Modal Categorical Inferences in Quarc

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Abstract

We investigate basic forms of inference involving modal notions and quantifiers, called modal categorical inferences. We do so by extending Quarc, a novel logic that assigns a primary role to quantified phrases, with modalities from the hexagon of opposition. We show that there are two possible readings of *de dicto* modalities (called symmetric and asymmetric, respectively), as opposed to the unique reading of *de re* modalities. We focus on the asymmetric reading of *de dicto* modalities and explore the logical relations that obtain between them. These are proven in a natural deduction system, accompanied by an appropriate syntax and semantics, and graphically represented via a dodecagon of opposition. Moreover, we show that the asymmetric reading, in contrast to the symmetric one, preserves all properties of the hexagon for basic modal notions. Thus, it provides a particularly attractive basis on which to further investigate quantified modal reasoning.

Keywords

Modal Categorical Inferences, Quantified Modal Logic, Quarc, Polygons of Opposition, Categorical Reasoning

1. Introduction

Many arguments in everyday reasoning involve an interaction of modal notions and quantification. Representing these arguments in a formal setting is known to be challenging, as witnessed by the existence of a plethora of rival accounts of quantified modal logic [1]. Although the choice of an account can be motivated by semantic issues (e.g. the interpretation of possible worlds or the identity of individuals across worlds), a more fundamental criterion to compare alternative accounts is how they formalize the syntactic structure of quantified modal statements. In the present work, we will employ a formalism that keeps track of quantified phrases and we will focus on the logical rendering of simple arguments involving modal categorical statements. Importantly, our syntactic focus will unveil that two alternative readings for some of these statements are available. As we will see, different motivations lead to adopting one reading or the other.

We call ϕ a modal categorical statement iff:

- it makes reference to two *categories* of individuals (e.g. *S* and *P*), one of which is used as a *domain of quantification*;
- it contains one *modality* from a given family;
- it does not make reference to specific individuals.

A modal categorical inference is an inference from a statement ϕ_1 to a statement ϕ_2 s.t. both ϕ_1 and ϕ_2 qualify as modal categorical statements. It can also be seen as a simple *argument* consisting of one premise (ϕ_1) and a conclusion (ϕ_2). Here is an example:

(*) It's necessary that every data controller is authorized to process user data. Therefore, it isn't contingent

that some data controller is authorized to process user data.

In (*), ϕ_1 is the first sentence and ϕ_2 the second; they are separated by the conclusion marker 'therefore'. The two categories of individuals correspond to the properties 'is a data controller' and 'is authorized to process user data';¹ the modalities involved are 'necessity' (in ϕ_1) and 'contingency' (in ϕ_2). Inferences of this kind are involved in more complex arguments, ranging from patterns of syllogism (with two premises) to tree-like argumentative structures. Thus, understanding how modal categorical inferences work is crucial to generally assess arguments including modalities and quantification.

We will be concerned with modalities from the *hexagon of opposition* in Fig. 1. Something will be said to be:

- necessary iff it holds in all cases;
- possible iff it holds in some cases;
- *impossible* iff it holds in no cases;
- avoidable iff it does not hold in some cases;
- *contingent* iff it holds in some cases and it does not hold in other cases;
- *absolute* iff either it holds in all cases or it holds in no cases.

For further analysis of the hexagon, see e.g. [2].²

Statement (**) is compatible with the three options below:

- I know that all user accounts are secure;
- I know *that* some user accounts are secure and some aren't;
- I know that no user account is secure.

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¹The analysis of the two categories of individuals involved in a modal categorical statement could be refined by representing some of these as *many-ary* predicates rather than *unary* predicates. Yet, our simplification is sufficient for the purposes of the present article. In the end, a predicate S applied to n terms, for n > 1, can always be transformed into a unary predicate S' by incorporating n - 1 terms in S'. For instance, the binary predicate 'is a friend of' as applied to the terms 'Sara' and 'John' may be transformed into the unary predicate 'is a friend of John' as applied to 'Sara'.

²The reason why we chose to work with the hexagon, rather than the *square of opposition* (without contingency and absoluteness), is that in many contexts of reasoning hexagon modalities play a crucial role. For instance, consider the following statement:

^(**) I know whether all user accounts are secure.

This compatibility is explained by the difference between knowing

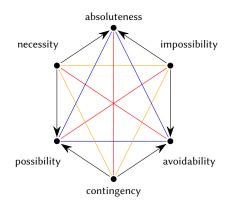


Figure 1: Hexagon of opposition for modalities. An arrow from x to y indicates *subalternation*: if x is true, then y is true too (while the opposite direction does *not* hold). An orange line between x and y indicates *contrariety*: if x is true, then y is false. A blue line between x and y indicates *subcontrariety*: if x is false, then y is true. A red line between x and y indicates *contradiction*: x is true if and only if y is false.

In this work we are interested in both the formal and the visual representation of modal categorical inferences. As far as the *visual representation* is concerned, it is well-known that polygons of opposition, which originated in medieval works on Aristotle's logic, facilitate the comprehension of logical relations and are therefore regarded as cognitively efficient devices for deductive reasoning [3].

As far as the *formal representation* is concerned, we want to employ a mathematical language that is as simple as possible and closely adheres to the structure of natural languages (in particular, English); thus, a language that is useful to move back and forth between formal and informal reasoning. In this regard, we can observe that the syntax of the Predicate Calculus (hereafter, PC) does not keep track of the fact that *quantified phrases* are used as arguments of predication in modal categorical statements. For instance, when we formalize argument (*) following PC's syntax,³ we need to render the first sentence as a construction involving *material implication* (\rightarrow), namely:

 $\Box \forall x (\texttt{DataController}(x) \to \texttt{Authorized}(x))$

Yet, there is no natural language expression corresponding to \rightarrow in (*). Moreover, due to the use of \rightarrow and of the individual variable *x*, we lose information about the fact that the quantified phrase 'every data controller' plays the role of argument of predication in (*).

By contrast, the formalism that we are going to employ, the Quantified Argument Calculus (shortened 'Quarc') [4], allows for a simple and explicit formalization of quantified phrases. More precisely, the quantifiers \forall and \exists are always used in combination with a unary predicate P and the result constitutes an argument of predication. For instance, we can use the expression

(VDataController)Authorized

to formalize 'every data controller is authorized to process user data'.⁴ As shown in [6], it is possible to define a translation mapping PC formulas into Quarc formulas thanks to the addition of a universal predicate T ('is a thing') to the language of Quarc, as well as a converse translation.⁵

When we combine a modal notion and a quantifier in a sentence, we distinguish between *de re* and *de dicto* combinations.⁶ In the former, the modalization applies to (categories of) individuals, whereas in the latter it applies to an entire sub-sentence. Consider the difference between 'someone is known to rob banks in this area' (*de re*) and 'it is known that someone robs banks in this area' (*de dicto*). Only the former statement, when uttered, conveys the information that we are aware of the identity of the robber. For a discussion of *de re* and *de dicto* combinations in modal PC, see [1].

Quarc keeps track of the distinction between *de dicto* and *de re* modalities as follows, where \Box denotes necessity:

De Dicto	$\Box(\forall DataController)$ Authorized	
De Re	(∀DataController) □ Authorized	

Modal versions of Quarc are contained in various works, including [4, 7, 8, 9]. Although a systematic comparison of the expressiveness of modal PC and of modal Quarc has not yet been offered in the literature, as long as we restrict our analysis to modal categorical statements, it is clearly possible to define a simple back-and-forth translation between the two formal languages, following the hints provided above. We will discuss this aspect further at the end of our article.

The analysis of modal categorical inferences in Quarc was first put forward in [10], where 24 combinations (12 *de dicto* + 12 *de re*) of quantifiers and hexagon modalities were identified. Here we will focus on *de dicto* combinations and provide an *alternative reading* of these, showing that it gives rise to a radically different family of modal categorical inferences. Our aim is to emphasize that while *de re* combinations have just one legitimate reading, there are two legitimate readings for *de dicto* combinations, the one proposed in [10] and the one proposed here.

Below is the reading of *de dicto* combinations offered in [10], where \diamond stands for possibility, \neg for negation, \land for conjunction and \lor for disjunction:

- **NU** (Necessary Universality): $\Box \forall SP$
- **NP** (Necessary Particularity): $\Box \exists SP$
- **PU** (Possible Universality): $\Diamond \forall SP$
- **PP** (Possible Particularity): $\Diamond \exists SP$
- **VU** (Avoidable Universality): $\Diamond \forall S \neg P$
- **VP** (Avoidable Particularity): $\Diamond \exists S \neg P$
- **IU** (Impossible Universality): $\Box \forall S \neg P$
- **IP** (Impossible Particularity): $\Box \exists S \neg P$
- **CU** (Contingent Universality): $\Diamond \forall SP \land \Diamond \forall S \neg P$
- **CP** (Contingent Particularity): $\Diamond \exists SP \land \Diamond \exists S \neg P$
- **BU** (Absolute Universality): $\Box \forall SP \lor \Box \forall S \neg P$
- **BP** (Absolute Particularity): $\Box \exists SP \lor \Box \exists S \neg P$

whether and knowing that. While knowing that can be interpreted as a form of necessity/impossibility (hence, a modality in the square of opposition), knowing whether should be interpreted as a form of absoluteness (hence, a modality *not* in the square of opposition).

³We stress that our idea of reducing many-ary predicates to unary predicates does not play any role in this argument.

⁴A similar syntactic treatment of quantified phrases can be found in *description logics* [5].

⁵The converse translation works for a fragment of Quarc. The original version of Quarc presented in [4] features additional syntactic devices that are not used in the present work (e.g. reorder of predicates or anaphora) and that are not straightforwardly translatable into the syntax of PC.

⁶We can also speak of *de re* and *de dicto modalities*, given that these combinations are named with reference to the scope of the modal notion.

The family of modal categorical inferences associated with this reading is graphically represented as a *dodecagon of opposition* in Fig. 2. Such a reading of *de dicto* combinations ensures *total symmetry* with respect to the modal categorical inferences based on *de re* combinations. As a matter of fact, the dodecagon of opposition for *de re* combinations offered in [10] is identical to the one in Fig. 2. For this reason, we will say that the aforementioned reading of *de dicto* combinations is the *symmetric reading* (with respect to *de re* combinations).

However, in the symmetric reading modalities involving negation are decomposed. Take VU (Avoidable Universality), rendered as $\Diamond \forall S \neg P$. The quantified phrase $\forall S$ occurs between \diamondsuit and \neg and thus, in a sense, the modality at issue is not purely *de dicto* (i.e. about the statement): while \diamond applies to a statement, ¬ applies to a category of individuals (predicate). Moreover, as a consequence of this, some fundamental properties of hexagon modalities fail. For instance, one would expect that something is avoidable iff it is not necessary. Yet, this does not hold under the symmetric reading. Consider again VU in the list above: assuming the standard duality of \square and $\diamondsuit,$ we can see that \mathbf{VU} is not equivalent to \neg **NU**, namely $\neg \Box \forall SP$, under the symmetric reading. Summing up, the symmetry between de re and de dicto combinations proposed in [10] is obtained at a certain cost.

The alternative reading of *de dicto* combinations that we will propose in this article avoids the aforementioned problems. The main idea behind this reading is anticipating the occurrence of negation in a formula so that it always appears (if at all) immediately after a modal operator. We will call it the *asymmetric reading* (with respect to *de re* combinations), since it gives rise to a dodecagon of opposition that is highly different from the one in Fig. 2. The new dodecagon is illustrated later in our article, in Fig. 3, once the analysis of the asymmetric reading is carried out in rigorous terms.

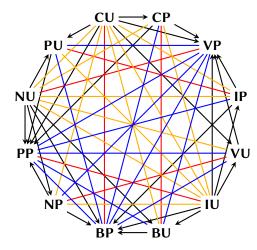


Figure 2: Dodecagon for the symmetric reading of *de dicto* combinations, taken from [10]. The same colour conventions used in the previous figure apply.

Our contribution paves the way to a systematic taxonomy of modal categorical inferences, which have not received much attention in the literature so far, despite their relevance for everyday reasoning. We conjecture that this might be due to the fact that PC is the underlying framework used by many researchers working in the area of modal logic and that the main ingredients of a modal categorical statement require a more complex representation in modal PC.

The article is organized as follows. In sections 2 and 3 we introduce the syntax and the semantics of modal Quarc, respectively. Section 4 contains a natural deduction calculus to build proofs. In section 5, we illustrate the modal categorical inferences associated with the new reading of *de dicto* combinations, together with the corresponding dodecagon of opposition. Finally, section 6 indicates directions for future research.

2. Syntax

We follow the basic set-up for the language of Quarc found in [11], but exclude some ingredients, namely anaphora, reorders and predicates beyond arity 1, since they are irrelevant for present purposes.

Definition 2.1 (Language). The *language* \mathcal{L} consists of the following symbols:

- (a) Denumerably many singular arguments a_1, a_2, a_3, \dots
- (b) Denumerably many predicate symbols of arity 1, P₁, P₂, P₃,
- (c) Connectives: \land , \lor , \rightarrow , \neg .
- (d) Quantifiers: \forall , \exists .
- (e) One modal operator: \Box .
- (f) Auxiliary symbols: brackets ((,)).

Remark 2.1. We shall use P, Q, R, S, etc. to denote arbitrary predicate symbols, and a, b, c, etc. to denote arbitrary singular arguments. The lower case letter q will be used to denote either of the quantifiers. The set of singular arguments of \mathcal{L} shall be denoted by SA, whereas the set of predicate symbols will be denoted by PRED.

Remark 2.2. Although singular arguments are not involved in categorical reasoning, they are needed to provide truthconditions for quantified statements, as we will see below. The requirement for denumerably many singular arguments is not a necessity, but would enable more straightforward soundness and completeness proofs. While these aspects are not relevant to the present investigation, we decided to define languages in this way for the purpose of presenting a logic that is functional beyond its applications in this paper.

Before defining the set of formulas, one further notion needs to be introduced.

Definition 2.2 (Quantified Arguments). Let $P \in PRED$. The expressions $\forall P$ and $\exists P$ are called *universally* and *particularly quantified arguments*, respectively.

Definition 2.3 (Formulas). The set of \mathcal{L} -formulas FORM is defined recursively as follows:

- (a) **Basic formulas**: Let $c \in SA$ and $P \in PRED$ be given. Then cP is a *basic formula*.
- (b) **Predicate negation**: Let c and P be as in (a). Then, $c\neg P$ is a formula.⁷
- (c) Connectives: If φ and ψ are formulas, then so are (φ ∨ ψ), (φ ∧ ψ) and (φ → ψ). If φ is a formula, then so is ¬φ.

⁷It is possible to extend both the syntax and the semantics to allow for finite strings of \Box and \neg in this position, but these are irrelevant for our purposes.

- (d) **Modals:** If ϕ is a formula, then so is $\Box \phi$.
- (e) Governed formulas: Let φ(c) be a formula containing an occurrence of some c ∈ SA, and let qP be a quantified argument. If there is no quantified argument to the left of c in φ(c), and there is no (proper) substring ψ in φ(c) s.t. ψ is a formula which contains c, then φ[qP/c] is a formula. It is said to be governed by that occurrence of qP.
- (f) Nothing else is a formula.

The notion of governance provides an analogue to the usual definitions of quantifier-scope and variable-binding in PC (cf. [4]).

Example 2.1. The expression $(\Box a_3 P_1 \land \forall P_5 P_8)$ is a formula of \mathcal{L} . However, it is not governed by $\forall P_5$. For a formula of the form $(\Box a_3 P_1 \land cP_8)$, for some $c \in SA$, does not meet the requirements of 2.3 (e). Rather, it was generated via 2.3(c) by combining the formula $\Box a_3 P_1$ and the governed formula $\forall P_5 P_8$. The latter *is* governed, since it was generated from a formula of the form cP_8 , satisfying the conditions of 2.3(e).

Remark 2.3. We shall usually omit the brackets in conjunctions, conditionals and disjunctions, as long as unique readability is preserved.

Remark 2.4. While it will not be crucial for the following discussion, we wish to remark that FORM has unique parsing. The proof for an extended syntax can be found in [11], a simple adaption of which proves the same result for FORM.

3. Semantics

While Quarc has been interpreted with both model-theoretic and truth-valuational semantics, we employ the former. Originally introduced in [6], and further developed in [7], the model-theoretic semantics enjoy greater familiarity and should thus be more intuitive. However, such semantics for Quarc differ, in some respects, from those for the predicate calculus. This further extends into the modal semantics originally presented in [7] and repeated below. A systematic analysis of model-theoretic semantics for modal Quarc can be found in [8].

Definition 3.1 (Frames). A *frame* \mathcal{F} is an ordered pair $\langle W, R \rangle$ of a non-empty set of possible worlds or indices W and a binary relation $R \subseteq W \times W$. If $w \in W$ stands in relation R to $v \in W$, we write wRv.

We will primarily be working with *serial* frames, i.e. where for each $w \in W$ there is some $v \in W$ s.t. wRv. Instead of providing a universal domain of quantification, the model-theoretic semantics of Quarc employ only *interpretation functions*. They specify for each predicate P an extension, which in turn functions as a 'local' domain of quantification. Additionally, each singular argument is mapped to some object under an interpretation – objects which constitute the members of the various predicate-extensions (cf. [6], [7] and [12]). We follow [7] in treating elements of extensions as tuples, whose last coordinate is an element $w \in W$ of a given frame $\mathcal{F} = \langle W, R \rangle$.

Definition 3.2 (Interpretations). Let a frame $\mathcal{F} = \langle W, R \rangle$ be given. An *interpretation* \mathcal{I} of \mathcal{L} on \mathcal{F} is a function whose domain is $SA \cup PRED$ and which satisfies the following conditions:

- 1. Each $c \in SA$ is assigned some object $\mathcal{I}(c)$. This assignment is independent of any $w \in W$.⁸
- 2. Each $P \in PRED$ is assigned a set $\mathcal{I}(P)$ consisting of pairs $\langle x, w \rangle$, where x is an object and $w \in W$. Furthermore, the following constraint must be satisfied for each $P \in PRED$: for all $w \in W$, there is an object x s.t. $\langle x, w \rangle \in \mathcal{I}(P)$.

Remark 3.1. We shall sometimes talk about all x that are P relative to a given $w \in W$, for some frame $\langle W, R \rangle$ and some interpretation \mathcal{I} . In that case, we shall write $|P|_w$ in order to denote the set $\{x : \langle x, w \rangle \in \mathcal{I}(P)\}$.

In order to assign the proper truth-conditions to formulas governed by a quantified argument qP, we need a way to extend an interpretation and the set SA so that we can name every object in $|P|_w$, for each possible world w. This is called an *expansion*.

Definition 3.3 (Expansions). Let \mathcal{I} be an interpretation of \mathcal{L} on a frame $\mathcal{F} = \langle W, R \rangle$. Let c be a singular argument (possibly already in \mathcal{L} , i.e. SA) and let x be an object. The $(c \rightarrow x)$ -expansion of \mathcal{I} , denoted by $\mathcal{I}_{c \rightarrow x}$, is an interpretation that satisfies the following conditions:

- 1. Either $c \in SA$, $\mathcal{I}(c) = x$ and $\mathcal{I}_{c \to x}(c) = x$, or $c \notin SA$ and $\mathcal{I}_{c \to x}(c) = x$.
- 2. For all $c' \in SA$ s.t. $c \neq c'$, $\mathcal{I}_{c \to x}(c') = \mathcal{I}(c')$.
- 3. For all $P \in PRED$, $\mathcal{I}_{c \to x}(P) = \mathcal{I}(P)$.

Remark 3.2. Observe that in the case that $c \in SA$ and yet $\mathcal{I}(c) \neq x$, there is no $(c \rightarrow x)$ -expansion for \mathcal{I} .

Definition 3.4 (Truth-Conditions). Let $\mathcal{F} = \langle W, R \rangle$ be a frame and \mathcal{I} an interpretation of \mathcal{L} on \mathcal{F} . The *truthconditions* for formulas in *FORM* on \mathcal{F} over \mathcal{I} are given by a function *FORM* × $W \rightarrow \{0, 1\}$, which obeys the rules (i)-(vi) below. If a formula ϕ is assigned the value 1 in w, we write $\mathcal{I}, w \vDash \phi$, and $\mathcal{I}, w \nvDash \phi$ if it is assigned 0. The conditions for a given ϕ are as follows:

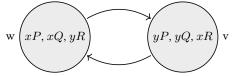
- (i) Basic formulas: Let cP be a basic formula. Then:
 I, w ⊨ cP iff (I(c), w) ∈ I(P).
- (ii) Predicate Negation: Let c¬P be a predicate negation. Then, I, w ⊨ c¬P iff I, w ⊨ ¬cP.⁹
- (ii) **Sentential Negation:** If ϕ is of the form $\neg \psi$, then $\mathcal{I}, w \models \neg \psi$ iff $\mathcal{I}, w \not\models \psi$.
- (iii) Connectives:
 - 1. If ϕ is of the form $\psi \land \chi$, then $\mathcal{I}, w \vDash \psi \land \chi$ iff $\mathcal{I}, w \vDash \psi$ and $\mathcal{I}, w \vDash \chi$.
 - 2. If ϕ is of the form $\psi \lor \chi$, then $\mathcal{I}, w \vDash \psi \lor \chi$ iff $\mathcal{I}, w \vDash \psi$ or $\mathcal{I}, w \vDash \chi$.
 - 3. If ϕ is of the form $\psi \to \chi$, then $\mathcal{I}, w \models \psi \to \chi$ iff $\mathcal{I}, w \nvDash \psi$ or $\mathcal{I}, w \models \chi$.
- (iv) **Modals:** If ϕ is of the form $\Box \psi$, then $\mathcal{I}, w \vDash \Box \psi$ iff for every $v \in W$ s.t. $wRv, \mathcal{I}, v \vDash \psi$.
- (v) **Universal quantification:** Let $\phi(\forall P)$ be governed by the universally quantified argument $\forall P$. Then: $\mathcal{I}, w \models \phi(\forall P)$ iff for every $x \in |P|_w, \mathcal{I}_{c \to x}, w \models \phi[c/\forall P]$.

⁸In that sense, singular arguments are rigid designators.

⁹This clause only applies to formulas of the required form. As soon as quantification is involved, this equivalence no longer holds, as is easily verified.

(vi) **Particular quantification**: Let $\phi(\exists P)$ be governed by the particularly quantified argument $\exists P$. Then: $\mathcal{I}, w \models \phi(\exists P)$ iff for some $x \in |P|_w, \mathcal{I}_{c \to x}, w \models \phi[c/\exists P]$.

Example 3.1. Consider the following interpretation \mathcal{I} on an equivalence frame with two worlds (reflexive arrows are suppressed):



with $x \neq y$ two random objects, and P, Q and $R \in PRED$. Only the true basic formulas (for the symbols under consideration) are listed in the diagram. Since each predicate has at least one member for each world, 3.2(2.) is satisfied. In $w, |P_w| = \{x\}$. Thus, in order to evaluate $\forall PQ$, we first pick some $c \notin SA$ (for convenience) and check the truth of cQ under the appropriate $(c \rightarrow x)$ -expansion. Since there is only once case to check, and $x \in |Q_w|$, the formula is true. Similarly, we can establish:

- $\mathcal{I}, w \models \exists PQ$
- $\mathcal{I}, w \neq \forall RP$
- $\mathcal{I}, v \neq \exists RQ$
- $\mathcal{I}, v \not\models \Box \forall RQ$
- etc.

Having defined truth for formulas, we can now proceed to define validity and entailment over a class of frames in the usual way:

Definition 3.5 (Entailment and Validity). We first define entailment, treating validity as a special case:

- Let $\mathcal{F} = \langle W, R \rangle$ be a frame. A set of formulas Γ entails a formula ϕ on \mathcal{F} iff for every interpretation \mathcal{I} and every world $w \in W$, if $I, w \models \gamma$ for every $\gamma \in \Gamma$, then $I, w \models \phi$. In such a case, we write $\Gamma \models_{\mathcal{F}} \phi$.
- Let *𝔅* be a class of frames. Then, a set of formulas Γ entails a formula φ on *𝔅* iff for every ∈ *𝔅*, Γ ⊨_𝔅 φ. In this case, we write Γ ⊨_𝔅 φ.
- A formula φ is a validity on a frame F iff if it is entailed by the empty set on F. It is a validity on a class of frames ℑ just in case the empty set entails it on ℑ. In the latter case, we write ⊨_ℑ φ.

4. Proof Theory

In this section, we introduce an unlabelled Gentzen-style natural deduction system. It will be a Quarc-analogue of the normal modal logic D. We choose an unlabelled system in order to once again remain closer to the linguistic form of quantified modal reasoning in natural language. We follow the basic set-up found in [13] and [14].

Definition 4.1 (Proofs). A *proof* is a rooted tree where every vertex is labelled with an element of *FORM*, and each edge is labelled with one of the rules of definition 4.2. The root is the *conclusion* of the proof, and its leaves are *assumptions* that are either *discharged* or *undischarged*, as specified by the rules. We say the conclusion *depends* on the undischarged assumptions. We explicitly allow empty assumption classes (cf. [14]).

With this standard graph-theoretic set-up, the logic $N_{\Box Q^-}^D$ can be introduced, where Q^- designates the simplified version of Quarc we are employing. The rules for the \Box -free fragment are taken from [11], while the rule \Box I is taken from [15]. Lastly, the rules $\Box \neg I$ and $\Box \neg E$ are original.¹⁰

Definition 4.2 $(N_{\Box Q^{-}}^{D})$. The logic $N_{\Box Q^{-}}^{D}$ is given by the following rules:

 Connectives: We adopt the standard introduction and elimination rules for ∧, ∨ and →.
 Since the symbol ⊥ is not part of the syntax, the following rules for ¬ are chosen:

$$\begin{array}{ccccc} [\phi]^i & [\phi]^i & [\neg\phi]^i & [\neg\phi]^i \\ \vdots & \vdots & \vdots & \vdots \\ \underline{\psi & \neg\psi} & \neg I_i & \underline{\psi & \neg\psi} & \neg E_i \end{array}$$

2. Predicate Negation:

$$\frac{c\neg P}{\neg cP}$$
 PS $\frac{\neg cP}{c\neg P}$ SP

3. **Quantification:** Let $\phi(qP)$ be a formula governed by the quantified argument qP. Let $\phi[c/qP]$ be the formula where the governing occurrence of qP has been replaced by the singular argument c.

The rules for universal quantification are as follows:

$$\begin{array}{c} [cP]^{i} \\ \vdots \\ \hline \phi[c/\forall P] \\ \hline \phi[\forall P] \end{array} \forall \mathbf{I}_{i}, * \end{array} \qquad \begin{array}{c} \phi[\forall P] \quad cP \\ \hline \phi[c/\forall P] \\ \hline \phi[c/\forall P] \end{array} \forall \mathbf{E} \end{array}$$

where the side condition \star requires c to not occur in any undischarged assumption or in $\phi(\forall P)$. Particular quantification has the following introduction rule:

$$\frac{cP}{\phi[\exists P]} \Rightarrow \frac{\phi[c/\exists P]}{\phi[\exists P]} \exists P$$

Since no $P \in PRED$ is ever empty, both quantifiers obey the following rule:

$$[cP]^{i}, [\phi[c/qP]]^{i}$$

$$\vdots$$

$$\phi[qP] \qquad \psi \qquad \text{Imp}_{i}, *$$

where the side condition \star requires *c* to not occur in any undischarged assumptions, ψ or $\phi(\forall P)$.

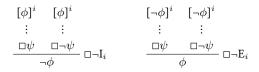
4. **Modality:** For the modal operator □, we have the following introduction rule:

$$\frac{\phi}{\Box\phi}$$
 \Box I, *

where the condition * says that if ϕ depended on the assumptions $\gamma_1, ..., \gamma_k$, then $\Box \phi$ now depends on the assumptions $\Box \gamma_1, ..., \Box \gamma_k$.

Lastly, given our interest in working with serial frames, the following rules capture the modal logic *D*:

¹⁰The authors are indebted to Elio La Rosa for suggesting such rules.



Remark 4.1. We allow multiple applications of \Box I, though this will never occur in the proofs in subsection 5.2. In any case, we keep track of the added \Box s by putting \Box into the superscript of an assumption γ . We add the following convention: if \Box^k is a string of *k*-many \Box s, then if we discharge γ^{\Box^k} , we discharge the formula $\Box^k \gamma$.

Remark 4.2. Caution must be exercised when mixing the rule \Box I with others that use assumption classes. For example, if we apply \neg I while having assumed ϕ , then, if ϕ underwent an application of \Box I on the left branch, but not on the right, we cannot derive $\neg \Box \phi$ via \neg I. For otherwise, the application of \neg I becomes incorrect, since ϕ does not have the prerequisite form on the right branch of the tree.

Definition 4.3 (Syntactic Consequence). Let Γ be a set of formulas and ϕ a formula. Then, Γ *proves* ϕ , or ϕ is a *syntactic consequence* of Γ , just in case there are finitely many $\gamma_1, ..., \gamma_k \in \Gamma$ s.t. there is a proof of ϕ with undischarged assumptions $\gamma_1, ..., \gamma_k$. In such a case, we write $\Gamma \vdash \phi$.¹¹ If $\Gamma = \emptyset$, ϕ is a *theorem* and we write $\vdash \phi$.

Remark 4.3. The preceding proof system has been stated in more generality than strictly required. For example, in $\forall I$, all $\phi(\forall P)$ will always be of the form $\forall PQ$ or $\forall P\neg Q$, since the syntax only contains unary predicates. Nevertheless, we would like to stress that the rules for the quantifiers also work with respect to more complex syntaxes, such as those including Quarc's other features, like anaphora, reordered predicates and *n*-ary predicates. Likewise, the rules PS and SP are not used in subsection 5.2. However, they would be required to established the relations of the symmetric *de dicto* formalizations mentioned in sections 1 (cf. Fig. 2) and 5.4.

Since it is not the focus of the present investigation, we merely mention the following result, without proof:

Theorem 4.1. $N_{\Box Q^-}^D$ is both strongly sound as well as complete with respect to the class of all serial frames.

We shall rely on the soundness of $N_{\Box Q^-}^D$ in subsection 5.2. A related result can be found in [12] and [9], the latter which features an extended proof system of the one featured in definition 4.2, including strong soundness and completeness with respect to substitutional semantics.¹²

5. The Dodecagon of Opposition for *De Dicto* Modalities

In this section, we establish the dodecagon of opposition for the *asymmetric de dicto* reading. We begin by detailing the latter reading of the twelve combinations of quantification and modality (subsection 5.1). After presenting these, we prove the resulting individual relations in $N_{\square Q^-}^D$ (subsection 5.2) and provide example counter-models for some pairs that lack certain relations (subsection 5.3). Finally, we discuss these results by comparing them to the symmetric *de dicto* formalizations (5.4) of Fig. 2.

5.1. The Combinations and the Resulting Dodecagon

Before introducing the combinations themselves, further modal notions need to be introduced. The language introduced in 2.1 has only one primitive modal operator, viz. \Box (necessity). We shall define the remaining modal notions in terms of \Box :

Definition 5.1 (Further Modalities). Let $\phi \in FORM$ be given. The following modal notions receive their own symbols:

- **Possibility:** $\Diamond \phi :\equiv \neg \Box \neg \phi$.
- Absoluteness: $\Delta \phi :\equiv \Box \phi \lor \Box \neg \phi$.
- Contingency: $\nabla \phi :\equiv \neg \Box \neg \phi \land \neg \Box \phi$.

The final two modal notions – *avoidability* and *impossibility* – do not receive a special symbol, but are instead directly rendered as $\neg \Box \phi$ and $\Box \neg \phi$, respectively.¹³

This definition yields the six modal notions introduced in section 1. We shall follow the terminology established there and denote the type of quantification with either U (universal) or P (particular)¹⁴, and use N, P, V, B, C and I for the six modal notions from section 1, respectively. This yields the same twelve names as in section 1, albeit with different logical profiles:

Definition 5.2 (Asymmetric Reading). Let S and P be two predicates of \mathcal{L} . The twelve *asymmetric de dicto* combinations are the following:

• NU: $\Box \forall SP$	• IU: $\Box \neg \forall SP$
• NP: $\Box \exists SP$	• IP : $\Box \neg \exists SP$
• PU: $\Diamond \forall SP$	• CU : $\nabla \forall SP$
• PP : $\Diamond \exists SP$	• CP : $\nabla \exists SP$
• VU: $\neg \Box \forall SP$	• BU : $\Delta \forall SP$
• VP : $\neg \Box \exists SP$	• BP : $\Delta \exists SP$

Remark 5.1. Thus, **BP** would be read as 'Either, necessarily, some S are P, or, necessarily, no S are P'. This is quite unlike its symmetric counterpart, which would be read as 'Either, necessarily some S are P, or, necessarily, some S are not P'.

Remark 5.2. In the following, if XY is a combination, we shall write $\neg XY$ for the negation of the whole corresponding formula.

Before presenting the main results, the Aristotelian relations themselves must be introduced. Given the setting of section 3, the relations are rendered as follows:

Definition 5.3 (Aristotelian Relations). Let \vDash denote entailment on the class of all serial frames (cf. 3.5). Then, a formula ψ is a *subalternate* of another formula ϕ iff $\phi \vDash \psi$ and it is not the case that $\psi \vDash \phi$. Furthermore, two formulas ϕ and ψ are:

• contraries iff $\phi \vDash \neg \psi$,

¹¹Subscripts for \vdash are suppressed, given that $N^D_{\square Q^-}$ is the only logic under consideration.

 $^{^{12}}$ It can be shown that the two semantics validate the same sequents, hence the result can be transferred.

¹³This is a common choice in the literature, as witnessed by [2, 16].

¹⁴The letter P also fulfills other functions in the present investigation. However, it should always be clear by context which role is intended.

- subcontraries iff $\neg \phi \vDash \psi$,
- contradictories iff they are both contaries and subcontaries, i.e. φ ⊨ ¬ψ and ¬ψ ⊨ φ. In such a case, we write φ == ¬ψ.

Given that there are twelve combinations, we must investigate a total of 66 pairs.¹⁵ In total, 16 pairs have no Aristotelian relation between them, with the remaining 50 instantiating one of the relations in definition 5.3. These findings are summarized in the following dodecagon:

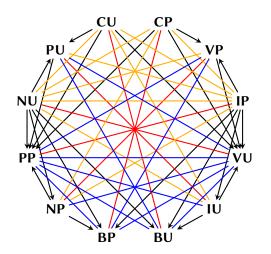


Figure 3: The dodecagon of opposition for the asymmetric reading. Colour conventions are as in previous figures.

Theorem 5.1 (Dodecagon of *Asymmetric De Dicto* Modalities). *The relations depicted in Fig. 3 obtain.*

There are eleven contraries, eleven subcontraries, six contradictories and 22 subalternations. We shall return to these facts in subsection 5.4.

5.2. Proofs of the Relations

In this subsection, we prove all 50 relations, namely all modal categorical inferences resulting from our asymmetric reading of de dicto modalities. Instead of proving them classified by type, we will generally prove them in order of increasing proof complexity. This allows for a more compact and straightforward presentation of the proofs. As mentioned before, we rely on the soundness of $N^D_{\Box Q^-}$ throughout this section. The proofs themselves are categorized into two types: simple proofs, which only require propositional reasoning or straightforward modal reasoning, and complex proofs. We consider each in turn.

5.2.1. Simple Proofs

Some of these relations hold purely by virtue of propositional reasoning. This holds for all contradictory pairs, in particular:

Fact 5.1. The following hold:

(i) $CP \Rightarrow \neg BP$	(iv) $NU = \neg VU$
(ii) $CU \rightleftharpoons \neg BU$	(v) $PP \rightrightarrows = \neg IP$
(iii) $PU \Rightarrow \neg IU$	(vi) $VP \rightrightarrows = \neg NP$

Proof. Given how the modalities are ultimately defined, proving all these claims becomes a matter of simple propositional reasoning:

- (i) and (ii) are instances of the *de Morgan* rules. For example, $CP \Rightarrow \neg BP$ becomes the claim $\neg \Box \neg \exists SP \land \neg \Box \exists SP \Rightarrow \neg (\Box \exists SP \lor \Box \neg \exists SP).$
- (iv) is an instance of double negation elimination/introduction: $\Box \forall SP \Rightarrow \neg \neg \Box \forall SP$.
- (iii), (v) and (vi) are instances of the schema $\phi \models \phi$. For example, $PU \models \neg IU$ is the claim that $\neg \Box$ $\neg \forall SP \models \neg \Box \neg \forall SP$.

Similarly, further relations are provable by applications of propositional reasoning. This applies to the following pairs:

Fact 5.2. The following hold:

(i) $IU \models BU$	(ix) $NU \vDash \neg CU$
(ii) $NU \models BU$	(x) $IU \vDash \neg CU$
(iii) $CU \models PU$	(xi) $CP \vDash \neg IP$
(iv) $CU \models VU$	(xii) $NP \vDash \neg CP$
(v) $NP \models BP$	(xiii) $\neg VU \vDash BU$
(vi) $CP \models PP$	(xiv) $\neg BP \vDash VP$
(vii) $CP \models VP$	$(xv) \neg PP \vDash BP$
(viii) $IP \models BP$	$(xvi) \neg PU \vDash BU$

Proof. We group the proofs by the type of relation they establish:

- The subalternations (i)-(viii) are established either by a single application of \lor I or \land E. For example, $IU \vDash BU$ is the claim that $\Box \neg \forall SP \vDash \Box \neg \forall SP \lor \Box \neg \forall SP$, and $CP \vDash VP$ is the claim that $\neg \Box \neg \exists SP \land \neg \Box \exists SP \vDash \neg \exists SP$.
- The contraries (ix)-(xii) are proven either by a singe application of ∧E as in the case of (xi) or a combination of ∧E and ¬I, as in the cases of (ix), (x) and (xii), since they are of the form φ ⊨ ¬(ψ ∧ ¬φ) or φ ⊨ ¬(¬φ ∧ ψ).¹⁶
- The remaining cases are subcontraries, and are proven by a combination of double negation elimination and ∨I (xiii), (xv) and (xvi) or by using the *de Morgan* rules and ∧E, as in (xiv). The latter is the claim that ¬(□∃SP ∨ □¬∃SP) ⊨ ¬ □ ∃SP, and the former are of the form ¬¬φ ⊨ φ ∨ ψ.

A number of the relations depicted in theorem 5.1 are the result of an application of $\Box \neg I$, as well as double negation elimination:

Fact 5.3. The following hold:

¹⁵We disallow repetitions for reasons of triviality, and order does not matter with respect to forming the pairs, due to the properties of Aristotelian relations.

¹⁶Observe that the sequent in (xi) coincides with the one in (vi).

Proof. Most of these pairs can be established with a single application of $\Box \neg I$. This holds for (i)-(vi).¹⁷ We prove (iv):

$$\frac{\Box \neg \forall SP \quad [\Box \forall SP]^1}{\neg \Box \forall SP} \Box \neg I_1$$

This result, in turn, establishes (viii):

$$\frac{\neg \neg \Box \neg \forall SP}{\Box \neg \forall SP} \text{ DNE}$$

$$\frac{\Box \neg \forall SP}{\neg \Box \forall SP} 5.3 \text{ (iv)}$$

Lastly, (vii) has the same proof as (viii), except with particular quantification instead of universal one. $\hfill \Box$

With these facts, thirty of the fifty relations have been established. We now turn to the more complex cases.

5.2.2. Complex Proofs

We begin by noticing the following fact:

Fact 5.4. $\forall SP \vDash \exists SP$

Proof.

$$\frac{\exists SP}{\exists SP} \frac{[aS]^1 \quad [aP]^1}{\exists SP} \operatorname{Imp}_1 \exists I$$

In the following, we treat this proof as an admissible rule with the name Sub ('subalternation').¹⁸ As an immediate corollary, we gain the entailment $\neg \exists SP \models \neg \forall SP$. The corresponding proof will also be treated as an admissible rule, named CSub ('contrapositive subalternation'). With these basic building blocks, we can establish the following:

Fact 5.5. The following hold:

(i) $NU \models NP$	(v) $VP \vDash \neg NU$
(ii) $NU \models PP$	(vi) $VP \vDash VU$
(iii) $NU \vDash \neg IP$	(vii) $NU \vDash \neg CP$
(iv) $NU \vDash BP$	(viii) $\neg VU \vDash NP$

Proof. We first prove (i):

$$\frac{\forall SP^{\Box}}{\exists SP} \operatorname{Sub}_{\Box I}$$

Recall that according to remark 4.1, we add \Box as a superscript to all undischarged assumptions after applying \Box I. Thus, the conclusion $\Box \exists SP$ now depends on the assumption $\Box \forall SP$. This proof can in turn be extended to yield (ii):

$$\underbrace{ \frac{\forall SP^{\Box}}{\Box \exists SP} 5.5(i)}_{\Box \neg \exists SP} \underbrace{ [\Box \neg \exists SP]^2}_{\Box \neg \exists SP} \Box \neg I_{SP}$$

¹⁷Moreover, the sequents in (i) and (v), as well as (ii) and (vi), coincide. ¹⁸Essentially the same proof also establishes $\phi(\forall P) \models \phi(\exists P)$, irrespective of the complexity of ϕ and the underlying syntax, so long as the quantified arguments $\forall P$ and $\exists P$ are governing ϕ . Additionally, (iii) is the same sequent as (ii). Certain propositional arguments further prove (iv)-(viii). First, (iv) is proven by applying \lor I to the conclusion of the proof in (i). Second, (v) is ultimately the same sequent as (vi): $\neg \Box \exists SP \vDash \neg \Box \forall SP$, which is the contraposition of (i). Third, (vii) has the following proof:

$$\frac{\forall SP^{\Box}}{\Box \exists SP} 5.5(\mathbf{i}) \qquad \frac{\left[\neg \Box \neg \exists SP \land \neg \Box \exists SP\right]^{2}}{\neg \Box \exists SP} \land \mathbf{E}$$

Finally, (viii) is established as follows:

DNE
$$\frac{\neg \neg \Box \forall SP}{\Box \forall SP} \qquad \frac{\frac{[\forall SP^{\Box}]^{2}}{\Box \exists SP}}{\Box \forall SP \rightarrow \Box \exists SP} \xrightarrow{\rightarrow I_{2}} S.5(i)$$
$$\frac{\neg \forall SP}{\Box \exists SP} \xrightarrow{\rightarrow I_{2}} SP$$

Observe that, given how $\Box I$ works, $\forall SP^{\Box}$ must first be discharged, for we cannot reason with $\Box \forall SP$ directly to $\Box \exists SP$.

From these proofs, we can further derive the following results:

Fact 5.6. Fact 5.5 further entails the following:

(i)
$$CP \models VU$$
 (iii) $\neg VU \models BP$
(ii) $\neg VU \models PP$

Proof. (i) follows directly from \land E applied to $\neg \Box \neg \exists SP \land$ $\neg \Box \exists SP$ and by proceeding as in 5.5(v)/(vi). (ii) follows from 5.5(viii) by applying $\Box \neg I$:

$$\frac{\neg \neg \Box \forall SP}{\Box \exists SP} 5.5(\text{viii}) \qquad [\Box \neg \exists SP]^2 \\ \neg \Box \neg \exists SP} \Box \neg I_2$$

To prove (iii), we use the proof of 5.6(i) as a starting point and apply \lor to its conclusion to yield $\Box \exists SP \lor \Box \neg \exists SP$. \Box

We now turn to the amodal entailment $\neg \exists SP \models \neg \forall SP$. It provides the basis for the remaining proofs:

Fact 5.7. The following entailments obtain:

(i) $IP \models IU$	(vi) $\neg IU \vDash PP$
(ii) $IP \models BU$	(vii) $CU \models PP$
(iii) $IP \models VU$ (iv) $IP \models \neg PU$	(viii) $CU \vDash \neg IP$
(v) $PU \models PP$	(ix) $\neg BU \models PP$

Proof. We once again prove (i) first:

$$\frac{\neg \exists SP^{\Box}}{\neg \forall SP} \operatorname{CSub}_{\Box \neg \forall SP} \Box I$$

Via an application of \lor I, we immediately yield (ii). Via $\Box \neg I$, we establish (iii):

$$\frac{\neg \exists SP^{\sqcup}}{\Box \neg \forall SP} 5.7(i) \qquad [\Box \forall SP]^1 \\ \neg \Box \forall SP \qquad \Box \neg I_1$$

From (i) and double negation introduction, we can prove (iv). (v) is established in the following way, where CP refers to the transformation of a conditional into its contrapositive:

$$\frac{\begin{bmatrix} \neg \exists SP^{\Box} \end{bmatrix}^{1}}{\Box \neg \forall SP} 5.7(i) \\
\frac{\hline \Box \neg \exists SP \rightarrow \Box \neg \forall SP}{\neg \Box \neg \forall SP \rightarrow \Box \neg \exists SP} \bullet I_{1} \\
\frac{\hline \neg \Box \neg \forall SP \rightarrow \neg \Box \neg \exists SP}{\neg \Box \neg \exists SP} \bullet P \\
\hline \neg \Box \neg \exists SP} \to E$$

Since (vi) is the same sequent as (v), we have therefore also established (vi). Furthermore, we can prove (vii) via applying $\wedge E$ to $\neg \Box \neg \forall SP \wedge \neg \Box \forall SP$ and then proceeding as in the proof of (v). (viii) is the same sequent as (vii), and (ix) is proven by first applying the *de Morgan* laws to $\neg BU$ to obtain CU (cf. 5.1(ii)), and then proceeding as in (vii). \Box

With these results, theorem 5.1 has been established:

Proof of theorem 5.1. Follows from facts 5.1, 5.2, 5.3, 5.5, 5.6 and 5.7. □

5.3. Sample Counter-Models for the Remaining Pairs

The remaining sixteen pairs do not instantiate any Aristotelian relation. Proving this is mostly straightforward. We showcase one pair in detail, as well as two further failures of entailment that explain a large portion of the remaining cases. We will always exhibit S5-counter-models in order to avoid the suspicion that the relevant relations do not obtain due to the underlying accessibility relation. The reflexive arrows are suppressed throughout.

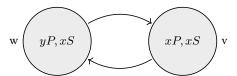
Example 5.1. Consider the pair IU/VP, i.e. $\Box \neg \forall SP$ and $\neg \Box \exists SP$.

$$IU \neq VP$$
:
w (yP, xS, yS) (xP, xS, yS) (xP, xS, yS)

where x and y are two distinct objects. We can see that for each world $i, i \in \{w, v\}, |S|_i \cap |P|_i \neq \emptyset$ and $|S|_i \notin |P|_i$. As such, $\Box \neg \forall SP$ and $\Box \exists SP$ are both true in w.

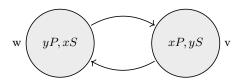
• $VP \neq IU$:

•



where x and y are once again dummy objects, distinct from each other. By construction: $|S|_w \cap |P|_w = \emptyset$ and $|S|_v \subseteq |P|_v$. It follows that $\neg \Box \neg \forall SP$ and $\neg \Box \exists SP$ are both true in w.

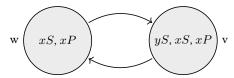
- $\neg IU \neq VP$: We keep w from the first and v from the second counter-model. In that case, $\Diamond \forall SP$ and $\Box \exists SP$ are both true in w.
- : $IU \neq \neg VP$ is established with the following counter-model:



In this case, both $\Box \neg \forall SP$ and $\neg \Box \exists SP$ are true in w, as is easily verified.

The fact that no relation obtains for this pair immediately explains why certain other entailments also fail to obtain. Thus, since IU does not entail VP, neither does $\Delta \forall SP$ entail $\nabla \exists SP$, even if $\Box \forall SP$ entails $\Diamond \exists SP$, as per 5.5(ii). In a similar vein, the following entailments also fail to hold:

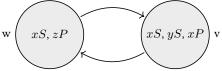
Example 5.2. $CU \neq VP$:



By construction, w makes $\Diamond \forall SP$ true, since $\forall SP$ is true in w. In v, y is S but not P, making $\forall SP$ false at v. Thus, $\neg \Box \forall SP$ is true at w. As a consequence, w makes $\nabla \forall SP$ true. However, $\exists SP$ is true in both worlds, hence w makes $\neg \Box \exists SP$ false.

As a final example, we demonstrate that IU does not entail BP, since it entails neither disjunct:

Example 5.3. $IU \neq BP$: Consider the following model:



In w, $\exists SP$ is false and $\neg \forall SP$ true. The latter formula is also true in v, but so is $\exists SP$. Thus, $\Box \forall SP$ is true in w, but neither $\Box \exists SP$ nor $\Box \neg \exists SP$.

With this last example, we also have a counter-model to verify that $\Box \neg \forall SP \neq \Box \exists SP$ (i.e. $IU \neq NP$), or that $\Delta \forall SP \neq \Delta \exists SP$ (i.e. $BU \neq BP$). Lastly, as these countermodels demonstrate, it is straightforward to invalidate the purported entailments of the remaining pairs, thus we omit the remaining counter-models.

5.4. Comparison to the Symmetric *De Dicto* Reading

We conclude this section by comparing the results of theorem 5.1 with the symmetric *de dicto* reading from [10] presented in section 1. The difference between the two lies in the placement of negation. Whereas the symmetric reading employs predicate negation, \neg shifts into sentential position in the asymmetric one. In the setting of the symmetric reading, the same 66 pairs of formulas can be investigated, which yield the dodecagon of Fig. 2.

Thus, with the symmetric reading, we end up with six contradictories, twelve contraries and subcontraries, and 24 subalternations, for a total of 54 pairs instantiating an Aristotelian relation. Thus, at first glance, by going from the asymmetric reading to the symmetric one, we only seem to gain four additional relations. However, this superficial glance obscures the vast differences between the two dodecagons.

In total, only 32 pairs do not change the relation they instantiate, whereas 34 do. The former do not include any contradictions, and the latter include two cases of subalternation where the direction of entailment is reversed. Ultimately, the 34 pairs that change their instantiated relation

Table 1	
List of pairs that differ between formalizations	

Pair	Sym.	Asym.	Pair	Sym.	Asym.
CP/BP	SubC	Contrad	CU/BU	Cont	Contrad
PU/IU	Cont	Contrad	NU/VU	Cont	Contrad
PP/IP	SubC	Contrad	VP/NP	SubC	Contrad
CP/IP	none	Cont	NP/CP	none	Cont
VU/BU	none	SubC	PU/BU	none	SubC
NP/IP	none	Cont	PU/VU	none	SubC
VP/NU	Contrad	Cont	VU/NP	Contrad	SubC
CP/VU	none	Subalt (→)	IP/BU	none	Subalt (→)
IP/VU	none	Subalt (→)	IU/PP	Contrad	SubC
IU/CP	Cont	none	CU/NP	Cont	none
BU/VP	SubC	none	PU/BP	SubC	none
BU/CP	Contrad	none	BP/CU	Contrad	none
CU/VP	Subalt (→)	none	CU/CP	Subalt (→)	none
IU/BP	Subalt (→)	none	IU/VP	Subalt (→)	none
BU/BP	Subalt (→)	none	IU/NP	Cont	none
PU/VP	SubC	none	IP/PU	Contrad	Cont
IP/IU	Subalt (←)	Subalt (\rightarrow)	VP/VU	Subalt (←)	Subalt (→)

contain twelve cases of pairs whose relation merely *shifts*, and 22 where the pair only instantiates a relation in *one* of the two readings. Thus, there are vast differences between the two dodecagons.

For clarity of reference, the 34 pairs that undergo change are detailed in table 1. In case a pair instantiates a subalternation, the arrow denotes the direction of entailment relative to the way the pair is listed in the left column. Subalternations are abbreviated with Subalt, contradictions with Contrad, contraries with Cont and subcontraries with SubC.

Instead of discussing all these pairs in detail, we would like to zero in on a few interesting facts. First, shifting the position of the negation can have the effect of a contraposition, as in the pairs IP/IU and VP/VU, where the direction of entailment reverses when transitioning from one setting to the other. Second, many of the relations in the symmetric setting obtain due to two underlying entailments:

Fact 5.8. The following hold:

1.
$$\neg(\forall SP) \vDash \exists S \neg P$$

2. $\neg(\exists SP) \vDash \forall S \neg P$

Proof. The proof for 1. proceeds indirectly:

$$\frac{\begin{bmatrix} [aS]^2 & \frac{[\neg aP]^1}{a \neg P} \\ \exists S \neg P \end{bmatrix}^3 \xrightarrow{[aS]^2} \exists S \neg P}{\exists S \neg P} \exists I \\ \neg (\forall SP) & \frac{aP}{\forall SP} \forall I_2 \\ \exists S \neg P & \neg E_3 \end{bmatrix}$$

whereas 2. is established straightforwardly:

$$\begin{array}{c} \underline{\neg \exists SP} & \underline{[aS]^2 & [aP]^1} \\ \hline \underline{\exists SP} & \neg \mathbf{I}_1 \\ \hline \\ \underline{\neg aP} & SP \\ \hline \hline \\ \underline{\neg aP} & \forall \mathbf{I}_2 \end{array}$$

Naturally, since predicate negation is completely absent in the asymmetric setting of definition 5.1, these entailments become irrelevant to establishing any of the relations in the latter context. As a consequence, many entailments are weakened or strengthened. Third, the shifting of negation and the resulting lack of predicate negation can also change the formula so much that entailments can be lost completely. For example, whereas $\Diamond \forall SP \land \Diamond \forall S \neg P \vDash \Diamond \exists SP$ (i.e. $CU \models VP$) holds in the symmetric setting, the corresponding pair $\Diamond \forall SP \land \Diamond \neg \forall SP$ and $\Diamond \neg \exists SP$ from definition 5.1 do not instantiate any relation (cf. example 5.2). Fourth, the fact that most changes happen to pairs that are contrary or subcontrary (or both) in one of the settings is unsurprising. For given that many subalternations in either context hold purely due to simple propositional or modal reasoning, they are insensitive to the subtleties regarding the difference between predicate and sentence negation. Thus, we find that out of the 32 pairs that do not undergo change when switching settings, 17 are subalternations. Finally, and most importantly, whereas the dodecagon in Fig. 2 does not preserve the hexagon of opposition, the asymmetric reading generates one that does (cf. Fig. 3), as can be read off from the figures.

6. Conclusion and Future Work

In this article we analyzed one of the basic building blocks of modal quantified reasoning, namely *modal categorical inferences*. We did this within Quarc, a formal framework that significantly departs from Predicate Logic in representing natural language sentences, especially due to the pivotal role it assigns to *quantified phrases*. We focused on *de dicto* combinations of modalities and quantifiers in modal categorical statements and proposed a new reading for them, which we called *asymmetric*, as opposed to the *symmetric* reading presented in [10]. The crucial differences between the symmetric and the asymmetric reading can be summarized as below:

 as shown in [10], the symmetric reading gives rise to the same dodecagon of opposition as the one obtained for *de re* modalities and yet violates some fundamental properties of hexagon modalities; • the asymmetric reading gives rise to a radically different dodecagon of opposition and yet preserves all fundamental properties of hexagon modalities.

Clearly, our results can be transferred to other formal frameworks, thanks to the use of translation functions. For instance, consider definition 5.2. The following simple procedure indicates how one can move from a formula ϕ of modal Quarc that represents a modal categorical statement to an equivalent formula of modal PC (for reading disambiguation, we use quotation marks to delimit a string of symbols):

- One must add the string of symbols 'x(' immediately after the quantifier in ϕ , thus getting ϕ_1 .
- If φ₁ contains ∀, then one must add the string of symbols 'x →' immediately after S (i.e. the predicate forming the quantified argument in φ), otherwise φ₁ contains ∃ and one must add the string of symbols 'x∧' immediately after S. In any of these cases, one gets φ₂.
- One must add the string of symbols 'x)' at the end of ϕ_2 , thus getting ϕ_3 .

It is easy to verify that ϕ_3 is a formula of modal PC. Following this procedure, if our starting point is e.g. $\phi = \diamondsuit \forall SP$ (which represents **PU**), we get $\phi_3 = \diamondsuit \forall x(Sx \rightarrow Px)$.

Looking into the future, a natural direction to follow is developing a theory of *modal categorical syllogism*. A syllogism of this kind consists of two premises ϕ and ψ and a conclusion χ , all of which qualify as modal categorical statements. Here is an example:

(* * *) I don't know whether all rooms on this floor have an access code. I know that all rooms with an access code are inaccessible to guests. Therefore, it's possible that all rooms on this floor are inaccessible to guests.

A simple formalization of (* * *) can be as follows, where R stands for 'is a room on this floor', C for 'has an access code' and A for 'is accessible to guests':

$$\{\diamondsuit \forall RC \land \diamondsuit \neg \forall RC, \Box \forall C \neg A\} \vDash \diamondsuit \forall R \neg A$$

An important task is identifying all possible triples of (forms of) modal categorical statements that give rise to a valid syllogism. Moreover, one can look at the interaction between *de re* and *de dicto* combinations, or at the interaction between symmetric and asymmetric *de dicto* combinations, to identify other patterns of valid syllogism. The example provided above already illustrates some noteworthy interaction, since it has the following structure, where *s* stands for 'symmetric reading' and *a* for 'asymmetric reading':

$\{\mathbf{CU}(a), \mathbf{IU}(s)\} \models \mathbf{VU}(s)$

Finally, one can analyse patterns of valid syllogism, where some categories of individuals are treated as n-ary relations (n > 1) rather than properties (i.e. unary relations), generalizing the definition of a modal categorical statement used here.

As far as modalities are concerned, one can take into account more complex statements formalized via at least two modal operators (e.g. 'it is necessarily possible that...', 'it is known that it is unknown that...') or a family of modalities different from the one in the hexagon of opposition. Extensive research has been done on generalizations of the modal square of opposition, including works on solid figures such as *cubes of opposition*, whose definition varies with authors [17, 18]. It would be interesting to extend our framework in order to cover similar families of modalities.

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