Algebraic (non) Relations Among Polyzetas

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Abstract

Two confluent rewriting systems in noncommutatives polynomials are constructed using the equations allowing the identification of the local coordinates (of second kind) of the graphs of the ζ polymorphism as being (shuffle or quasi-shuffle) characters and bridging two algebraic structures of polyzetas.

In each system, the left side of each rewriting rule corresponds to the leading monomial of the associated homogeneous in weight polynomial while the right side is canonically represented on the algebra generated by irreducible terms which encode an algebraic basis of the algebra of polyzetas.

These polynomials are totally lexicographically ordered and generate the kernels of the ζ polymorphism meaning that the free algebra of polyzetas is graded and the irreducible polyzetas are transcendent numbers, algebraically independent.

Keywords

Polylogarithms, Hamonic Sums, Polyzetas, Rewritting Systems

1. Introduction

For any $r \geq 1$ and $(s_1, \ldots, s_r) \in \mathbb{N}_{\geq 1}$, for any $z \in \mathbb{C} \setminus \widetilde{\{0,1\}}$ and $n \geq 1$, let

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>\ldots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_r^{s_r}} \quad \text{and} \quad \mathrm{H}_{s_1,\ldots,s_r}(n) := \sum_{n_1>\ldots>n_r>0}^{n} \frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{1}$$

which are respectively called *polylogarithm* and *harmonic sum*.

Let $\mathcal{H}_r$ be $\{(s_1, \ldots, s_r) \in \mathbb{N}^r_{\geq 1}, s_1 > 1\}$. Then, for any $(s_1, \ldots, s_r)$ belonging to $\mathcal{H}_r$, by a Abel's theorem, the following limits exist and are called *polyzetas*[1] [9, 10]

$$\zeta(s_1, \ldots, s_r) := \lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{n\to+\infty} \mathrm{H}_{s_1,\ldots,s_r}(n) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1}\ldots n_r^{-s_r}. \tag{2}$$

Euler earlier studied polyzetas, in particular $\{\zeta(s_1, s_2)\}^{r\geq 1}_{s_1>1,s_2\geq 1}$ in classic analysis. He stated that $\zeta(6, 2)$ can not be expressed on $\zeta(2), \ldots, \zeta(8)$ and proved [6]

$$\zeta(2, 1) = \zeta(3) \quad \text{and} \quad \zeta(s, 1) = \frac{1}{2}\Big(s\zeta(s + 1) - \sum_{j=1}^{s-2} \zeta(j + 1)\zeta(s - j)\Big), s > 1. \tag{3}$$

The $\{\zeta(s_1, \ldots, s_r)\}^{r\geq 1}_{s_1>1,s_2,\ldots,s_r\geq 1}$ are also called *multi zeta values* (MZV for short) [13] or *Euler-Zagier sums* [2] and the numbers r and $s_1 + \ldots + s_r$ are, respectively, *depth* and *weight* of $\zeta(s_1, \ldots, s_r)$. One can also found in their biographies some recent applications of these special values in algebraic geometry, Diophantine equations, knots invariants of Vassiliev-Kontsevich, modular forms, quantum electrodynamic, ….

Many new linear relations for polyzetas are detected using LLL type algorithms in high performance computing and the truncations of $\{\zeta(s_1, \ldots, s_r)\}^{r\geq 1}_{s_1>1,s_2,\ldots,s_r\geq 1}$, *i.e.* $\{\mathrm{H}_{s_1,\ldots,s_r}(n)\}^{r\geq 1}_{s_1>1,s_2,\ldots,s_r\geq 1}$ [1, 2]. In this approach, the main problem is to detect with near certainty which polyzetas can not be

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[1]Polyzeta is the contraction of polymorphism and of zeta (see (5)–(6) below).

expressed on $\{\zeta(2),\ldots,\zeta(s+k)\}$ and are qualified as *new constants* (as for Euler's $\zeta(6,2)$) [2]. Such polyzetas could be $\mathbb{Q}$-algebraically independent on these zeta values (see Example 1 below) and the polyzetas could be transcendent numbers (see [9, 10] for proof). Checking linear relations among $\{\zeta(s_1,\ldots,s_r)\}^{r\geq 1}_{\substack{s_1>1,s_2,\ldots,s_r\geq 1\\ 2\leq s_1+\ldots+s_r\leq 12}}$, Zagier stated that the $\mathbb{Q}$-module generated by MZV is graded (see [9, 10] for proof) and guessed (see [7, 8, 12] for other algebraic checks)

Conjecture 1 ([13]). *Let* $d_k := \dim \mathcal{Z}_k$ *and* $\mathcal{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\zeta(w)\}^{r\geq 1}_{\substack{s_1>1,s_2,\ldots,s_r\geq 1\\ s_1+\ldots+s_r=k}}$, *for* $k\geq 1$. *Then* $d_1=0, d_2=d_3=1$ *and* $d_k=d_{k-3}+d_{k-2}$, *for* $k\geq 4$.

Studying Conjecture 1, in continuation with [3, 5] by a symbolic approach, this work provides more explanations and consequences regarding the algorithm **LocalCoordinateIdentification**, partially implemented in [3] and briefly described in [4].

It applies an Abel like theorem concerning the generating series of $\{\mathrm{H}_{s_1,\ldots,s_r}\}^{r\geq 1}_{s_1,\ldots,s_r\geq 1}$ (resp. $\{\mathrm{Li}_{s_1,\ldots,s_r}\}^{r\geq 1}_{s_1,\ldots,s_r\geq 1}$) [5], over the alphabet $Y=\{y_k\}_{k\geq 1}$ (resp. $X=\{x_0,x_1\}$) generating the free monoid $(Y^*,1_{Y^*})$ (resp. $(X^*,1_{X^*})$) with respect to the concatenation (denoted by conc and omitted when there is no ambiguity), the set of Lyndon words $\mathcal{L}ynY$ (resp. $\mathcal{L}ynX$) and the set of polynomials, $\mathbb{Q}\langle Y\rangle$ (resp. $\mathbb{Q}\langle X\rangle$). This theorem exploits the indexations of polylogarithms and harmonic sums in (1) by words, *i.e.* [9, 10]

$$\mathrm{Li}_{x_0^r}(z)=\log^r(z)/r!,\quad \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}=\mathrm{Li}_{s_1,\ldots,s_r},\quad \mathrm{H}_{y_{s_1}\ldots y_{s_r}}=\mathrm{H}_{s_1,\ldots,s_r}. \tag{4}$$

It follows that the isomorphism of algebras $\mathrm{H}_{\bullet}:(\mathbb{Q}\langle Y\rangle, ⧢)\longrightarrow(\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*},\times)$ (resp. $\mathrm{Li}_{\bullet}:(\mathbb{Q}\langle X\rangle,\sqcup\!\sqcup)\longrightarrow(\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*},\times)$), mapping u (resp. v) to H_u (resp. Li_v), induce the following *surjective* polymorphism [9, 10]

$$\zeta:\begin{array}{c}(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1,\sqcup\!\sqcup,1_{X^*})\\ (\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle, ⧢,1_{Y^*})\end{array}\longrightarrow\!\!\!\!\rightarrow(\mathcal{Z},\times,1), \tag{5}$$

$$\begin{array}{c}x_0x_1^{s_1-1}\ldots x_0x_1^{s_k-1}\\ y_{s_1}\ldots y_{s_k}\end{array}\longmapsto\zeta(s_1,\ldots,s_r), \tag{6}$$

where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra generated by polyzetas (not linearly free [13]) and the product ⧢ (resp. $\sqcup\!\sqcup$) is defined, for any $u,v,w\in Y^*$ (resp. X^*) and $y_i,y_j\in Y$ (resp. $x,y\in X$), by

$$w ⧢ 1_{Y^*}=1_{Y^*} ⧢ w=w \text{ and } y_iu ⧢ y_jv=y_i(u ⧢ y_jv)+y_j(y_iu ⧢ v)+y_{i+j}(u ⧢ v), \tag{7}$$

$$(\text{resp. } w\sqcup\!\sqcup 1_{X^*}=1_{X^*}\sqcup\!\sqcup w=w \text{ and } xu\sqcup\!\sqcup yv=x(u\sqcup\!\sqcup yv)+y(xu\sqcup\!\sqcup v)). \tag{8}$$

The graphs of the ζ polymorphism in (5)–(6) are expressed as ⧢ (resp. $\sqcup\!\sqcup$)-group like series as follows [9, 10]

$$Z_\gamma=e^{\gamma y_1}\prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow}e^{\zeta(\Sigma_l)\Pi_l}\quad\text{and}\quad Z_{\sqcup\!\sqcup}=\prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow}e^{\zeta(S_l)P_l}, \tag{9}$$

where $\{\Pi_w\}_{w\in Y^*}$ (resp. $\{P_w\}_{w\in X^*}$) is the PBW-Lyndon basis (of the Lie polynomilas $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{P_l\}_{l\in\mathcal{L}ynX}$) basis) in duality with $\{\Sigma_w\}_{w\in X^*}$ (resp. $\{S_w\}_{w\in X^*}$) (containing the basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ (resp. $\{S_l\}_{l\in\mathcal{L}ynY}$)), on the ⧢ (resp. $\sqcup\!\sqcup$)-bialgebra [9, 10]. Finally, the identification of their local coordinates (of second kind in the group of group like series) in the equations bridging the lagebraic structures of polyzetas, *i.e.* [5]

$$Z_\gamma=e^{\gamma y_1-\sum_{k\geq 2}\zeta(k)(-y_1)^k/k}\pi_Y Z_{\sqcup\!\sqcup}\quad\text{and}\quad Z_{\sqcup\!\sqcup}=e^{-\gamma x_1+\sum_{k\geq 2}\zeta(k)(-x_1)^k/k}\pi_X Z_\gamma, \tag{10}$$

provides the algebraic relations among $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ (resp. $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$), independent on γ, leading to the algebraic bases for $\mathrm{Im}\,\zeta$ and the homogenous polynomials generating $\ker\zeta$ [9, 10] (see [3] for examples), with[2] the morphism of monoids $\pi_Y:X^*x_1\longrightarrow Y^*$ (resp. $\pi_X:Y^*\longrightarrow X^*x_1$) maps y_k to $x_0^{k-1}x_1$ (resp. $x_0^{k-1}x_1$ to y_k).

[2]There are one-to-one correspondences over the above monoids and that generated by $\mathbb{N}_{\geq 1}$, *i.e* $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1\rightleftharpoons_{\pi_X}^{\pi_Y}y_{s_1}\ldots y_{s_r}\in Y^*\leftrightarrow(s_1,\ldots,s_r)\in\mathbb{N}^*_{\geq 1}$

2. Rewriting among $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ and among $\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$

For convenience, $\mathcal{X}$ denotes X or Y and if $\mathcal{X}=X$ then $\mathrm{gDIV}=X$ and $\mathrm{CONV}=x_0X^*x_1$ else $\mathrm{gDIV}=\{y_1\}$ and $\mathrm{CONV}=(Y\setminus\{y_1\})Y^*$. It follows that $\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}\subset\mathrm{CONV}$.

Expressing, *i.e* replacing "$=$" by "$\to$", the relations among polyzetas in [3] become the rewriting rules among polyzetas and yield the following increasing sets of irreducible polyzetas (see Example 1 below)

$$\mathcal{Z}_{irr}^{\mathcal{X},\le 2}\subset\cdots\subset\mathcal{Z}_{irr}^{\mathcal{X},\le p}\subset\cdots\subset\mathcal{Z}_{irr}^{\mathcal{X},\infty} \tag{11}$$

and their images by a section of ζ (see Example 2 below)

$$\mathcal{L}_{irr}^{\mathcal{X},\le 2}\subset\cdots\subset\mathcal{L}_{irr}^{\mathcal{X},\le p}\subset\cdots\subset\mathcal{L}_{irr}^{\mathcal{X},\infty}, \tag{12}$$

such that the following restriction is an isomorphism of algebras [9, 10]

$$\zeta:\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]\longrightarrow\mathbb{Q}[\mathcal{Z}_{irr}^{\mathcal{X},\infty}]=\mathcal{Z}. \tag{13}$$

Note that one also has

$$\mathcal{L}_{irr}^{\mathcal{X},\infty}=\bigcup_{p\ge 2}\mathcal{L}_{irr}^{\mathcal{X},\le p}\quad\text{and}\quad\mathcal{L}_{irr}^{\mathcal{X},\infty}=\bigcup_{p\ge 2}\mathcal{L}_{irr}^{\mathcal{X},\le p}. \tag{14}$$

Now, let us describe the algorithm **LocalCoordinateIdentification** below which brings aditional results to [3]. It provides the rewriting systems $(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1,\mathcal{R}_{irr}^X)$ and $(\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle,\mathcal{R}_{irr}^Y)$ which are without critical pairs, noetherian, confluent and precisely contains the above sets (see (11)–(12)) and, on the other hand, the set of homogenous in weight polynomials, belonging to $\mathbb{Q}[\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}]$, which are image by a section of the surjective ζ polymorphism from $\{\zeta(Q_l)=0\}_{l\in\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}}$. It is denoted by $\mathcal{Q}_\mathcal{X}$:

$$\mathcal{Q}_\mathcal{X}=\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}} \tag{15}$$

and generates the shuffle or quasi-shuffle ideal $\mathcal{R}_\mathcal{X}$ inside $\ker\zeta$ as follows

$$\mathcal{R}_\mathcal{X}:=\mathrm{span}_\mathbb{Q}\mathcal{Q}_\mathcal{X}\subseteq\ker\zeta. \tag{16}$$

For any $p\ge 2$ and $l\in\mathcal{L}yn^p\mathcal{X}:=\{l\in\mathcal{L}yn\mathcal{X}|(l)=p\}$, any nonzero homogenous in weight polynomial (belonging to $\mathcal{Q}_\mathcal{X}$) $Q_l=\Sigma_l-\Upsilon_l$ (resp. $Q_l=S_l-U_l$) is led by Σ_l (resp. S_l) being transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\le p}]$ and $\Upsilon_l=Q_l-\Sigma_l$ (resp. $U_l=Q_l-S_l$) is canonically represented in $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\le p}]$. Then let $\Sigma_l\to\Upsilon_l$ and $S_l\to U_l$ be the rewriting rules, respectively, of

$$\mathcal{R}_{irr}^Y:=\{\Sigma_l\to\Upsilon_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}\quad\text{and}\quad\mathcal{R}_{irr}^X:=\{S_l\to U_l\}_{l\in\mathcal{L}ynX\setminus X}. \tag{17}$$

On the other hand, the following assertions are equivalent (see Example 2 below)

1. $Q_l=0$
2. $\Sigma_l\in\mathcal{L}_{irr}^{Y,\le p}$ (resp. $S_l\in\mathcal{L}_{irr}^{X,\le p}$),
3. $\Sigma_l\to\Sigma_l$ (resp. $S_l\to S_l$).

In the other words, the ordering over $\mathcal{L}yn\mathcal{X}$ induces the ordering over $\mathcal{L}_{irr}^{\mathcal{X},\infty},\mathcal{R}_\mathcal{X},\mathcal{R}_{irr}^\mathcal{X}$ and, in the systems $(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1,\mathcal{R}_{irr}^X)$ and $(\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle,\mathcal{R}_{irr}^Y)$,

1. each irreducible term, in $\mathcal{L}_{irr}^{\mathcal{X},\infty}$, is an element of the algebraic basis $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ of $(\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle,\ \uplus\ )$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynX\setminus X}$ of $(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1,\sqcup\!\sqcup)$),
2. each rewriting rule, in $\mathcal{R}_{irr}^\mathcal{X}$, admits the left side being transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$ and the right side being canonically represented in $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$. The difference of these two sides belongs to the ordered ideal $\mathcal{R}_\mathcal{X}$ of $\mathbb{Q}[\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}]$.

LocalCoordinateIdentification

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$\mathcal{Z}_{irr}^{\mathcal{X},\infty}$ := {}; $\mathcal{L}_{irr}^{\mathcal{X},\infty}$ := {}; $\mathcal{R}_{irr}^{\mathcal{X}}$ := {}; $\mathcal{Q}_{\mathcal{X}}$ := {};
for $p$ ranges in $2,\ldots,\infty$ do
    for $l$ ranges in the totally ordered $\mathcal{L}yn^p\mathcal{X}$ do
      identify $\langle Z_\gamma|\Pi_l\rangle$ in $Z_\gamma = B(y_1)\pi_Y Z_{\sqcup\!\sqcup}$ and $\langle Z_{\sqcup\!\sqcup}|P_l\rangle$ in $Z_{\sqcup\!\sqcup} = B(x_1)^{-1}\pi_X Z_\gamma$;
      by elimination, obtain equations on $\{\zeta(\Sigma_{l'})\}_{l'\in\mathcal{L}yn^pY, l'\preceq l}$ and on $\{\zeta(S_{l'})\}_{l'\in\mathcal{L}yn^pX, l'\preceq l}$;
      express[3] the equations led by $\zeta(\Sigma_l)$ and by $\zeta(S_l)$ as rewriting rules;
      if $\zeta(\Sigma_l)\to\zeta(\Sigma_l)$ then $\mathcal{Z}_{irr}^{Y,\infty} := \mathcal{Z}_{irr}^{Y,\infty}\cup\{\zeta(\Sigma_l)\}$ and $\mathcal{L}_{irr}^{Y,\infty} := \mathcal{L}_{irr}^{Y,\infty}\cup\{\Sigma_l\}$
        else $\mathcal{R}_{irr}^{Y} := \mathcal{R}_{irr}^{Y}\cup\{\Sigma_l\to\Upsilon_l\}$ and $\mathcal{Q}_Y := \mathcal{Q}_Y\cup\{\Sigma_l-\Upsilon_l\}$;
      if $\zeta(S_l)\to\zeta(S_l)$ then $\mathcal{Z}_{irr}^{X,\infty} := \mathcal{Z}_{irr}^{X,\infty}\cup\{\zeta(S_l)\}$ and $\mathcal{L}_{irr}^{X,\infty} := \mathcal{L}_{irr}^{X,\infty}\cup\{S_l\}$
        else $\mathcal{R}_{irr}^{X} := \mathcal{R}_{irr}^{X}\cup\{S_l\to U_l\}$ and $\mathcal{Q}_X := \mathcal{Q}_X\cup\{S_l-U_l\}$
    end_for
end_for
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With the notations introduced in (11)–(17), on also has[4]

Proposition 1 ([9, 10]). 1. $\mathcal{R}_{\mathcal{X}} = \ker\zeta$ *and* $\mathbb{Q}[\mathcal{Z}_{irr}^{\mathcal{X},\infty}] = \mathcal{Z} = \mathrm{Im}\,\zeta$.

2. $\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus X}] = \mathcal{R}_X \oplus \mathbb{Q}[\mathcal{L}_{irr}^{X,\infty}]$ *and* $\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}] = \mathcal{R}_Y \oplus \mathbb{Q}[\mathcal{L}_{irr}^{Y,\infty}]$.

Proof –

1. Let $Q\in\ker\zeta, \langle Q|1_{\mathcal{X}^*}\rangle = 0$. Then $Q = Q_1 + Q_2$ (with $Q_2\in\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$ and $Q_1\in\mathcal{R}_{\mathcal{X}}$). Hence, decomposing in $\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$) and reducing by $\mathcal{R}_{irr}^{\mathcal{X}}$, it follows that $Q \equiv_{\mathcal{R}_{irr}^{\mathcal{X}}} Q_1\in\mathcal{R}_{\mathcal{X}}$ and then the expected result.
   Let $w\in$ CONV. Decomposing in $\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$) and reducing by $\mathcal{R}_{irr}^{\mathcal{X}}$, $w\in\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$. Applying (13) and (5)–(6), $\zeta(w)\in\mathbb{Q}[\mathcal{Z}_{irr}^{\mathcal{X},\infty}] = \mathcal{Z}$ and $\mathcal{Z} = \mathrm{Im}\,\zeta$. Extending by linearrity, it follows the expected result.
2. For any $w\in$ CONV, decomposing in $\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$) and reducing by $\mathcal{R}_{irr}^{\mathcal{X}}$, $\zeta(w)\in\mathbb{Q}[\mathcal{Z}_{irr}^{\mathcal{X},\infty}]$. By linearity, if $P\in\mathbb{Q}[\{S_l\}_{l\in\mathcal{L}ynX\setminus X}]$ (resp. $\mathbb{Q}[\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}]$) and $P\notin\ker\zeta\supseteq\mathcal{R}_{\mathcal{X}}$ then $\zeta(P)\in\mathbb{Q}[\mathcal{Z}_{irr}^{\mathcal{X},\infty}]$.
   On the other hand, if $Q\in\mathcal{R}_{\mathcal{X}}\cap\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$ then, by (16), $\zeta(Q) = 0$ and then, by (13), $Q = 0$ yielding the expected result.

□

Theorem 1 ([9, 10]). *The* $\mathbb{Q}$*-algebra* $\mathcal{Z}$ *is freely generated by* $\mathcal{Z}_{irr}^{\mathcal{X},\infty}$ *and* $\mathcal{Z} = \mathbb{Q}1\oplus\bigoplus_{k\ge2}\mathcal{Z}_k$.

Proof – By (13) and Proposition 1, $\mathcal{Z}$ is freely generated by $\mathcal{Z}_{irr}^{\mathcal{X},\infty}$ and $\ker\zeta$, being generated by the homogenous in weight polynomials $\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}}$, is graded. With the notations in Conjecture 1, being isomorphic to $\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta$ and to $\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1/\ker\zeta$, $\mathcal{Z}$ is also graded. □

Corollary 1 ([9, 10]). *Let* $P\in\mathcal{L}_{irr}^{\mathcal{X},\infty}$*. Then* $\zeta(P)$ *is a transcendent number.*

Proof – Let $P\in\mathbb{Q}\langle\mathcal{X}\rangle$ and $P\notin\ker\zeta$, being homogenous in weight, or $P\in$ CONV. Since $\mathcal{Z}_k\mathcal{Z}_{k'}\subset\mathcal{Z}_{k+k'}$ $(k,k'\ge1)$ then each monomial $(\zeta(P))^k$ $(k\ge1)$ is of different weight and then, by Theorem 1, $\zeta(P)$ could not satisfy, over $\mathbb{Q}$, an algebraic equation $T^k + a_{k-1}T^{k-1}+\ldots = 0$ meaning that $\zeta(P)$ is a transcendent number. Since any $P\in\mathcal{L}_{irr}^{\mathcal{X},\infty}$ is homogenous in weight then it follows the expected result. □

[3] This step and the following ones are not yet been achieved by the implementation in [3].
[4] See also [11] for further information.

Example 1 (irreducible polyzetas, [3]).

$$
\begin{aligned}
\mathcal{Z}_{irr}^{X,\le 12} &= \{\zeta(S_{x_0x_1}), \zeta(S_{x_0^2x_1}), \zeta(S_{x_0^4x_1}), \zeta(S_{x_0^6x_1}), \zeta(S_{x_0x_1^2x_0x_1^4}), \zeta(S_{x_0^8x_1}),\\
&\quad \zeta(S_{x_0x_1^2x_0x_1^6}), \zeta(S_{x_0^{10}x_1}), \zeta(S_{x_0x_1^3x_0x_1^7}), \zeta(S_{x_0x_1^2x_0x_1^8}), \zeta(S_{x_0x_1^4x_0x_1^6})\}.\\
\mathcal{Z}_{irr}^{Y,\le 12} &= \{\zeta(\Sigma_{y_2}), \zeta(\Sigma_{y_3}), \zeta(\Sigma_{y_5}), \zeta(\Sigma_{y_7}), \zeta(\Sigma_{y_3y_1^5}), \zeta(\Sigma_{y_9}), \zeta(\Sigma_{y_3y_1^7}),\\
&\quad \zeta(\Sigma_{y_{11}}), \zeta(\Sigma_{y_2y_1^9}), \zeta(\Sigma_{y_3y_1^9}), \zeta(\Sigma_{y_2^2y_1^8})\}.
\end{aligned}
$$

Example 2 (Rewriting on $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ and $\{S_l\}_{l\in\mathcal{L}ynX\setminus X}$, irreducible terms).

	Rewriting on $\{\Sigma_l\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$	*Rewriting on* $\{S_l\}_{\mathcal{L}ynX\setminus X}$
3	$\Sigma_{y_2y_1} \to \frac{3}{2}\Sigma_{y_3}$	$S_{x_0x_1^2} \to S_{x_0^2x_1}$
4	$\Sigma_{y_4} \to \frac{2}{5}\Sigma_{y_2}^{\uplus 2}$	$S_{x_0^3x_1} \to \frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_3y_1} \to \frac{3}{10}\Sigma_{y_2}^{\uplus 2}$	$S_{x_0^2x_1^2} \to \frac{1}{10}S_{x_0x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_2y_1^2} \to \frac{2}{3}\Sigma_{y_2}^{\uplus 2}$	$S_{x_0x_1^3} \to \frac{2}{5}S_{x_0x_1}^{\sqcup\!\sqcup 2}$
5	$\Sigma_{y_3y_2} \to 3\Sigma_{y_3}\Sigma_{y_2} - 5\Sigma_{y_5}$	$S_{x_0^3x_1^2} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_4y_1} \to -\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{2}\Sigma_{y_5}$	$S_{x_0^2x_1x_0x_1} \to -\frac{3}{2}S_{x_0^4x_1} + S_{x_0^2x_1}S_{x_0x_1}$
	$\Sigma_{y_2^2y_1} \to \frac{3}{2}\Sigma_{y_3}\Sigma_{y_2} - \frac{25}{12}\Sigma_{y_5}$	$S_{x_0^2x_1^3} \to -S_{x_0^2x_1}S_{x_0x_1} + 2S_{x_0^4x_1}$
	$\Sigma_{y_3y_1^2} \to \frac{5}{12}\Sigma_{y_5}$	$S_{x_0x_1x_0x_1^2} \to \frac{1}{2}S_{x_0^4x_1}$
	$\Sigma_{y_2y_1^3} \to \frac{1}{4}\Sigma_{y_3}\Sigma_{y_2} + \frac{5}{4}\Sigma_{y_5}$	$S_{x_0x_1^4} \to S_{x_0^4x_1}$
6	$\Sigma_{y_6} \to \frac{8}{35}\Sigma_{y_2}^{\uplus 3}$	$S_{x_0^5x_1} \to \frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}$
	$\Sigma_{y_4y_2} \to \Sigma_{y_3}^{\uplus 2} - \frac{4}{21}\Sigma_{y_2}^{\uplus 3}$	$S_{x_0^4x_1^2} \to \frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3} - \frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_5y_1} \to \frac{2}{7}\Sigma_{y_2}^{\uplus 3} - \frac{1}{2}\Sigma_{y_3}^{\uplus 2}$	$S_{x_0^3x_1x_0x_1} \to \frac{4}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3}$
	$\Sigma_{y_3y_1y_2} \to -\frac{17}{30}\Sigma_{y_2}^{\uplus 3} + \frac{9}{4}\Sigma_{y_3}^{\uplus 2}$	$S_{x_0^3x_1^3} \to \frac{23}{70}S_{x_0x_1}^{\sqcup\!\sqcup 3} - S_{x_0^2x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_3y_2y_1} \to 3\Sigma_{y_3}^{\uplus 2} - \frac{9}{10}\Sigma_{y_2}^{\uplus 3}$	$S_{x_0^2x_1x_0x_1^2} \to \frac{2}{105}S_{x_0x_1}^{\sqcup\!\sqcup 3}$
	$\Sigma_{y_4y_1^2} \to \frac{3}{10}\Sigma_{y_2}^{\uplus 3} - \frac{3}{4}\Sigma_{y_3}^{\uplus 2}$	$S_{x_0^2x_1^2x_0x_1} \to -\frac{89}{210}S_{x_0x_1}^{\sqcup\!\sqcup 3} + \frac{3}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_2^2y_1^2} \to \frac{11}{63}\Sigma_{y_2}^{\uplus 3} - \frac{1}{4}\Sigma_{y_3}^{\uplus 2}$	$S_{x_0^2x_1^4} \to \frac{6}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3} - \frac{1}{2}S_{x_0^2x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_3y_1^3} \to \frac{1}{21}\Sigma_{y_2}^{\uplus 3}$	$S_{x_0x_1x_0x_1^3} \to \frac{8}{21}S_{x_0x_1}^{\sqcup\!\sqcup 3} - S_{x_0^2x_1}^{\sqcup\!\sqcup 2}$
	$\Sigma_{y_2y_1^4} \to \frac{17}{50}\Sigma_{y_2}^{\uplus 3} + \frac{3}{16}\Sigma_{y_3}^{\uplus 2}$	$S_{x_0x_1^5} \to \frac{8}{35}S_{x_0x_1}^{\sqcup\!\sqcup 3}$

$$
\begin{aligned}
\mathcal{L}_{irr}^{X,\le 12} &= \{S_{x_0x_1}, S_{x_0^2x_1}, S_{x_0^4x_1}, S_{x_0^6x_1}, S_{x_0x_1^2x_0x_1^4}, S_{x_0^8x_1},\\
&\quad S_{x_0x_1^2x_0x_1^6}, S_{x_0^{10}x_1}, S_{x_0x_1^3x_0x_1^7}, S_{x_0x_1^2x_0x_1^8}, S_{x_0x_1^4x_0x_1^6}\}.\\
\mathcal{L}_{irr}^{Y,\le 12} &= \{\Sigma_{y_2}, \Sigma_{y_3}, \Sigma_{y_5}, \Sigma_{y_7}, \Sigma_{y_3y_1^5}, \Sigma_{y_9}, \Sigma_{y_3y_1^7}, \Sigma_{y_{11}}, \Sigma_{y_2y_1^9}, \Sigma_{y_3y_1^9}, \Sigma_{y_2^2y_1^8}\}.
\end{aligned}
$$

3. Conclusion

Thanks to a Abel like theorem and the equation bridging the algebraic structures of the $\mathbb{Q}$-algebra $\mathcal{Z}$ generated by the polyzetas [5], the algorithm **LocalCoordinateIdentification** provides the algebraic relations[5] among the local coordinates, of second kind on the groups of group-like series, of the noncommutative series $Z_{\sqcup\!\sqcup}$ (*i.e.* $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$) and $Z_{\uplus}$ (*i.e.* $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$). These relations constitute two confluent rewriting systems in which the irreducible terms, belonging to $\mathcal{Z}_{irr}^{\mathcal{X},\infty}$, represent the algebraic generators for $\mathcal{Z}$ and, on the other hand, the $\sqcup\!\sqcup$-ideal $\mathcal{R}_X$ and the $\uplus$-ideal $\mathcal{R}_Y$ represent the kernels of the ζ polymorphism (Proposition 1). These ideals are generated by the polynomials, totally ordered and homogenous in weight, $\{Q_l\}_{l\in\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}}$ and are interpreted as the confluent rewriting systems in which the irreducible terms belong to $\mathcal{L}_{irr}^{\mathcal{X},\infty}$ and, in each rewriting rule of $\mathcal{R}_{irr}^{\mathcal{X}}$, the left side

[5]These are different from those among $\{\zeta(l)\}_{l\in\mathcal{L}yn\mathcal{X}\setminus\mathrm{gDIV}}$ obtained by "double shuffle relations" [8], for which Conjecture 1 holds, up to weight 10.

is the leading monomial of $Q_l, l \in \mathcal{L}yn\mathcal{X} \setminus$ gDIV and is transcendent over $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$ while the right side is canonically represented on $\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}]$. It follows that $\zeta(\mathbb{Q}[\mathcal{L}_{irr}^{\mathcal{X},\infty}])$, *i.e.* $\mathcal{Z}$, as being isomorphic to $\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1/\mathcal{R}_X$ and to $\mathbb{Q}1_{Y^*} \oplus (Y \setminus \{y_1\})\mathbb{Q}\langle Y\rangle/\mathcal{R}_Y$, is $\mathbb{Q}$-free and graded (Theorem 1) and then irreducible polyzetas, being $\mathbb{Q}$-algebraic independent, are transcendent numbers (Corollary 1). By these results, up to weight 12, Conjecture 1 holds (see also[6] [7, 12]), *i.e.* $\mathcal{Z}_{irr}^{\mathcal{X},\leq 12}$ is $\mathbb{Q}$-algebraically free (Example 2).

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[6]All these implementations base on the "double shuffle relations" and provide linear relations.