (2020) Conference Record - Asilomar Conference on Signals, Systems and Computers, 2020-November, pp. 693 – 698. doi: 10.1134/S0001434618050231.
[13] S. Rezaei Aghdam, A. Nooraiepour, T.M. Duman An Overview of Physical Layer Security With Finite-Alphabet Signaling (2019) IEEE Communications Surveys and Tutorials, 21 (2), pp. 1829 – 1850. doi: 10.1007/s11253-022-02084-4.
[14] A. Tadayon, M. Stojanovic Iterative Sparse Channel Estimation and Spatial Correlation Learning for Multichannel Acoustic OFDM Systems (2019) IEEE Journal of Oceanic Engineering, 44 (4), pp. 820 – 836. doi: 10.1007/s10559-023-00530-0
[15] R. Schafer, L. Rabiner A digital signal processing approach to interpolation. Proceedings of IEEE. 1973. Vol. 6, no. 61. P. 692–702. doi: 10.15330/cmp.15.1.286-294.
[16] M. Kazemi, H. Aghaeinia, T.M. Duman Discrete-Phase Constant Envelope Precoding for Massive MIMO Systems (2017) IEEE Transactions on Communications, 65 (5), , pp. 2011 – 2021. doi: 10.1007/s10559-023-00579-x.
[17] I.A. Tomić, Đ. Lukić, M. Davidović, D. Drajić ,D., Ivaniš P. Statistical Analysis of CQI Reporting and MIMO Utilization for Downlink Scheduling in Live LTE Mobile Network (2020) Telfor Journal, 12 (1), pp. 8 – 12. doi: 10.1007/s10559-023-00561-7.
[18] L. E. Franks Signal theory. Stroutsburg : Dowlen & Culver Inc., 1981. 323 p. doi: 10.12928/telkomnika.v17i4.12168.
[19] S. V. Vaseghi Advanced digital signal processing and noise reduction. 3rd ed. Chichester : John Wiley & Sons Ltd., 2006. 479 p. doi: 10.17762/ijcnis.v13i2.5008.
[20] C. L. Byrne Signal processing A mathematical approach. 2nd ed. Massachusetts : CRC Press, 2015. 436 p. doi: 10.20535/SRIT.2308-8893.2021.1.03.
[21] A. B. Carlson, P. B. Crilly COMMUNICATION SYSTEMS an introduction to signals and noise in electrical communication. 5th ed. New York : McGraw-Hill, 2010. 953 p. ISSN 16130073.
[22] N. Karakoc, T.M. Duman Rate Selection for Wireless Random Access Networks over Block Fading Channels (2020) IEEE Transactions on Communications, 68 (3), pp. 1604 – 1616. doi: 10.30534/ijatcse/2019/26862019.
[23] M. Dabirnia, A. Korhan Tanc, S. Sharifi, T.M. Duman Code Design for Discrete Memoryless Interference Channels (2018) IEEE Transactions on Communications, 66 (8), pp. 3368 – 3380. doi: 10.1109/apuavd.2017.8308787.
[24] Z. Hu, V. Mukhin, Y. Kornaga, O. Barabash, O. Herasymenko Analytical Assessment of Security Level of Distributed and Scalable Computer Systems. International Journal of Intelligent Systems and Applications, 2016. Vol. 8. No. 12. Hong Kong: MECS Publisher, 2016. P. 57–64. doi: 10.5815/ijisa.2016.12.07.
[25] Yu.I. Kharkevych, "Approximation Theory and Related Applications," Axioms, 2022, vol. 11 (12): 736. doi: 10.3390/axioms11120736.
[26] R. W. Hamming Digital filters. New Jersey : Prentice-Hall, International, 1977. 224 p.
[27] J. Boiko, I. Pyatin, O. Eromenko, O. Barabash (2020) Methodology for Assessing Synchronization Conditions in Telecommunication Devices. Advances in Science, Technology and Engineering Systems Journal (ASTESJ), 2020, Vol. 5, No 2. ISSN: 2415-6698. P. 320–327. doi: 10.25046/aj050242.
[28] A. Dulny, P. Heinisch, A. Hotho, A. Krause GrINd: Grid Interpolation Network for Scattered Observations/ Würzburg. 19 p. (Preprint. University Würzburg).
[29] JI. Cen, X. Chen, M. Xu, Q. Zou Deep finite volume method for high-dimentional partial differential equations. Guangzhou. 16 p. (Preprint)
[30] X. Zhang, J. Helwig, Y. Lin, Y. Xie, C. Fu, S. Wojtowytsch, Sh. Ji SineNet: Learning teporal dynamics in time-dependent partial differential equations. Texas. 42 p.