A Sequent Calculus Representation of Lorenzen Dialogue Extended with Why-Because Dialogue

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Abstract
To apply proof theory to the investigation of dialogues, we introduce a sequent calculus representation of Lorenzen dialogue. Then, as a first step in extending our study of Lorenzen dialogue to the study of argumentation more generally, we extend Lorenzen dialogue by introducing why-because moves. We show that a sequence of why-because moves corresponds to a successive applications of cut-rule in a sequent calculus proof. Then, by combining the correspondence between Lorenzen dialogue and sequent calculus GK, we show that our dialogue corresponds to sequent calculus GK+cut.

Keywords
Lorenzen dialogue, Sequent calculus, Proof theory

1. Introduction

In the 1950s, Lorenzen [11, 10] proposed a dialogical interpretation of logical connectives. Lorenzen and his followers formulated the rules of logical dialogue to characterize provability in intuitionistic/classical logic as the existence of winning strategies in the dialogue. Since then, further researches have been developed. See [16] for a concise survey.

Apart from those studies on Lorenzen dialogue as another semantics of logics than the usual set-theoretical semantics, Lorenzen dialogue has also received attention from the viewpoint of argumentation theory. See, for example, [7, 18]. From the viewpoint of argumentation theory, Lorenzen dialogue has been investigated, not as a semantics for certain logic, but as a formal dialectical system.

In Lorenzen dialogue, as well as related dialogues investigated in argumentation theory, a dialogue is defined as a sequence of moves or locutions, such as attack and defense, and such moves are associated with each other, for example, by reference numbers. However, such a description of a dialogue as a sequence of moves is not suitable for structural analysis of dialogue. From the viewpoint of logic, this is similar to the description of proofs in Hilbert-style, which is not suited for structural analysis of proofs. Cf. [14].

In this article, we introduce a sequent calculus representation of Lorenzen dialogue. Sequent calculus, along with natural deduction, is the most popular formalization in logic, and it led to a significant development in proof theory. Furthermore, sequent calculus provides a basis for both automated theorem proving and logic programming. Our sequent calculus representation of

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dialogues enables the application of the proof theoretical results and techniques to the study of dialogues; investigation of typical or normal forms of dialogues and the normalization theorem, structural analysis of dialogues, characterization of the structure of dialogues, automated dialogue construction, and so on. Cf. [14].

In Section 2, we review a variant of sequent calculus called Kleene's sequent calculus GK [17, 13]. It is well-known that GK corresponds to Lorenzen dialogue for classical logic [8, 5, 12]. In Section 3, we review Lorenzen dialogue for classical logic following [4, 5]. We introduce the sequent calculus representation GKD of Lorenzen dialogue given in [12]. We show that GKD is Lorenzen dialogue.

The original Lorenzen dialogue is a dialogue for the interpretation of basic logical connectives, and as such is inadequate for exploring general arguments that do not fit within the framework of those logical connectives. Kacprzak and Yaskorska [9] extended Lorenzen dialogue to Lorenzen-Hamblin Natural Dialogue LHND by integrating Lorenzen dialogue and Hamblin dialogue. In this article, as a first step in extending our study of Lorenzen dialogue to the study of argumentation more generally, we introduce our sequent calculus representation to a restricted fragment of LHND, where only why and because/since moves, as well as moves of Lorenzen dialogue, are considered. Caminada and Podlaszewski [2] investigated essentially the same dialogue as our why-because dialogue, without Lorenzen dialogue, to build a connection between studies on formal dialogues and on the argumentation framework. We introduce our sequent calculus representation GKDwb of Lorenzen dialogue extended with why-because dialogue. We show that a sequence of why-because moves corresponds to a successive applications of cut-rule in a sequent calculus proof. Then, by combining the correspondence between Lorenzen dialogue and sequent calculus GK, we show that dialogues of GKDwb corresponds to proofs of sequent calculus GK + cut.

2. Kleene's sequent calculus GK

Formulas are defined inductively as usual. Formulas are denoted by $A, B, C, \ldots$, and atoms are denoted by $Q, R, \ldots$. For simplicity, we consider only propositional connectives $\neg$ (negation) and $\wedge$ (conjunction).

We introduce a variant of Gentzen's sequent calculus [6] for classical logic called Kleene's sequent calculus GK [17, 13]. In sequent calculus, the basic component is a sequence of formulas called a *sequent* instead of a formula. A **sequent** has the form $A_1, \ldots, A_k \vdash B_1, \ldots, B_l$, which corresponds to the formula $A_1 \wedge \cdots \wedge A_k \to B_1 \vee \cdots \vee B_l$. Although $A_1, \ldots, A_k$ or $B_1, \ldots, B_l$ is normally defined as a "sequence" of formulas, we define it as a "multiset" of formulas, that is, a finite sequence, modulo the ordering of occurrences of formulas. For example, we identify the following two sequents: $A, A, B \vdash C, D$ and $A, B, A \vdash D, C$. We denote multisets of formulas by the Greek capital letters for $\Gamma, \Delta, \Pi, \Lambda, \ldots$. For any sequent $\Gamma \vdash \Delta$, Γ is called the *antecedent* of the sequent, while Δ is called the *succedent* of the sequent. See [17, 14] for sequent calculus.

Definition 2.1. The **inference rules of** GK are the following $\neg l$ and $\neg r$ (inferences on $\neg A$), and $\wedge l_i$ ($i = 1, 2$) and $\wedge r$ (inferences on $A_1 \wedge A_2$).

$$\frac{\Gamma, \neg A \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg l \qquad \frac{\Gamma, A \vdash \Delta, \neg A}{\Gamma \vdash \neg A, \Delta}\ \neg r$$

$$\frac{A_i, \Gamma, A_1 \wedge A_2 \vdash \Delta}{\Gamma, A_1 \wedge A_2 \vdash \Delta}\ \wedge l_{i_{(i\,=\,1,\,2)}} \qquad \frac{\Gamma \vdash A_1, \Delta, A_1 \wedge A_2 \quad \Gamma \vdash A_2, \Delta, A_1 \wedge A_2}{\Gamma \vdash A_1 \wedge A_2, \Delta}\ \wedge r$$

Any sequent of the form $\Gamma, Q \vdash Q, \Delta$, where Q is an atom, is called an **axiom**.

In each rule, Γ and Δ are collectively called the **context**. A proof, which is denoted by $\pi, \pi_1, \pi_2, \ldots$, is a finite tree constructed using inference rules, whose leaves are axioms. See [17] for a formal definition.

In section 4, we investigate $\mathsf{GK} + cut$ by adding the following rule of cut to GK, which is known as an admissible rule in GK [17, 13].

$$\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ (A)$$

In the above form of cut, the two occurrences of A is called the cut-formula.

cut-rule can be regarded as a generalization of *modus ponens*; $\vdash A$ and $A \vdash C$ implies $\vdash C$. Furthermore, this rule can be interpreted as logical reasoning that uses an appropriate “lemma” ($\vdash A$), intermediate steps that may be easier to prove, to prove conclusion ($\vdash C$).

3. Sequent calculus representation of Lorenzen dialogue

In Section 3.1, we review Lorenzen dialogue following Felscher [3, 4] and Fermüller [5]. Then, in Section 3.2, we introduce a sequent calculus representation GKD of Lorenzen dialogue given in [12].

3.1. Lorenzen dialogue

Lorenzen dialogue is introduced as a two players’ dialogue between *Proponent* and *Opponent*. We use X, Y $(X \neq Y)$ to denote P (*Proponent*) or O (*Opponent*). The rule of a dialogue is defined based on the following argumentation form, which describes how a composite formula may be attacked and, how, if possible, this attack may be defended.

The **argumentation forms** of Lorenzen dialogue for the connectives $\wedge$ and $\neg$ are defined as follows (read from bottom up). Y’s assertion is a formula C obtained previously by Y’s defense on C or Y’s attack on $\neg C$.

$\wedge$-**argumentation form**	$\neg$-**argumentation form**
Y’s defense by asserting A_i	No defense is possible
X’s attack by choosing $i = 1$ or 2	X’s attack by asserting A
Y’s assertion of $A_1 \wedge A_2$	Y’s assertion of $\neg A$

We express an X’s attack upon $A_1 \wedge A_2$ by choosing A_i (for $i = 1, 2$) as $(Xa; \wedge i)$, an X’s attack upon $\neg A$ by asserting A as $(Xa; A)$, an X’s defense by asserting A as $(Xd; A)$.

An X’s **move** is $(Xa; \wedge i; j)$, $(Xa; A; j)$, or $(Xd; A; j)$, where j is a natural number called the reference number, which denotes for what number of the move is attack/defense in a sequence of moves. We denote moves as $m, m_1, m_2, \ldots$.

The attack of j-th move is said to be **open** at k-th move with $j < k$, if there is no l-th move with $j < l \leq k$ which carries a defense to the attack of j-th move according to the appropriate argumentation form.

Definition 3.1. A sequence of moves $m_1, m_2, m_3, \ldots$, where we refer k-th move by m_k, is a **dialogue** for a formula A if it satisfies the following conditions.

(D00) The first move m_1 is $(Pd; A; 0)$, and thereafter, P's and O's moves appear alternately.

(D01) If m_k ($k > 1$) is $(Xa; \wedge i; j)$, or $(Xa; A; j)$ for $j < k$, then m_k is an X's attack upon the assertion in m_j according to the appropriate argumentation form.

(D02) If m_k ($k > 1$) is $(Xd; A; j)$ for $j < k$, then m_k is an X's defense to the attack in m_j according to the appropriate argumentation form.

(D10) P may assert an atom only after it has been previously asserted by O.

(D11) If, at m_{k-1}, there are several open attacks suitable to be defended at m_k of O, then only the *latest* of them may be defended at m_k of O.

(D12) A P's attack may be defended by O at most once.

(D13) A P's assertion may be attacked by O at most once.

A dialogue for A is **won by** P if it is finite, ends with P's move and if the rules do not permit O to continue with another move.

Definition 3.2. A **strategy for** A is a tree with moves as nodes, which satisfies the following conditions.

1. Each branch is a dialogue for A;
2. Every O's move has at most one successor node;
3. Every P's move has all successor nodes for each possible next O's move.

A **winning strategy for** A is a strategy for A, where every dialogue in it is won by P.

The above dialogue is that for classical logic. Felscher [3, 4] demonstrated that a dialogue for intuitionistic logic is obtained by imposing (D11) and (D12) not only on O's moves but also on P's moves.

3.2. Sequent calculus representation GKD of Lorenzen dialogue

We review the sequent calculus representation GKD of Lorenzen dialogue introduced in [12]. We introduce *d-sequents*, which are a slight modification of the usual sequents. In a d-sequent, we distinguish formulas to be defended using brackets [ ] from formulas to be attacked. The introduction of d-sequents renders the reference numbers in Lorenzen dialogue unnecessary.

Definition 3.3. A **d-sequent** is of the form $[\Pi], \Gamma \vdash \Lambda, [\Delta]$ for any multiset of formulas $\Pi, \Gamma, \Lambda, \Delta$.

$[\Pi]$ is formulas to be defended by O; Γ is formulas to be attacked by P; Λ is formulas to be attacked by O; and $[\Delta]$ is formulas to be defended by P.

Although we can define our GKD in the same way as Lorenzen dialogue of the previous section, we here introduce our dialogue GKD in the style of sequent calculus. From the proof theoretical viewpoint, a winning strategy corresponds to a proof, and a dialogue in such a winning strategy corresponds to a branch in the proof. Based on the correspondence, we define the notion of a strategy prior to a dialogue. Thus, in the following definition of moves, O's attack on $A_1 \wedge A_2$ (denoted as $Oa\wedge_{12}$) is defined including branching as it is in $\wedge r$-rule in GK.

Definition 3.4. **Moves** of X's attack on $A_1 \wedge A_2$ ($Xa\wedge_i$), X's attack on $\neg A$ ($Xa\neg$), X's defense on A (Xd) are defined as follows.

$$\frac{[\Pi], \Gamma \vdash \Lambda, [A_1, \Delta, A_1 \wedge A_2] \quad [\Pi], \Gamma \vdash \Lambda, [A_2, \Delta, A_1 \wedge A_2]}{[\Pi], \Gamma \vdash A_1 \wedge A_2, \Lambda, [\Delta]} \; Oa\wedge_{12}$$

$$\frac{[\Pi, A_i], \Gamma, A_1 \wedge A_2 \vdash \Lambda, [\Delta]}{[\Pi], \Gamma, A_1 \wedge A_2 \vdash \Lambda, [\Delta]} \; Pa\wedge_i (i = 1, 2)$$

$$\frac{[\Pi], \Gamma, A \vdash \Lambda, [\Delta, \neg A]}{[\Pi], \Gamma \vdash \neg A, \Lambda, [\Delta]} \; Oa\neg \qquad \frac{[\Pi], \Gamma, \neg A \vdash A, \Lambda, [\Delta]}{[\Pi], \Gamma, \neg A \vdash \Lambda, [\Delta]} \; Pa\neg$$

$$\frac{[\Pi], A, \Gamma \vdash \Lambda, [\Delta]}{[\Pi, A], \Gamma \vdash \Lambda, [\Delta]} \; Od \qquad \frac{[\Pi], \Gamma \vdash \Lambda, A, [\Delta]}{[\Pi], \Gamma \vdash \Lambda, [A, \Delta]} \; Pd$$

The above moves are read from the bottom up. For $Oa\wedge_{12}$, when we focus one of the two upper sequents, we refer the move by $Oa\wedge_1$ or $Oa\wedge_2$. In the given definition of moves, if we ignore contexts $\Pi, \Gamma, \Delta, \Lambda$, we find that they correspond to the argumentation forms of Lorenzen dialogue: X's $\neg$-argumentation form corresponds to $Xa\neg$; X's $\wedge$-argumentation form corresponds to the pair of moves $Xa\wedge_i$ followed by Yd.

Definition 3.5. A **strategy** for a sequent $\Gamma \vdash \Delta$ is a tree constructed using moves satisfying the following conditions.

1. The root of the strategy is the d-sequent $\Gamma \vdash [\Delta]$.
2. In each branch, the first (i.e., the bottom of the tree) is P's move, and thereafter, the moves of O and P alternately appear.
3. Pd on Q or $Pa\neg$ on $\neg Q$ for an atom Q is possible only when the given d-sequent is of the form $[\Pi], \Sigma, Q \vdash \Lambda, [\Phi]$, where $Q \in \Phi$ or $\neg Q \in \Sigma$.

Every branch in a strategy is called a **dialogue**. A strategy for $\Gamma \vdash \Delta$ is **winning** if it is finite, and every leaf thereof is of the form $\Sigma, Q \vdash Q, [\Phi]$ for an atom Q.

Example 3.6. The following left is a winning strategy for $\neg(\neg Q \wedge Q)$ (which is equivalent to $Q \vee \neg Q$) in Lorenzen dialogue, and the following right is that in GKD. Both read from bottom to top.

7. $(Pa; Q; 6)$
6. $(Od; \neg Q; 5)$
5. $(Pa; \wedge 1; 2)$
4. $(Od; Q; 3)$
3. $(Pa; \wedge 2; 2)$
2. $(Oa; \neg Q \wedge Q; 1)$
1. $(Pd; \neg(\neg Q \wedge Q); 0)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg Q, Q, \neg Q \wedge Q \vdash Q, [\neg(\neg Q \wedge Q)]}{\neg Q, Q, \neg Q \wedge Q \vdash [\neg(\neg Q \wedge Q)]}\ Pa\neg}{[\neg Q], Q, \neg Q \wedge Q \vdash [\neg(\neg Q \wedge Q)]}\ Od}{Q, \neg Q \wedge Q \vdash [\neg(\neg Q \wedge Q)]}\ Pa\wedge_1}{[Q], \neg Q \wedge Q \vdash [\neg(\neg Q \wedge Q)]}\ Od}{\neg Q \wedge Q \vdash [\neg(\neg Q \wedge Q)]}\ Pa\wedge_2}{\vdash \neg(\neg Q \wedge Q)}\ Oa\neg}{\vdash [\neg(\neg Q \wedge Q)]}\ Pd$$

Although the original Lorenzen dialogue is defined for a formula A, it can be regarded as our dialogue for the sequent $\vdash A$. Condition (3) of Definition 3.5 is the same as (D10) of the original Lorenzen dialogue.

By a P's attack ($Pa\wedge_i$ or $Pa\neg$), the attacked O's assertion remains in the same position in the upper sequent; hence, P may attack more than once for the same assertion of O, in the same way as Lorenzen dialogue. In contrast, by an O's attack ($Oa\wedge_i$ or $Oa\neg$), the attacked P's assertion (eg., $A_1 \wedge A_2$ in the lower sequent $[\Pi], \Gamma \vdash A_1 \wedge A_2, \Lambda, [\Delta]$ of $Oa\wedge_{12}$) is moved inside [ ] which are formulas to be defended by P (eg., $[\Pi], \Gamma \vdash \Lambda, [A_i, \Delta, A_1 \wedge A_2]$). Thus, O can attack a P's assertion at most once, and the condition (D13) of Lorenzen dialogue is included in our description of moves of Oa.

By an O's defense Od, the defended formula A in the lower sequent $[\Pi, A], \Gamma \vdash \Lambda, [\Delta]$ is moved to the position of O's assertion $[\Pi], A, \Gamma \vdash \Lambda, [\Delta]$. Thus, O can defend a P's attack at most once, and the condition (D12) of Lorenzen dialogue is included in our Od. Although, at first glance, Pd appears to be the same as Od, but it is not. By Oa (eg., $A_1 \wedge A_2$ in the upper sequent $[\Pi], \Gamma \vdash \Lambda, [A_i, \Delta, A_1 \wedge A_2]$ of $Oa\wedge_{12}$), the attacked formula itself ($A_1 \wedge A_2$) is duplicated inside [ ] to be defended by P. Thus, when Pd on A is applied to an O's attack $[\Pi], \Gamma \vdash \Lambda, A, [\Delta]$, the formula A is contained in Δ. That is, P may defend an O's attack several times.

We find that the restriction (D11) on O's moves also holds in GKD as follows.

Lemma 3.7. *In any strategy, the O's move is uniquely determined.*

Proof. Beginning with a d-sequent of the form $\Gamma \vdash [\Delta]$, P's possible move is Pd, $Pa\neg$, or $Pa\wedge_i$. (1) By Pd or $Pa\neg$, we obtain a d-sequent of the form $\Gamma' \vdash A, [\Delta']$. Then, the next possible O's move is only an attack on A; as such, we return to a d-sequent of the form $\Gamma'' \vdash [\Delta'']$. (2) By $Pa\wedge_i$, we obtain $[A], \Gamma' \vdash [\Delta']$. The next possible O's move is only Od, and we return to a d-sequent of the form $\Gamma'' \vdash [\Delta'']$. Thus, in a strategy, no choice of moves exists for O. ∎

In particular, Od is possible for only the latest Pa; hence, (D11) holds in our GKD. Based on the above discussion and Lemma 3.7, we show that GKD is Lorenzen dialogue.

Proposition 3.8. *Every dialogue of* GKD *is a Lorenzen dialogue.*

Proof. We show that a given dialogue of GKD for $\vdash A$, i.e., a branch in a strategy for $\vdash [A]$, is a Lorenzen dialogue. The first P's move Pd is common; hence, (D00) holds in GKD.

As for (D01), $Oa\wedge_i$ in GKD corresponds to the move $(Oa; \wedge i; k)$ in Lorenzen dialogue. Note that when $Oa\wedge_i$ is possible in GKD, there exists P's assertion $A_1 \wedge A_2$, that is, $A_1 \wedge A_2$ is already asserted by P before the $Oa\wedge_i$. The same applies to the other attack moves of O and P; hence, (D01) also holds in GKD.

As for (D02), Od on A in GKD corresponds to the move $(Od; A; k)$ in Lorenzen dialogue. Note that when Od on A is possible in GKD, A is already inside [ ] formulas to be defended by O, and hence, A should be moved inside [ ] before the Od. The same applies to Pd, and hence, (D02) also holds in GKD.

(D10)∼(D13) hold in GKD as discussed previously and by Lemma 3.7. ∎

Thus, we obtain the equivalence between our GKD and Lorenzen dialogue via the completeness theorem of Felscher [3, 4] and the correspondence between GKD and GK given in [12]. Although we do not enter into the detail here, we briefly review the correspondence. Not all GK-proofs can be expressed as winning strategies in GKD. The reason is due to the impossibility of the defense move in the ¬-argumentation form in Lorenzen dialogue. Together with the restriction on O's moves, in Lorenzen dialogue, O should attack on A immediately after a P's attack on $\neg A$. From the viewpoint of sequent calculus, this restriction indicates that an inference on A should be applied immediately after an application of $\neg l$ on $\neg A$. However, any proof in GK can be transformed into a proof satisfying this restriction. Thus, with such a transformation of proofs in GK, we obtain the correspondence between GK-proofs and winning strategies of GKD. Specifically, $\wedge l$ and $\wedge r$ in GK correspond to the following pairs of P's and O's moves in GKD.

$$\dfrac{A_i, A_1 \wedge A_2, \Gamma \vdash \Delta}{A_1 \wedge A_2, \Gamma \vdash \Delta}\ \wedge l_i \quad \Longleftrightarrow \quad \dfrac{\dfrac{A_i, A_1 \wedge A_2, \Gamma \vdash [\Delta]}{[A_i], A_1 \wedge A_2, \Gamma \vdash [\Delta]}\ Od}{A_1 \wedge A_2, \Gamma \vdash [\Delta]}\ Pa\wedge_i$$

$$\dfrac{\Gamma \vdash \Delta, A_1 \wedge A_2, A_1 \quad \Gamma \vdash \Delta, A_1 \wedge A_2, A_2}{\Gamma \vdash \Delta, A_1 \wedge A_2}\ \wedge r \quad \Longleftrightarrow \quad \dfrac{\dfrac{\Gamma \vdash [A_1, A_1 \wedge A_2, \Delta] \quad \Gamma \vdash [A_2, A_1 \wedge A_2, \Delta]}{\Gamma \vdash A_1 \wedge A_2, [\Delta]}\ Oa\wedge_{12}}{\Gamma \vdash [\Delta, A_1 \wedge A_2]}\ Pd$$

4. Extension of Lorenzen dialogue with why-because dialogue

In Section 4.1, we introduce our why- and because-moves, and discuss their counterparts in sequent calculus. Then, in Section 4.2, we define our WB-dialogue, and Lorenzen dialogue extended with WB-dialogue GKDwb. We show the correspondence between our dialogue GKDwb and sequent calculus GK + cut.

4.1. Why- and because-moves

We introduce O's why-move and P's because-move. On the one hand, by a why-move, O requires grounds for a formula provided by P. On the other hand, by a because-move, P provides grounds for a formula asked by O. Essentially the same dialogue is introduced in [2]. Let us illustrate how our WB-dialogue (why-because dialogue) is constructed.

Example 4.1. A WB-dialogue is expressed by a sequence of d-sequents connected by $\circ$ as follows.

1. **P's claim:** A_1
   Firstly, P chooses a formula A_1 and claims it.
2. **O's why:** $\vdash [A_1]$
   By the why-move, O requires grounds for A_1, which is expressed by the d-sequent $\vdash [A_1]$. By the O's why-move, A_1 becomes a formula to be defended by P.
3. **P's because:** $A_2, A_3 \vdash [A_1]$
   P provides grounds A_2 and A_3 for A_1. By the because-move, P puts A_2 and A_3 on the left-hand side of the given d-sequent $\vdash [A_1]$ of (2). If O agrees that A_2 and A_3 hold, then the dialogue ends. Otherwise, O further requires grounds for A_2 or A_3 by a why-move.
4. **O's why:** $(\vdash [A_2]) \circ (A_2, A_3 \vdash [A_1])$
   O requires grounds for A_2. By the why-move, O extends the given d-sequent $A_2, A_3 \vdash [A_1]$ of (3) by connecting another d-sequent $\vdash [A_2]$.
5. **P's because:** $(A_4 \vdash [A_2]) \circ (A_2, A_3 \vdash [A_1])$
   P provides ground A_4 for A_2.
6. **O's why:** $(\vdash [A_3]) \circ (A_4 \vdash [A_2]) \circ (A_2, A_3 \vdash [A_1])$
   O requires grounds for A_3.
7. **P's because:** $(A_5 \vdash [A_3]) \circ (A_4 \vdash [A_2]) \circ (A_2, A_3 \vdash [A_1])$
   P provides ground A_5 for A_3.

In this way, a WB-dialogue is expressed by a sequence of d-sequents extended to the left. Except for the first P's claim, a pair of an O's why-move and a P's because-move constitutes a d-sequent of the form $\Gamma \vdash [A]$. We call the above sequence of d-sequents connected with $\circ$ a *multi-d-sequent* (*md-sequent* for short).

From the viewpoint of sequent calculus, $\circ$ corresponds to *cut*-rule; the md-sequent $(A_4 \vdash [A_2]) \circ (A_2, A_3 \vdash [A_1])$ corresponds to the following application of *cut*.

$$\frac{A_4 \vdash A_2 \quad A_2, A_3 \vdash A_1}{A_4, A_3 \vdash A_1}\ (A_2)$$

Thus, if O agrees that A_4 and A_5 hold, then the above WB-dialogue is finished, and it is interpreted as the following successive applications of *cut* called a *cut-string*.

$$\dfrac{\vdash A_5 \quad \dfrac{\vdash A_4 \quad \dfrac{A_5 \vdash A_3 \quad \dfrac{A_4 \vdash A_2 \quad A_2, A_3 \vdash A_1}{A_4, A_3 \vdash A_1}\,(A_2)}{A_5, A_4 \vdash A_1}\,(A_3)}{A_5 \vdash A_1}\,(A_4)}{\vdash A_1}\,(A_5)$$

There is agreement between O and P on A_4 and A_5, but not on the sequents $A_5 \vdash A_3$, and $A_4 \vdash A_2$, and $A_2, A_3 \vdash A_1$. Whether or not O accepts them is examined by Lorenzen dialogues for them further. O first chooses such a d-sequent which constitutes the given WB-dialogue, and then P starts a Lorenzen dialogue for the d-sequent designated by O. Note that the d-sequent $\Gamma \vdash [A]$ designated by O is that whose validity is challenged by O. That is, O considers that A is false even if Γ is true. Thus, as it is in GKD, Γ is a set of formulas (tentatively) asserted by O, and $[A]$ is a formula to be defended by P.

4.2. Sequent calculus representation GKDwb of Lorenzen dialogue with why-because dialogue

We define our WB-dialogue, and then, by combining with our GKD, we define Lorenzen dialogue with why-because dialogue GKDwb. Although the general form of a d-sequent is of the form $[\Pi], \Gamma \vdash \Lambda, [\Delta]$, it is sufficient for WB-dialogue to consider a particular form $\Gamma \vdash [A]$, where $[\Pi]$ and Λ are empty, and $[\Delta]$ is restricted to a single formula.

Definition 4.2. An **md-sequent** $\mathcal{M}$ is a sequence of d-sequents of the form $\Gamma \vdash [A]$, which are connected by $\circ$, defined inductively as follows.

1. Any d-sequent of the form $\Gamma \vdash [A]$ is an md-sequent.
2. Let $\Gamma \vdash [A]$ be a d-sequent, and $\mathcal{M}$ be an md-sequent such that A appears on the antecedent of a d-sequent that constitutes $\mathcal{M}$. Then, $((\Gamma \vdash [A]) \circ \mathcal{M})$ is an md-sequent.

We call every d-sequent that constitutes an md-sequent $\mathcal{M}$ a **component sequent** of $\mathcal{M}$.

For appropriate d-sequents $S_1, \ldots, S_n$, an md-sequent is of the form $(S_n \circ (\cdots (S_3 \circ (S_2 \circ S_1)) \cdots))$; hence, we usually omit the parentheses of an md-sequent, and write as $S_n \circ \cdots \circ S_2 \circ S_1$.

Definition 4.3. O's why-move and P's because-move are defined as follows.

O's why-move: For any md-sequent $\mathcal{M}$, where a component sequent $A, \Gamma \vdash [C]$ appears, O extends $\mathcal{M}$ to the md-sequent $((\vdash [A]) \circ \mathcal{M})$ by connecting the d-sequent $\vdash [A]$.

In particular, when $\mathcal{M}$ is empty (at the beginning of a dialogue), O puts the d-sequent $\vdash [A]$ for a formula A (chosen by P).

P's because-move: For any md-sequent $\mathcal{M}$, where a component sequent $\vdash [A]$ with the empty context appears, P replaces it by $\Gamma \vdash [A]$ by adding any non-empty formulas Γ to the component sequent.

To set up the end of a dialogue, we introduce a set of formulas $\mathcal{B}$ called the *basis* of a WB-dialogue, that O and P agree are true. By its nature, O cannot ask why to every formula $B \in \mathcal{B}$. When all antecedents not asked by O of all component sequents of a WB-dialogue are belongs to $\mathcal{B}$, the dialogue ends. However, neither P nor O wins at that point, and who wins the dialogue is determined by Lorenzen dialogues for all component d-sequents of the WB-dialogue that follow.

Definition 4.4. Let $\mathcal{B}$ be a given set of formulas called the **basis**. A **WB-dialogue** is a sequence of moves satisfying the following conditions.

1. P first chooses a formula A and claims it, and thereafter, moves of O and P alternately appear.
2. Every occurrence of a formula is asked by an O's why-move at most once.
3. O cannot apply a why-move to any formula $B \in \mathcal{B}$.
4. A dialogue ends when O cannot continue the dialogue with another why-move.

A WB-dialogue is that for A, when P's first claim is A.

We identify a WB-dialogue and the resulting md-sequent $\mathcal{M}$.

Note that the above condition (2) is defined for occurrences of a formula. The same formula may be provided as ground by P several times, and in such a case, O should ask why for all occurrences of that formula, not once collectively.

P may end a WB-dialogue at any time by choosing formulas of $\mathcal{B}$ as required grounds. We consider the purpose of a WB-dialogue is the investigation of appropriate grounds or lemmas, by P and O working cooperatively, to establish a given proposition.

We define GKDwb by combining GKD and WB-dialogue.

Definition 4.5. A **dialogue of** GKDwb **for** A consists of the following two stages.

1. The first stage consists of a WB-dialogue $\mathcal{M}$ for A.
2. The second stage consists of dialogues of GKD for all component sequents of $\mathcal{M}$ chosen by O.

A dialogue of GKDwb is won by P if all dialogues of GKD at the second stage are won by P.

Definition 4.6. A **strategy of** GKDwb **for** A is a sequence of strategies of GKD for all component sequents of a WB-dialogue $\mathcal{M}$ for A constructed at the first stage of the dialogue.

A strategy of GKDwb is winning if every strategy of GKD thereof is winning.

Note that since we consider a set of strategies of GKD as a strategy of GKDwb, the validity of formulas in GKDwb is reduced to that in GKD. Thus, the valid formulas are common in GKD and GKDwb, while GKDwb has a higher expressive power than GKD.

We investigate a correspondence between our dialogue GKDwb and our sequent calculus GK + cut. We extend GK + cut by introducing non-logical axioms of the form $\vdash B$ for all $B \in \mathcal{B}$ for a given basis $\mathcal{B}$. We first show that there is a one-to-one correspondence between a WB-dialogue and a successive applications of cut called a *cut-string* in GK + cut.

Definition 4.7. In $\mathsf{GK} + cut$, a path of a proof-tree that consists only of successive applications of cut is called a **cut-string**.

In the following, for any multisets $\Gamma_1, \ldots, \Gamma_n$ of formulas, by $\bigcup \Gamma_n$, we denote $\Gamma_1 \cup \cdots \cup \Gamma_n$, where $\cup$ is the multiset union. $\Gamma_1 \setminus \{A_1, \ldots, A_n\}$ is the multiset difference. For example, $\{A, A, B\} \cup \{A, C\} = \{A, A, A, B, C\}$ and $\{A, A, B\} \setminus \{A\} = \{A, B\}$.

Definition 4.8. In the construction of a WB-dialogue, except for the first P's claim, every successive pair of an O's why and a P's because moves is called an OP**-pair**. Furthermore, a sequence of OP-pairs, in the order in which they appear in the given WB-dialogue, is called an OP**-sequence**.

Lemma 4.9. *Let any d-sequent $\Gamma \vdash [A]$ in* GKDwb *be identified with a sequent $\Gamma \vdash A$ in* GK*. Then, for any OP-sequence $\mathcal{M}$, there exists a cut-string consisting of all component sequents of $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be an OP-sequence $(\Gamma_n \vdash [A_n]) \circ \cdots \circ (\Gamma_2 \vdash [A_2]) \circ (\Gamma_1 \vdash [A_1])$. We show that $\mathcal{M}$ corresponds to the following cut-string by induction on the length l the number of OP-pairs of the given OP-sequence.

$$\dfrac{\Gamma_n \vdash A_n \quad \dfrac{\vdots}{\bigcup \Gamma_{n-1} \setminus \{A_2, \ldots, A_{n-1}\} \vdash A_1}}{\bigcup \Gamma_n \setminus \{A_2, \ldots, A_n\} \vdash A_1}\ (A_n) \qquad \text{where the top is } \dfrac{\Gamma_2 \vdash A_2 \quad \Gamma_1 \vdash A_1}{\bigcup \Gamma_2 \setminus \{A_2\} \vdash A_1}\ (A_2)$$

(Base step) When $l = 2$, the given OP-sequence is of the form $(\Gamma_2 \vdash [A_2]) \circ (\Gamma_1 \vdash [A_1])$, and it corresponds to the following cut.

$$\dfrac{\Gamma_2 \vdash A_2 \quad \Gamma_1 \vdash A_1}{(\Gamma_1 \cup \Gamma_2) \setminus \{A_2\} \vdash A_1}\ (A_2)$$

(Induction step) When $l > 2$, the given OP-sequence is of the form $(\Gamma_l \vdash [A_l]) \circ (\Gamma_{l-1} \vdash [A_{l-1}]) \circ \cdots \circ (\Gamma_1 \vdash [A_1])$, and it corresponds to the following cut-string, by applying the induction hypothesis IH.

$$\dfrac{\Gamma_l \vdash A_l \quad \begin{matrix} \vdots\ \text{IH} \\ \bigcup \Gamma_{l-1} \setminus \{A_2, \ldots, A_{l-1}\} \vdash A_1 \end{matrix}}{\bigcup \Gamma_l \setminus \{A_2, \ldots, A_l\} \vdash A_1}\ (A_l)$$

Note that by the definition of the OP-sequence, we have $A_l \in \bigcup \Gamma_{l-1} \setminus \{A_2, \ldots, A_{l-1}\}$; hence, the above cut (A_l) is applicable.

When the given OP-sequence is a WB-dialogue, the above $\bigcup \Gamma_n \setminus \{A_2, \ldots, A_n\}$ is the formulas $\{B_1, \ldots, B_m\}$ of the basis $\mathcal{B}$. Hence, by applying cut with non-logical axioms, we obtain the following cut-string.

$$\dfrac{\dfrac{\vdash B_1 \quad \bigcup\Gamma_n \setminus \{A_2,\ldots,A_n\} \vdash A_1}{\bigcup\Gamma_n \setminus \{A_2,\ldots,A_n\} \setminus \{B_1\} \vdash A_1}\,(B_1)}{\begin{array}{c}\vdots\\ \vdash B_m \qquad\qquad B_m \vdash A_1\end{array}}$$

$$\dfrac{\vdash B_m \qquad\qquad B_m \vdash A_1}{\vdash A_1}\,(B_m)$$

■

The converse of the lemma is shown in the same way, by induction on the length of a given cut-string.

Note that a strategy of GKDwb is defined as a sequence of strategies of GKD. As discussed in the end of Section 3.2, there is a correspondence between winning strategies of GKD and sequent calculus proofs in GK. Thus, by combining the above Lemma 4.9, we obtain the following theorem.

Theorem 4.10. *If there exists a winning strategy of* GKDwb *for A, then there exists a proof of* $\vdash A$ *in* $\mathsf{GK} + cut$.

Example 4.11. Let us illustrate the following dialogue of GKDwb, which is a modification of the example given in [15, 9].

1. P: My car is safe.
2. O: Why is your car safe?
3. P: Because my car has an airbag, and if my car does not have an airbag then it isn't safe.
4. O: Why if your car does not have an airbag then it isn't safe?
5. P: Because if my car does not have any airbag then I may die in an accident, and if I may die in an accident then my car isn't safe.

Let "my car is safe" be S, "my car has an airbag" be A, and "I may die in an accident" be B. Let the basis $\mathcal{B}$ be $\{\neg A \to B, B \to \neg S, A\}$. Then, the above dialogue is represented by the following WB-dialogue.

$$(\neg A \to B, B \to \neg S \vdash [\neg A \to \neg S]) \circ (\neg A \to \neg S, A \vdash [S])$$

By conducting Lorenzen dialogues further, we obtain the following dialogue of GKDwb. Since we consider only $\wedge$ and $\neg$ in this article, we translate the above formulas as follows. $\neg A \to B \equiv \neg(\neg A \wedge \neg B)$, $B \to \neg S \equiv \neg(B \wedge S)$, $\neg A \to \neg S \equiv \neg(\neg A \wedge S)$. In the following dialogue, to save the space, we omit formulas duplicated in an application of a move in GKD.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(P\text{ wins})}{A, \neg(B \wedge S) \vdash A}}{A, \neg A, \neg(B \wedge S) \vdash}\,Pa\neg}{\neg A, \neg(B \wedge S) \vdash \neg A}\,Oa\neg}{\neg A, \neg(B \wedge S) \vdash [\neg A]}\,Pd}{[\neg A], \neg(B \wedge S) \vdash [\neg A]}\,Od}{\neg(B \wedge S), \neg A \wedge S \vdash [\neg A]}\,Pa\wedge$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{(P\text{ wins})}{\neg A \wedge S, B \vdash B}}{\neg A \wedge S, B \vdash [B]}\,Pd}{\neg A \wedge S \vdash \neg B, [B]}\,Oa\neg}{\neg A \wedge S \vdash [B, \neg B]}\,Pd \qquad \dfrac{\dfrac{\dfrac{\overset{(P\text{ wins})}{S \vdash S, [\neg B]}}{S \vdash [S, \neg B]}\,Pd}{[S] \vdash [S, \neg B]}\,Od}{\neg A \wedge S \vdash [S, \neg B]}\,Pa\wedge}{\neg A \wedge S \vdash B \wedge S, [\neg B]}\,Oa\wedge}{\neg(B \wedge S), \neg A \wedge S \vdash [\neg B]}\,Pa\neg$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(B \wedge S), \neg A \wedge S \vdash [\neg A] \qquad \neg(B \wedge S), \neg A \wedge S \vdash [\neg B]}{\neg(B \wedge S), \neg A \wedge S \vdash \neg A \wedge \neg B}\,Oa\wedge}{\neg(\neg A \wedge \neg B), \neg(B \wedge S), \neg A \wedge S \vdash}\,Pa\neg}{\neg(\neg A \wedge \neg B), \neg(B \wedge S) \vdash \neg(\neg A \wedge S)}\,Oa\neg}{(\neg(\neg A \wedge \neg B), \neg(B \wedge S) \vdash [\neg(\neg A \wedge S)])}\,Pd \qquad \circ$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(P\text{ loses})}{A, A \vdash [S]}}{A \vdash \neg A, [S]}\,Oa\neg}{A \vdash [\neg A, S]}\,Pd \qquad \overset{(P\text{ loses})}{A \vdash [S, S]}}{A \vdash \neg A \wedge S, [S]}\,Oa\wedge}{(\neg(\neg A \wedge S), A \vdash [S])}\,Pa\neg$$

Since P loses the dialogue for $\neg(\neg A \wedge S), A \vdash [S]$, and there exists no winning strategy thereof, the first P's claim S (“My car is safe.”) cannot be justified.

5. Future work

Since our investigation in this article is a first step toward applying proof theory to the study of argumentation, there are many possible future work. Among others, we will extend our sequent calculus representation to the full fragment of Lorenzen-Hamblin Natural Dialogue LHND [9]. We will further investigate an extension of our framework to express defeasible reasoning as it is in [1], where Arieli and Straßer investigated defeasible reasoning by using an enhanced sequent calculus.

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