# **Embedding Temporal Description Logic Ontologies by Cone-based Geometric Models**

Mena Leemhuis

*Free University of Bozen-Bolzano, Universitätsplatz 1, 39100 Bozen/Bolzano, Italy*

#### **Abstract**

The embedding of description logic ontologies into low dimensional vector spaces is, e.g., in the context of knowledge graph embedding. an established option to do for example link prediction or concept membership prediction with the help of background information in form of an ontology. However, it is not only of interest to model expressive description logics such as  $\widehat{ALC}$  but also to model temporal aspects, thus the evolution of concepts over time geometrically. Therefore, there is a need to model operators such as, e.g., eventually or next, thus temporal description logics geometrically. In this paper, an approach for embedding an expressive Boolean temporal description logic based on the embedding of concepts as closed convex cones is presented and it is proven that an ontology is satisfiable in the classical sense if and only if it is satisfiable in a geometric model based on the presented embedding. This model is a first step towards a learning approach able to model background information in form of an expressive temporal description logic ontology.

#### **Keywords**

Temporal Description Logics, Neuro-symbolic AI, Knowledge Graph Embedding, Convex Cones

# **1. Introduction**

Though subsymbolic learning approaches gained importance due to good-quality results in the last years, they lack important features such as explainability and trustworthiness. This led to the research area of Neuro-Symbolic AI [\[1\]](#page--1-0) which is based on the idea of combining subsymbolic and symbolic approaches to use both information on similarity of instances and the ability to do deductive reasoning on the symbolic level. One way of tackling this neuro-symbolic combination is pursued in the area of knowledge graph embedding (KGE), where knowledge graphs (thus,  $(subject, predicate, object)$ -triples such as  $(alice, loves, bob)$  are embedded into a lowdimensional vector space by modeling instances (thus  $subject$  and  $object$ ) as points in this space and relations (thus  $predicates$ ) as geometric operations between these points. This enables to do for example link prediction, thus the prediction of new triples based on given ones. One prominent example is TransE [\[2\]](#page--1-1) where relations are represented as vector translations. However, not only instance information but also information on concepts and their interaction can be modeled, e.g., to predict only relations fulfilling some background knowledge statements, for example enforcing that the relation *is capital of* needs to have the object being a country. Therefore, it is necessary to be able to embed expressive background logic ontologies.

Those approaches are based on the idea of embedding concepts as convex sets in a vector space and logical operations between concepts as geometrical operations between those sets, e.g., representing concept conjunction as set intersection. An instance belonging to a concept is then modeled as a point in the convex set representing this concept. For those instances, e.g., relations between them or their concept membership can be predicted. The approaches are, e.g., able to model the description logic (DL)  $\mathcal{EL}^{++}$ , e.g., by representing concepts as boxes [\[3\]](#page--1-2) or spheres [\[4\]](#page--1-3). Some approaches are even able to model full concept negation and disjunction, thus the DL  $ALC$ , e.g., based on subspaces

[\[5\]](#page--1-4) or closed convex cones [\[6,](#page--1-5) [7\]](#page--1-6). The approach of Özçep et al. [\[6,](#page--1-5) [7\]](#page--1-6) shows that  $ALC$ -ontologies are satisfiable if and only if they are satisfiable by a geometric model based on closed convex cones.

However, concepts are normally not static but evolve over time. A cured person is, for example, a person who has recovered from an illness. Therefore, to model the concept of cure, it is necessary to model both the concepts of healthiness and of illness and a temporal combination thereof. To model such ideas, temporal logic can be used and, in the context of ontologies, this leads to the area of several different temporal description logics (see, e.g., [\[8\]](#page--1-7) for a survey).

This directly leads to the question whether existing approaches for embedding  $ALC$  can be extended to model also temporal aspects and especially whether such a model has the same expressivity as the DL interpretation, thus, whether a temporal DL-ontology is satisfiable if and only if the corresponding geometric model is satisfiable.

Though, temporal knowledge graph embeddings are a widely studied topic (see, e.g., [\[9\]](#page--1-8) for a survey), to the best of the author's knowledge, there are no approaches incorporating background information in form of an expressive temporal description logic ontology.

The basic idea is to extend the cone-based embedding of Özçep et al. [\[6,](#page--1-5) [7\]](#page--1-6) by modeling the passing of time as an increasing distance to the point of origin. Thus, on each sphere with the point of origin as center, the concepts of one time point are modeled. An instance is represented as a ray and thus the intersections of the ray with different spheres can be considered to determine the concept membership of the instance and especially the change of the concept membership of this instance. Thus both are modeled, the operators of classical description logic (by considering concepts at the same distance) and the temporal operators (by considering the concepts being on a ray starting at the point of origin).

In contrast to classical KGE-approaches, the focus lies here on representing concepts and their temporal aspects and not on relations. Thus, a Boolean temporal  $ALC$ ontology is considered. The main result is that it is actually possible to model temporal DL-ontologies via those models based on cones, namely that a temporal DL-ontology is satisfiable if and only if it is satisfiable in such a geometric model.

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 $\bigcirc$  [menahildegard.leemhuis@unibz.it](mailto:menahildegard.leemhuis@unibz.it) (M. Leemhuis)

[https://orcid.org/0000-0003-1017-8921](https://orcid.org/https://orcid.org/0000-0003-1017-8921) (M. Leemhuis)

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After a short introduction to the description logic  $\mathcal{ALC}$ and to the temporal description logic  $LTL_{\text{ALC}}$  in Section [2,](#page-1-0) in Section [3,](#page-1-1) the cone embedding of Özçep et al. [\[6,](#page-5-0) [7\]](#page-6-0) is introduced. In Section [4,](#page-2-0) the extension of the cone embedding to the temporal case is presented and the expressivity of this approach is discussed in Section [5.](#page-4-0) The paper ends with a short conclusion.

## <span id="page-1-0"></span>**2. Preliminaries**

In the following, the description logic  $ALC$  and the temporal description logic  $LTL_{\text{ALC}}$  is shortly introduced.

### **2.1. Description Logic**

We are going to work with the description logic of Boolean  $ALC$ , i.e.,  $ALC$  without considering roles [\[10\]](#page-6-1). We assume that there is a DL vocabulary given by a set of constants  $N_c$ and a set of concept names  $N_C$ . The  $ALC$  concepts (concept descriptions) over  $N_C$  are described by the grammar

$$
C \longrightarrow A \mid \perp \mid \top \mid \neg C \mid C \sqcap C \mid C \sqcup C
$$

where  $A \in N_C$  is an atomic concept and C stands for arbitrary concepts. A classical ALC interpretation is a pair  $(∆, (·)<sup>I</sup>)$  consisting of a set  $∆$ , called the *domain*, and an *interpretation function*  $(\cdot)^{\mathcal{I}}$  which maps constants to elements in  $\Delta$ , concept names to subsets of  $\Delta$ , and role names to subsets of  $\Delta \times \Delta$ . The semantics of arbitrary concept descriptions for a given interpretation  $\mathcal I$  is as follows:

• 
$$
T^{\mathcal{I}} = \Delta
$$
  
\n•  $L^{\mathcal{I}} = \emptyset$   
\n•  $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$   
\n•  $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$   
\n•  $(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$ 

An *ontology*  $\mathcal O$  is defined as a pair  $\mathcal O = (\mathcal T, \mathcal A)$  of a *terminological box (TBox)*  $T$  and an *assertional box (ABox)*  $A$ . A TBox consists of *general inclusion axioms (GCIs)*  $C \sqsubseteq D$  (" $C$ is subsumed by  $D^{\prime\prime}$ ) with concept descriptions  $C, D$ . An ABox consists of a finite set of *assertions*, i.e., facts of the form  $C(a)$  for  $a \in N_c$ . An interpretation  $\mathcal I$  *models* a GCI  $C \sqsubseteq D$ , for short  $\mathcal{I} \models C \sqsubseteq D$ , iff  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ . An interpretation *I* models an ABox axiom  $C(a)$ , for short  $\mathcal{I} \models C(a)$ , iff  $a^{\mathcal{I}} \in C^{\mathcal{I}}$ . An interpretation is a *model of an ontology*  $({\cal T}, {\cal A})$  iff it models all axioms appearing in  ${\cal T} \cup {\cal A}$ . An ontology  $O$  *entails* a (TBox or ABox) axiom  $ax$ , for short  $\mathcal{O}$   $\models$  ax, iff all models of  $\mathcal{O}$  are also models of ax.

Each TBox  $T$  generates a Boolean algebra, the so-called *Lindenbaum-Tarski algebra*, as follows: For concepts  $C, D$ let ∼ be the relation defined by  $C \sim D$  iff  $\mathcal{T} \models C \sqsubseteq D$ and  $\mathcal{T} \models D \sqsubseteq C$ . Relation  $\sim$  is an equivalence relation inducing for each concept  $C$  an equivalence class  $[C]$ . Define operations ⊓, ⊔, ¬ on the equivalence classes by setting  $[C|\sqcap[D] = [C \sqcap D], [C] \sqcup [D] = [C \sqcup D]$  and  $\neg [C] = [\neg C]$ which can be shown to fulfill the axioms of a Boolean algebra.

### **2.2. Temporal Description Logic**

There are several different temporal description logics, for a survey, see, e.g., [\[8\]](#page-6-2). Here,  $LTL_{ALC}$  [\[11\]](#page-6-3) is considered, as it is a widely known, expressive temporal description logic. Here, a slight adaptation is used, namely roles are

not considered, thus the consideration is restricted to the Boolean part. As temporal operators, ∘ (at the next moment),  $\diamond$  (eventually),  $\Box$  (always in the future) and  $\mathcal U$  (until) are used. The temporal  $ALC$  concept descriptions over  $N_C$ are described by the grammar for  $ALC$  extended with the following

$$
C \longrightarrow \cdots | \circ C | \circ C | \Box C | CUC
$$

Note that  $\diamond C$  is used as shorthand for  $\top \mathcal{U} C$  and  $\square C$  as shorthand for  $\neg \diamond \neg C$ .

A *temporal interpretation*  $\mathcal{J} = (\Delta, (\cdot)^{\mathcal{J}})$  is based on a non-empty domain  $\Delta$  and an interpretation function  $\mathcal J$  that maps every concept name  $A \in N_C$  to a subset  $A^{\mathcal{J}} \subseteq \mathbb{N} \times \Delta$ and every individual name  $a \in N_c$  to an element  $a^{\mathcal{J}} \in \Delta$ .  $(n, d) \in A^{\mathcal{J}}$  describes that d is an instance of A at time point *n*. Constants are considered as *rigid*, meaning they are interpreted the same way at every time point. Additionally, the *constant domain assumption* is assumed to be valid, meaning that constants are not destroyed or created over time. Though, there are different interpretations possible, here the standard assumption of a bounded past and an unbounded future is used, where the time flow is discrete. Thus, the time is represented as  $(N, <)$ . Alternatively, it is possible to interpret a temporal interpretation as an infinite sequence  $\mathcal{J}(0), \mathcal{J}(1), \ldots$  of (non-temporal) interpretations based on the same domain ∆.

The semantic is defined based on the semantic for  $\cal{ALC}$ with the extension of

- $(\circ C)^{\mathcal{J}} = \{(n, d) | (n + 1, d) \in C^{\mathcal{J}}\}\$
- $(CUD)^{\mathcal{J}} = \{(n, d) | \exists m \geq n((m, d) \in D^{\mathcal{J}} \land$  $(k, d) \in C^{\mathcal{J}}$  for  $n \leq k < m$  }.

A temporal interpretation  $\mathcal J$  is a model of a concept  $C$  if  $C$ is satisfied at time point 0, i.e.,  $(0, d) \in C^{\mathcal{J}}$  for some  $d \in \Delta$ . An interpretation  $\mathcal J$  is a *model* of a TBox  $\mathcal T$  if and only if  $C^{\mathcal{J}} \subseteq \overline{D}^{\mathcal{J}}$  for all  $C \sqsubseteq D$  in  $\mathcal{T}$ .

# <span id="page-1-1"></span>**3. Cone Embeddings**

The geometric interpretation that is used as basis for the embedding approach introduced in this paper was presented by Özçep et al [\[6,](#page-5-0) [7\]](#page-6-0) and is based on closed convex cones and a special case of it, the axis-aligned cones. A *closed convex cone X* is a non-empty set for which if  $v, w \in X$ , then also  $\lambda v + \mu w \in X$  for all  $\lambda, \mu \geq 0$ . In the following, the term "cone" refers to closed convex cones. A *polar cone* X° of a closed convex cone  $X$  is defined as

$$
X^{\circ} = \{ v \mid \text{for all } w \in X : \langle v, w \rangle \le 0 \},
$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual scalar product. Now, the DLinterpretation can be interpreted geometrically: The domain  $\Delta$  is interpreted as  $\mathbb{R}^n$  and a concept interpretation  $C^{\mathcal{I}}$  is interpreted as cone C, the negation of this concept  $(\neg C)^{\mathcal{I}}$ as its polar  $C^{\circ}$ , the conjunction of two concepts as set intersection between the respective cones and disjunction via de Morgan. An instance belonging to a concept is then represented as a point in the respective cone. Based on this interpretation, it can be shown that each set of cones in  $\mathbb{R}^n$  closed under set-intersection and polarity leads to an orthologic [\[7\]](#page-6-0). It is possible to restrict the cones to so-called *axis-aligned cones* (al-cones for short). Özçep et al [\[7\]](#page-6-0) show that an  $ALC$ -ontology is satisfiable if and only if there is



<span id="page-2-1"></span>**Figure 1:** Left: cone model for TBox { $A \sqsubseteq B$ }; middle: al-cone model for TBox { $A \sqsubseteq B$ }; right: example for a geometric model based on a sphere.

an al-cone model of that ontology. An al-cone is defined as follows:

$$
X \text{ is al-cone } : \Leftrightarrow
$$
  
 
$$
X = X_1 \times \cdots \times X_n, X_i \in \{ \mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \{ 0 \} \},
$$

where  $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$  and  $\mathbb{R}_- = \{x \in \mathbb{R} \mid x \geq 0\}$  $x \leq 0$ . Thus, al-cones can be considered as unions of neighboring hyperoctants. So, in  $n$  dimensions we have  $4^n$ possible al-cones. Conjunction, disjunction and negation are defined analogously to the case of closed convex cones, as al-cones are a specialization.

**Example 1.** *An example for a cone-model can be seen in Figure [1](#page-2-1) on the left, an example for an al-cone model in Figure [1](#page-2-1) in the middle. In both of them, the TBox*  ${A \sqsubset B}$  *is modeled. First consider the cone model: The dashed cone on the upper* left is the interpretation of the concept A, fulfilling the TBox*axiom, as it is a subset of the gray cone, the interpretation of . The negation of concept is interpreted as the polar cone* of cone  $A<sup>\mathcal{I}</sup>$ , thus the cone containing all rays having an angle *of* 90° *or more to all rays in in*  $A<sup>�}</sup>$ . Conjunction is interpreted as set intersection, thus  $(B \sqcap \neg A)^{\mathcal{I}}$  is the gray, dashed cone *in the top right. Disjunction can be determined via de Morgan, thus, e.g.,*  $(A \sqcup \neg B)^{\mathcal{I}}$  *would be*  $(\neg(\neg A \sqcap B))^{\mathcal{I}}$ *, thus the polar of the gray, dashed cone in the upper right, therefore the convex hull of the cones of*  $A<sup>\mathcal{I}</sup>$  and  $(\neg B)^{\mathcal{I}}$ . ABox-instances *are interpreted as points in the space, therefore an instance* with  $A(a)$  would be placed in the cone  $A<sup>\mathcal{I}</sup>$  as  $a<sup>\mathcal{I}</sup>$ .

*The al-cone example in Figure [1](#page-2-1) in the middle represents the same TBox, however, based on a restricted cone model not based on arbitrary but axis-aligned cones. There, is interpreted as positive -axis and is interpreted as upper right quadrant. The polar al-cones are defined the same as* for the case of classical cones, thus, for example,  $(\neg B)^{\mathcal{I}}$  is the *lower left quadrant.*

*In both cases,* ⊥ *is interpreted as point of origin, as there all cones intersect, leading to a contradiction, thus to*  $\perp^{\mathcal{I}}$ .

An al-cone interpretation can be defined formally as follows:

<span id="page-2-4"></span>**Definition 1.** *[\[6,](#page-5-0) Definition 1] A* Boolean al-cone interpreta- $\text{tion } \mathcal I \text{ is a structure } (\Delta, (\cdot)^{\mathcal I}) \text{ where } \Delta \text{ is } \mathbb R^n \text{ for some } n \in \mathbb N,$ and where  $(\cdot)^{\mathcal{I}}$  maps each concept symbol  $A$  to some al-cone *and each constant*  $\bar{a}$  *to some element in*  $\Delta \setminus {\{\vec{0}\}}$ *. An* al-cone interpretation for arbitrary Boolean ALC concepts *is defined recursively as*  $(\top)^{\mathcal{I}} = \Delta$ ,  $(\bot)^{\mathcal{I}} = \{\vec{0}\}$ ,  $(C \cap D)^{\mathcal{I}} = \vec{0}$  $(C^{\mathcal{I}} \cap D^{\mathcal{I}}, (\neg C)^{\mathcal{I}} = C^{\circ},$  and  $(C \sqcup D)^{\mathcal{I}} = (\neg(\neg C \sqcap \neg D))^{\mathcal{I}}$ .

*The notions of an al-cone being a model and that of entailment are defined as in the classical case (but using al-cone interpretations).*

This then leads to the following proposition:

<span id="page-2-2"></span>**Proposition 1.** [\[6,](#page-5-0) Proposition 2] Boolean ALC-ontologies *are classically satisfiable if and only if they are by a geometric* model on some finite  $\mathbb{R}^n$  based on al-cones of the form  $b_1 \times$  $\cdots \times b_n$  with  $b_i \in \{ \{0\}, \mathbb{R}_+, \mathbb{R}_-, \mathbb{R} \}$  for  $i \in \{1, ..., n\}$ .

The aim of this paper is to show a similar result for temporal description logics.

### <span id="page-2-0"></span>**4. Embedding of Temporal DL**

When considering those cone models, it gets apparent that on each ray (a ray through a point  $x \in \mathbb{R}^n$  contains all  $\lambda x$  for  $\lambda \geq 0$ ) only one concept is represented and that instances placed on such a ray are indistinguishable on a conceptual level. Therefore, without loss of expressivity it is possible to model an al-cone model instead of in  $\mathbb{R}^n$ on the unit  $n$ -sphere (an unit  $n$ -sphere can be defined as  $S^n = \{x \in \mathbb{R}^n : ||x|| = 1\}$  and thus gain the space to model temporal aspects.

**Example 2.** *In Figure [1](#page-2-1) on the right an example for a geometric model based on a sphere can be seen for a simple ontology* with only one concept A. This concept is represented as a *sphere part with radius one in the upper right quadrant. It is based on the al-cone of the upper right quadrant where each* vector in this al-cone is set to unit length.  $A<sup>\mathcal{I}</sup>$  can easily be *extended into an al-cone by considering the convex closure of it. The easiest way to determine the negation of is to consider the al-cone extension of*  $A<sup>T</sup>$  (thus, the upper right *quadrant), taking the polarity (the lower left quadrant) and then intersect it with a sphere of radius one, leading to the sphere part that can be seen in the figure. An instance with*  $A(a)$  can then be placed on the sphere part representing  $A$ *with*  $||a^{\mathcal{I}}|| = 1$ .

It can be shown that both the al-cone model and the sphere model have in fact the same expressivity.

<span id="page-2-3"></span>**Proposition 2.** *Boolean ALC-ontologies are satisfiable by a geometric model on some finite* R *based on al-cones of the form*  $X_1 \times \cdots \times X_n$  *with*  $X_i \in \{ \{0\}, \mathbb{R}_+, \mathbb{R}_-, \mathbb{R} \}$ *for*  $i \in \{1, \ldots, n\}$  *if and only if they are satisfiable by a geometric model on some finite*  $S<sup>m</sup>$  (with  $m \leq 2n$ ) where

*concepts are represented as intersection between al-cones and the unit sphere.*

*Proof.*  $\rightarrow$  The concept representation can be directly transformed to a spherical representation without loss of expressivity, as for a concept  $C^\mathcal{I}$  for each  $x \in C^{\mathcal{I}}, \frac{x}{||x||} \in C^{\mathcal{I}}$ . Therefore, it is possible to interpret a concept  $C^\mathcal{I}=\{x\mid x\in C^{\mathcal{I},\mathrm{cone}}\&||x||=1\}$ 1}, where  $C^{\mathcal{I},\text{cone}}$  represents the original concept representation as al-cone. It can be trivially seen that the represented TBox-axioms do not change. The instances can then be placed on the unit sphere in the same manner as it is done for the construction of the al-cone model in the proof of Proposition [1](#page-2-2) (as can be seen in [\[6\]](#page-5-0)). The only problem arises if a concept lies on an axis, as then only one instance can be represented as belonging to this concept. This can, however, be solved by using the same construction principle as in the proof of Proposition 8 of [\[7\]](#page-6-0), namely increasing the number of dimensions by creating a geometric mod of size  $\mathbb{R}^{2n}$  based on two concatenations of the original model in  $\mathbb{R}^n$ . Thus, if a concept is, e.g., placed at point  $(1, 0)$ , then two dimensions can be added having the same concept memberships as the first two dimension. Then, this concept is no longer at point  $(1, 0)$  but on the circle segment between  $(1, 0, 0, 0)$  and  $(0, 0, 1, 0)$  and thus allows for placing arbitrary many different instances on this segment.

A difference to the cone-based model is that  $\bot^{\mathcal{I}}$  is no longer represented as  $\{0\}$  but as  $\emptyset$  which, however, does not influence the satisfiability.

← Assume a geometric model based on a sphere  $S^m$  for  $m>0$  is given. As each concept  $C^\mathcal{I}$  is based on the intersection of the unit sphere with an al-cone, it is trivially possible to extend the representation on the sphere to an al-cone again such that for  $C^{1,cone}=$  $\{\lambda x \mid x \in C^{\mathcal{I}}\&\lambda \geq 0\}$  (thus by considering the convex hull).

The interpretation of instances does not change.

 $\Box$ 

Note that the increase of dimensions depends on the structure of the cone-based model and thus, the doubling of dimensions is only an upper bound.

When reducing an al-cone model to a sphere model, this opens up the opportunity to model spheres of different radii in a vector space, thus modeling different geometric models in the same vector space, as Proposition [2](#page-2-3) is trivially adaptable to a sphere of a different radius. This directly leads to the possibility of modeling temporal description logics: As each temporal interpretation can be modeled as a sequence  $\mathcal{J}(0), \mathcal{J}(1), \ldots$  of non-temporal interpretations, it is possible to model each  $\mathcal{J}(i)$  on a sphere with radius  $i + 1$  in the vector space  $(i + 1$  is considered, as  $\mathcal{J}(0)$  is interpreted as model on the unit sphere). To follow the assumption that instances are rigid, each instance is represented as a ray, interfering with all spheres, thus with concepts of all time points. Then, an instance has an interpretation for each time point, thus for each distance  $1, 2, \ldots$  to the point of origin.

Now, the classical operators need to be adjusted to model the temporal case and the temporal operators need to be defined. The domain  $\Delta$  is, in contrast to the  $\mathcal{ALC}$ -case not represented as  $\mathbb{R}^n$  but as the set of unit vectors in  $\mathbb{R}^n,$ 

thus  $\Delta = \{x \mid ||x|| = 1 \& x \in \mathbb{R}^n\}$ . The tuple  $(i, d)$ , thus an instance at a specific time point is then represented as  $(i + 1) \cdot d$  for  $d \in \Delta$  and a concept  $C^{\mathcal{J}}$  is represented as  $C^{\mathcal{J}} = \{(i+1) \cdot d \mid d \in C^{\mathcal{J}(i)}\}$ . Then, the interpretation of ∘ and  $U$  can be straightforwardly adapted.

<span id="page-3-0"></span>
$$
(\circ C)^{\mathcal{J}} = \{(i+1) \cdot d \mid (i+2) \cdot d \in C^{\mathcal{J}}\}
$$
  
\n
$$
(CUD)^{\mathcal{J}} = \{(i+1) \cdot d \mid \exists m \ge i : (m+1) \cdot d \in D^{\mathcal{J}}
$$
  
\n
$$
\& (k+1) \cdot d \in C^{\mathcal{J}} \text{ for } i \le k < m)\}
$$
\n(1)

The temporal interpretation of each time step leads to the representation of concepts as sphere parts. Now the question arises how the concepts look like when considering them in a combination of all time steps. The main observation is that the resulting sets are neither closed convex cones nor al-cones anymore, as for a point in a concept not necessarily the ray through this point is contained in that concept. It needs to be examined in a practical setting whether there are concepts which are rigid or at least constant in several successive time steps to be able to use the advantages of convexity as much as possible. However, if only one time step is considered, then it is again possible to consider closed convex cones, resp. al-cones by using the construction mentioned in the proof of Proposition [2.](#page-2-3)

This directly leads to the definition of a temporal cone interpretation, following and extending the ideas of the classical cone interpretation of Definition [1.](#page-2-4)

<span id="page-3-2"></span>**Definition 2.** *A* temporal al-cone interpretation  $J$  is a *structure*  $(\Delta, (\cdot)^{\mathcal{J}})$  *where*  $\Delta$  *is the set of unit vectors in*  $\mathbb{R}^n$ for some  $n \in \mathbb{N}$  and where  $(\cdot)^{\mathcal{J}}$  maps each concept symbol *for each radius to a union of intersections of al-cones and the sphere with radius i for*  $i \in \{1, ..., m\}$  *for*  $m \in \mathbb{N}$  *or for*  $i \in \mathbb{N}$  and each constant to an element of  $\Delta$ *.* 

- $(A \sqcap B)^{\mathcal{J}} = A^{\mathcal{J}} \cap B^{\mathcal{J}}$
- $(\neg A)^{\mathcal{J}}$  defined based on the polarity of  $A^{\mathcal{J}}$  inter*sected with a sphere of radius*  $||x||$  *for*  $x \in A$
- ∘, *as denoted in Equation* [\(1\)](#page-3-0) *and*
- ◇, □ *and* ⊔ *interpreted via the other operators as stated above*

The construction is in the following illustrated by an example taken from [\[11\]](#page-6-3).

<span id="page-3-1"></span>**Example 3.** *As an example, a TBox is modeled which contains the statement that any non-EU country has to be first an EU-member candidate before it can be an EU-member:*

$$
\neg EU\_member \sqcap \diamond EU\_member \sqsubseteq \\ \diamond (EU\_candidate \; U \; EU\_member)
$$

*One exemplary model can be seen in Figure [2.](#page-4-1) The first model for time step 0 is on the sphere with radius 1. At this time point, there aren't any EU-candidates but some EU-members. Then, to fulfill the axiom, there must be a point in time where a non-EU-member is included into the EU, this is modeled here in the second time step (in the al-cone model concatenated with a sphere with radius three) in the upper right quadrant. To model the subsumption mentioned in the axiom, it is necessary to incorporate the concept of a EU-candidate. Whereas at time step 0, there weren't any EU-candidates, in the first time step, the instances in the upper right quadrant became EU-candidates. Those instances are in the second time step EU-members, thus, the axiom is fulfilled for the upper right quadrant. The other three quadrants do not fulfill the*

*premise of the subsumption, as either the instances are not a non-member of the EU or they aren't becoming a member eventually. In this low dimensional example, there are concepts represented by a point at a time step, meaning that only one instance can be placed in this concept. This can be solved by increasing the dimension of the model, as described in the proof of Proposition [2](#page-2-3) and is omitted here for readability reasons.*

One basic property of cone-based models is their ability to model unknown or indeterminable information: the socalled *faithfulness* [\[6\]](#page-5-0) of the model. This term denotes that, e.g., in Figure [1](#page-2-1) in the middle an instance placed in the lower right quadrant can belong to  $A$  or to  $\neg A$ . Thus, if in an ABox no information on the membership to  $A$  of this instance is given, then it is not necessary to chose one of the two, but it is possible to model this missing knowledge. This ability can also be used in the cone-based model in the temporal description logic setting, as will be argued in detail below based on the previous example.

**Example 4** (Example [3](#page-3-1) continued)**.** *As stated before, the upper right quadrant of the model in Figure [2](#page-4-1) fulfills the premise and the conclusion of the stated axiom. The axes represent some atomic concepts. The other three quadrants, however, allow for the incorporation of faithfulness: The most obvious case is the lower left quadrant, there it is at no time point known whether a state is member of the EU or not, however, it is known that the state is definitely not a candidate (assume for example that someone has created the ABox knowledgeable about EU-candidates and the plans which states will be candidates, but not as knowledgeable about the history of the EU and the actual members). The lower right quadrant denotes states which are at the moment neither member nor candidate but for which it is at least thinkable of that they are maybe later on a candidate and a member. The upper left quadrant denotes states where it is known for some point in the future that they will be EU-members, however, it is not known whether they are members already or need to be candidates first.*

This example illustrates the wide possibilities of using faithfulness in the context of temporal models. Due to the restricted size of the model, it is not possible to model each possibility for unknown information, it is however, possible to focus on specific possibilities which should be modeled.

These temporal cone models can be used as basis for a learning approach. One option would be to extend the approach to the handling of relations (and thus to leave the Boolean case) and to do temporal knowledge graph embedding. The cone models in the form presented here also enable for embedding approaches by themselves, e.g., having a training set with instances, their attributes and their concept memberships and a test set only with instances and their attributes to predict the concepts. Additionally, possibly a temporal DL-ontology is given. Then, an embedding can be learned, e.g., via a neural network, where the optimizing function is on the one hand based on the axioms of the ontology and on the other hand based on the correct placement of instances. By choosing a sufficient dimension of the cone model, it is possible to find a trade of between faithfulness on the one site and recall on the other.



<span id="page-4-1"></span>**Figure 2:** Example for a cone-based temporal geometric model, here restricted to three time steps modeling the axiom mentioned in Example [3.](#page-3-1) For better readability,  $C<sup>i</sup>$  is used as a shorthand for  $C^{\mathcal{J}(i)}$ .

# <span id="page-4-0"></span>**5. Expressivity**

The construction principle mentioned in the last section is usable for modeling background knowledge in form of an ontology in a learning approach, as there it is mostly appropriate to model an approximation of the ontology. However, when there is a need for modeling the ontology exactly, it is necessary to prove that each classically satisfiable ontology is also satisfiable when considering the proposed embedding. In the following, a theorem proves this statement for  $LTL_{\mathcal{ALC}}$  ontologies and temporal embeddings as proposed in Definition [2.](#page-3-2) The proof for the case of Boolean  $ALC$ in [\[7\]](#page-6-0) without temporal aspects is based on the fact that the ontology consists of a finite number of atomic concepts which can be placed on the half axes of the geometric model (thus, e.g., on  $\mathbb{R}_+ \times \{0\} \times \cdots \times \{0\}$ ) and thus are the basis for the other concepts. This is in this case not possible, as here not only one interpretation  $\mathcal J$  needs to be considered but a number of interpretations for different time points  $\mathcal{J}(0), \mathcal{J}(1), \ldots$  To solve this problem, the ultimately periodic model property for temporal logic is used to restrict the number of time steps needed for the model.

<span id="page-4-2"></span>**Theorem 1** (Ultimately Periodic Model Property)**.** *[\[12,](#page-6-4) [13\]](#page-6-5) A LTL* formula  $\phi$  is satisfiable if and only if there is an ultimately *periodic model*  $\psi$  such that  $\phi$  is fulfilled, thus there is  $m, i \in \mathbb{N}$ *such that there is a model where*  $\mathcal{J}(k) = \mathcal{J}(k + m)$  *for all*  $k > i$  and period m, where *i* is a finite starting index.

With the help of this theorem, it is possible to prove the following main result of this paper, stating the strong connection between temporal cone embeddings and  $LTL_{\mathcal{ALC}}$ ontologies.

<span id="page-4-3"></span>Theorem 2. A Boolean LTL<sub>ACC</sub>-ontology with constant do*mains is satisfiable iff it has a geometric model on some finite* R *based on a temporal al-cone-interpretation as introduced in Definition [2.](#page-3-2)*

*Proof.*  $\rightarrow$  The ultimately periodic model property of Theorem [1](#page-4-2) can be used as a basis for the geometric model based on cones. Based on this theorem, it is enough to model a finite set of interpretations  $\mathcal{J}(0), \ldots, \mathcal{J}(i)$  up to the starting index *i* and additionally the first period, thus,  $\mathcal{J}(i+1), \ldots, \mathcal{J}(i+1)$  $m$ ). After the time point  $i + m$ , the information on the period can be used to define all following time points based on the preceding ones, thus a further modeling is not necessary. Therefore, there are only finite many atomic concepts possible. Then it is possible to create an intermediate al-cone model where analogously to the proof of Proposition [1](#page-2-2) each concept is placed on one half-axis. The instances can then be interpreted as points in this space as done in Proposition [1.](#page-2-2) This al-cone model can then be modified to lead to a temporal coneinterpretation as follows: First, for each atomic concept, a temporal representation (thus based on a representation for each time step) is created following the rules of Equation [\(1\)](#page-3-0). Then, the point  $x$  representing an instance  $a$  in the intermediate model is changed to a ray through the point of origin, meaning  $a^{\mathcal{I}} = {\lambda x \mid \lambda \geq 0}.$ 

 $\leftarrow$  Proposition [2](#page-2-3) in combination with Proposition [1](#page-2-2) shows that each sub-model  $\mathcal{J}(i)$  for  $i > 0$  represents a satisfiable ALC-ontology. As a temporal DLinterpretation can be interpreted as combination of classical interpretations  $\mathcal{J}(i)$  for  $i \geq 0$  and the operators modeled can be straightforwardly interpreted as classical operators, the classical interpretation is also satisfiable.

 $\Box$ 

Analogously to the case of cone models introduced in Section [3,](#page-1-1) for the temporal case it is also possible to use instead of al-cones cones as basis for the concept representation and thus increase the expressivity beyond  $LTL_{\text{ALC}}$ .

# **6. Related Work**

The embedding of temporal description logic is an important research topic in KGE, as the triples stored in knowledge graphs, e.g., (alice, loves, bob) can be extended with temporal information, as not all triples are valid at every time point. If alice and bob are separated today, the triple can be extended to (alice, loves, bob, 2023). There are several KGE-approaches handling the embedding of this temporal information (see, e.g., an extension of the classical KGEapproach TransE, TTransE [\[14\]](#page-6-6) or [\[9\]](#page-6-7) for a survey). They are, however, mostly only based on link prediction and don't incorporate concept information, e.g., in form of ontologies. One related approach, though not incorporating ontological information, has been presented by Dasgupta et al [\[15\]](#page-6-8), modeling time points explicitly, however, not like in this approach based on the distance to the point of origin but by modeling each time point as an individual hyperplane. Thus, an al-cone-like structure is used, however, not for modeling conceptual information but time points. One approach able to model first order logic is TFLEX [\[16\]](#page-6-9), however, it still does not incorporate concepts as set of a specific geometric structure (e.g., convex sets). Zhang [\[17\]](#page-6-10) introduces an embedding based on a similar principle of modeling time steps based on distances, however, does not model concepts explicitly. There are several other approaches for embedding ontologies geometrically, e.g., modeling  $\mathcal{EL}^{++}$  with the help of boxes [\[3\]](#page-5-1) or spheres [\[4\]](#page-5-2) or  $ALC$  based on subspaces [\[5\]](#page-5-3). However, they do not incorporate temporal information.

# **7. Conclusion**

In this paper, I have presented a model showing that for temporal description logics similar approaches as for classical description logics can be used to embed them into a vector space, here demonstrated based on an adaptation of a model based on closed convex cones. This is a first step towards a learning approach which is not only able to model temporal aspects regarding instances but also modeling their conceptual behavior, thus it enables designing a learning approach respecting background knowledge information. This model opens up the opportunity to be extended to metric temporal description logics (see, e.g., the work of Gutiérrez-Basulto et al [\[18\]](#page-6-11)) due to the possibility of representing distances geometrically. Another possible extension of this approach regards the consideration of roles. There are many KGEapproaches incorporating temporal information, therefore, it would be interesting to consider whether some of the existing approaches of modeling relations can be used to extend this approach and whether then Theorem [2](#page-4-3) can be extended to full  $ALC$ . Leaving the context of description logics, it would also be interesting to examine the expressivity of a model not based on al-cones but on adaptations of closed convex cones in general.

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