# A Note on Parabolic and Linear One-Factorizations of the Complete Graph $K_{p+1}$

György Kiss[1,2], Gábor Korchmáros[3], Federico Romaniello[4] and Valentino Smaldore[5,*]

[1]Department of Geometry and HUN-REN-ELTE Geometric and Algebraic Combinatorics Research Group, Eötvös Loránd University, Pázmány s. 1/c, Budapest, 1117, Hungary

[2]FAMNIT, University of Primorska, Glagoljaška 8, Koper, 6000, Slovenia

[3]Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Potenza, 85100, Italy

[4]Dipartimento delle Culture Europee e del Mediterraneo, Università degli Studi della Basilicata, Via Lanera 20, Matera, 75100, Italy

[5]Dipartimento di Tecnica e Gestione dei Sistemi Industriali, Università degli Studi di Padova, Stradella S. Nicola 3, Vicenza, 361000, Italy

**Abstract**

One-factorizations of the complete graph $K_n$ have wide applications, as an example they are often used for scheduling round-robin tournaments with $n$ teams. In this note, we characterize parabolic and linear one-factorizations of complete graphs $K_{p+1}$, when $p$ is an odd prime. This class of one-factorizations arises from the geometry of conics and lines in the affine plane $AG(2, p)$. We also include Magma computations for the cases $p \leq 19$.

**Keywords**

One-factorization, parabolas, affine lines

## Declaration

This work is written in memory of Angelo Sonnino. His ideas on one-factorizations were very inspiring for this note. We hope that the present work provides something he would find interesting to read. The fourth author is also grateful for his mentorship during the writing of Master's thesis, in 2019.

## 1. Introduction

Some of the most important sport competitions have a final classification which depends on a round-robin phase. In such tournaments, each participant plays at least once against any other team, and the final classification considers all results. Then tournaments as *Serie A* or *NBA* need efficient algorithms to compute all the possible match schedules, see [1, 2]. For example, some tournament uses *Berger's algorithm* [3] (developed by the chess player *Johann Berger*) which divides the $n$ players into two equal sides, from $1$ to $\frac{n}{2}$ and from $\frac{n}{2}+1$ to $n$; starts from the first pairing $\{1, n\}, \{2, n-1\}, \ldots, \{\frac{n}{2}, \frac{n}{2}+1\}$; and ends by giving some combinatorial argument to obtain all the other pairings. In our approach, we use one-factorizations of complete graphs. More precisely, a one-factorization of the complete graph $K_n$ corresponds to a pairing in a round-robin tournament with $n$ teams playing. In this note, we characterize parabolic and linear one-factorizations of complete graphs $K_{p+1}$, with $p$ an odd prime number. In Section 2, we give preliminaries on graph theory and one-factorizations. In Sections 3 and 4, we give characterization results for parabolic and linear one-factorizations, where a one-factor is said to be *parabolic* or *linear* if it is represented by a parabola or a line, respectively. This note concludes with an Appendix containing computational results for the parabolic one-factorizations of $K_{p+1}$, $p = 13, 17, 19$, and the Magma code used by the authors.

## 2. Preliminaries

A graph $G = (V, E)$ is an incidence structure consisting of a set $V$ of objects called vertices and a set $E$ of object called edges. An edge $e \in E$ is denoted in the form $e = \{x, y\}$, where the vertices $x, y \in V$. Two vertices $x$ and $y$ connected by the edge $e = \{x, y\}$, are said to be adjacent. In what follows, neither multiple edges between the same pair of vertices nor loops, i.e. edges of the form $\{x, x\}$, are admitted. When every vertex has an equal number of edges incident to it, the graph is said to be regular. In particular, we will deal with a special class of regular graphs, in which each pair of vertices is connected by an edge. They are called complete graphs

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*Corresponding author.
kissgy@cs.elte.hu (Gy. Kiss); gabor.korchmaros@unibas.it (G. Korchmáros); federico.romaniello@unibas.it (F. Romaniello); valentino.smaldore@unipd.it (V. Smaldore)
0000-0003-3312-9575 (Gy. Kiss); 0000-0002-2776-5754 (G. Korchmáros); 0000-0003-1166-3179 (F. Romaniello); 0000-0003-3433-3164 (V. Smaldore)

and are usually denoted by $K_n$, where $n$ is the number of vertices. Let $K_n$ be the complete graph with an even number $n$ of vertices. A *one-factor* is a partition of its vertex set into $\frac{n}{2}$ disjoint edges. A *one-factorization* of $K_n$ is a partition of its edge set into $n-1$ disjoint one-factors. For other notation or definitions on graphs not stated here, we refer the reader to [4]. Our approach to the problem of constructing one-factorizations of complete graphs is geometric, as in [5, 6, 7, 8, 9], and is based on techniques that have also been used for multigraphs, see [10, 11, 12, 13].

## 3. Parabolic one-factorizations

We adopt the same notation and terminology introduced in a paper by Korchmáros, Pace and Sonnino [5], and then used in subsequent paper by Kiss, Pace and Sonnino [6]. The reader may refer to [14] for notation and terminology on finite geometry not explicitly stated here. Let $p$ be an odd prime. Fix a projective frame in $PG(2,p)$ with homogeneous coordinates $(X_0 : X_1 : X_2)$, and consider $PG(2,p)$ as $AG(2,p) \cup \ell_\infty$ where $\ell_\infty$ has equation $X_0 = 0$. As usual, the points of $AG(2,p)$ are written as $(X,Y)$ with $X = \frac{X_1}{X_0}$ and $Y = \frac{X_2}{X_0}$. In $AG(2,p)$, let $\mathcal{P}_a$ be the parabola with affine equation $Y = X^2 + a$, where $a$ varies in $\mathbb{Z}_p$, and $V_\infty = (0:0:1)$ the point at infinity of the line $X_1 = 0$. We remark that, in the projective closure of $AG(2,p)$, any two parabolas $\mathcal{P}_a$ and $\mathcal{P}_b$, with $a \neq b$, meet only at the point $V_\infty$. Let $P_i^k$ denote the affine point $(i + \frac{k}{2}, i^2 + ik)$ and $P_i^\infty$ be the point $(0 : 1 : 2i)$ on the line at infinity $\ell_\infty$. The following result is easy to check:

**Lemma 3.1.** *For a fixed $k$, the points $P_0^k, P_1^k, \ldots, P_{p-1}^k$ are on the parabola $\mathcal{P}_{-\frac{k^2}{4}}$.*

**Definition 3.2.** *Let $R = (u,v)$ be a point. The* symptome *of $R$ is defined as $\mu_R = u^2 - v$.*

*Let $\ell$ be a non-vertical line with equation $Y = mX + b$. The* symptome *of $\ell$ is defined as $\lambda_\ell = m^2 + 4b$.*

We recall some basic properties of the parabolas $\mathcal{P}_a$. It is straightforward to check the following lemma:

**Lemma 3.3.** *Let $R = (u,v)$ be a point and $\ell\colon Y = mX + b$ be a non-vertical line. Then*

- *$R$ is an external (internal) point of $\mathcal{P}_0$ if and only if $\mu_R$ is a square (non-square) element in $\mathrm{GF}(p)$.*
- *$\ell$ is a tangent (secant or external line) to $\mathcal{P}_a$ if and only if $\lambda_\ell - 4a$ is $0$ (a square or a non-square element in $\mathrm{GF}(p)$).*
- *The pole of $\ell$ with respect to $\mathcal{P}_0$ is the point $L = (\frac{m}{2}, -b)$.*

The vertices of the complete graph $K_{p+1}$ correspond to the points of $\mathcal{P}_0 \cup V_\infty$, while the edges of $K_{p+1}$ correspond to the points of type $P_i^k$, with $k = 1, 2, \ldots, \frac{p-1}{2}, \infty$. Thus the set of edges of $K_{p+1}$ corresponds to the set of points

$$\mathcal{E} = \left( \bigcup_{k=1}^{\frac{p-1}{2}} \mathcal{P}_{-\frac{k^2}{4}} \right) \cup (\ell_\infty \setminus V_\infty).$$

These points are called *external* w.r.t. $\mathcal{P}_0$.

**Definition 3.4.** *A one-factor represented by a parabola $\mathcal{P}_a$ is a set of $\frac{p-1}{2}$ points of type $P_j^k$ on $\mathcal{P}_a$, together with a suitable point at infinity. A one-factor so defined is referred to as a* parabolic one-factor.

**Definition 3.5.** *A* one-factor represented by a secant line *$\ell$ of $\mathcal{P}_0$ is a set consisting of $\frac{p-1}{2}$ points of $\mathcal{E}$ on $\ell$, plus the pole of $\ell$ with respect to $\mathcal{P}_0$.*

*A* one-factor represented by an external line *$\ell$ of $\mathcal{P}_0$ is a set consisting of $\frac{p+1}{2}$ points of $\mathcal{E}$ on $\ell$.*

**Definition 3.6.** *A one-factorization of $K_{p+1}$ is called a* parabolic one-factorization *if $p-1$ of its one-factors are represented by parabolas and one of its one-factors is represented by a line.*

In [6] the authors proved the existence of an infinite family of parabolic one-factorization.

**Theorem 3.7.** *[6, Theorem 3.4] Let $p$ be an odd prime. Then the complete graph $K_{p+1}$ has a parabolic one-factorization.*

*Proof.* The proof is constructive. The set

$$F_0 = \left\{ P_{-\frac{k}{2}}^k : k = 1, 2, \ldots, \frac{p-1}{2} \right\} \cup \{P_0^\infty\}$$

is a one-factor represented by the line secant line of $\mathcal{P}_0$ of equation $X = 0$, and $P_0$ is the pole. Now we define the sets

$$G_k = \left\{ P_{\frac{k}{2}+2jk}^k : j = 0, 1, \ldots, \frac{p-3}{2} \right\} \cup \left\{ P_{-\frac{k}{2}}^\infty \right\},$$
$$H_k = \left\{ P_{\frac{k}{2}+(2j+1)k}^k : j = 0, 1, \ldots, \frac{p-3}{2} \right\} \cup \left\{ P_{\frac{k}{2}}^\infty \right\}.$$

By Lemma 3.1 $G_k \setminus \left\{ P_{-\frac{k}{2}}^\infty \right\}$ and $H_K \setminus \left\{ P_{\frac{k}{2}}^\infty \right\}$ are disjoint subsets of the parabola $\mathcal{P}_{-\frac{k^2}{4}}$, and both $G_k$ and $H_k$ are one-factors represented by the parabola $\mathcal{P}_{-\frac{k^2}{4}}$. □

**Definition 3.8.** *A one-factorization of $K_{p+1}$ is called an* almost parabolic one-factorization *if at least one of its one-factors is represented by $\mathcal{P}_{-\frac{k^2}{4}}$ for all $k \in \{1, 2, \ldots \frac{p-1}{2}\}$, and all other of its one-factors are represented by lines.*

For $p < 11$ exhaustive computer search shows that each almost parabolic one-factorization of $K_{p+1}$ is parabolic. In the rest of the paper we will assume $p \geq 11$.

**Lemma 3.9.** *The number of one-factors represented by lines in an almost parabolic one-factorization is either one, or at least* $\lceil \frac{p+1}{4} \rceil$.

*Proof.* If more than one one-factors are represented by lines, then there exist parabolas $\mathcal{P}_{-\frac{k^2}{4}}$ which represent only one one-factor. Hence $p - \frac{p-1}{2} = \frac{p+1}{2}$ of its points are covered by the lines represented the other one-factors. Any line meets $\mathcal{P}_{-\frac{k^2}{4}}$ in at most two points, so we need at least $\lceil \frac{p+1}{4} \rceil$ lines to cover these points. ☐

The following Lemma is a straightforward corollary of [6, Theorem 3.5].

**Lemma 3.10.** *Let $\mathcal{L}$ denote the set of points of type $P_i^k$ belonging to the one-factors represented by lines in an almost parabolic one-factorization. Suppose that a one-factor is represented by the line $X = 0$. Then for each $k \in \{1, 2, \ldots \frac{p-1}{2}\}$ either $\mathcal{L} \subset G_k$ or $\mathcal{L} \subset H_k$.*

**Proposition 3.11.** *An almost parabolic one-factorization contains at most two one-factors which are represented by vertical lines.*

*Proof.* Suppose to the contrary that it contains at least three one-factors which are represented by vertical lines. We may assume that the equations of the corresponding lines are $X = 0$, $X = u$, $X = v$ and $u - v = k' \in \{1, 2, \ldots \frac{p-1}{2}\}$.

We also may assume that the line $X = v$ intersects $G_{k'}$. Then $v = \frac{k'}{2} + 2jk'$ where $j = 0, 1, \ldots, \frac{p-3}{2}$. Hence

$$u = v + k' = \frac{k'}{2} + (2j+1)k' \text{ where } j = 0, 1, \ldots, \frac{p-3}{2}.$$

So the line $X = u$ intersects $H_{k'}$ contradicting Lemma 3.10. ☐

**Lemma 3.12.** *In* $\mathrm{GF}(p)$ *let $T$ denote the set*

$$T = \left\{(4j^2 + 4j)k^2 \colon j = 0, 1, \ldots, \frac{p-1}{2},\right.$$
$$\left. k = 1, 2, \ldots, \frac{p-1}{2}\right\}.$$

*Then* $|T| = p$.

*Proof.* First, we show that the cardinality of the set

$$U = \left\{4j^2 + 4j \colon j = 0, 1, \ldots, \frac{p-1}{2}\right\}$$

is $\frac{p+1}{2}$. If $4j_1^2 + 4j_1 = 4j_2^2 + 4j_2$, then $(j_1 - j_2)(j_1 + j_2 + 1) = 0$. So for $j_1 \neq j_2$ we have $4j_1^2 + 4j_1 = 4j_2^2 + 4j_2$, because the sum in the second factor is not 0.

The set $U$ obviously contains 0. Moreover, we claim that $U$ contains both square and non-square elements. Suppose to the contrary, that all of the non-zero products $j(j+1)$ are either squares or non-squares. If 2 is a square, then $1 \cdot 2$ is a square, hence $2 \cdot 3$ is also a square which implies that 3 is a square. In the same way, step by step, we get that all of the elements $4, \ldots, \frac{p-1}{2}, \frac{p+1}{2}$ are squares. Hence there are at least $\frac{p+1}{2}$ square elements, a contradiction. If 2 is a non-square, then $1 \cdot 2$ is a non-square, hence $2 \cdot 3$ is also a non-square, so 3 is a square. But 4 is also a square, hence $3 \cdot 4$ is a square, a contradiction again.

Now we show that the set

$$S = \left\{k^2 \colon k = 1, 2, \ldots, \frac{p-1}{2}\right\}$$

equals to the set of square elements of $\mathrm{GF}(p)$. The cardinality of $S$ is $\frac{p-1}{2}$, because $k_1^2 = k_2^2$ implies $(k_1 - k_2)(k_1 + k_2) = 0$, and the second factor is never 0, because $k_1 + k_2 \leq p - 1$.

Choose elements $u_1, u_2 \in U$ such that $u_1$ is a square and $u_2$ is a non-square. Then $u_1 U \cap u_2 U = \emptyset$, hence

$$|\{0\} \cup u_1 U \cup u_2 U| = 1 + \frac{p-1}{2} + \frac{p-1}{2} = p,$$

the statement is proved. ☐

**Proposition 3.13.** *If an almost parabolic one-factorization contains a one-factors which is represented by a vertical line, then it cannot contain one-factors represented by non-vertical lines.*

*Proof.* We may assume that the equation of the corresponding vertical line is $X = 0$. Suppose to the contrary that it contains the line $\ell \colon Y = mx + b$. We claim that there exists at least one $k \in \{1, 2, \ldots, \frac{p-1}{2}\}$ such that $\ell \cap G_k \neq \emptyset \neq \ell \cap H_k$. By Lemma 3.12, there exist $j$ and $k$ such that $\lambda_\ell = (4j^2 + 4j)k^2$. Then

$$\lambda_\ell + k^2 = (4j^2 + 4j)k^2 + k^2 = ((2j+1)k)^2,$$

so, by Lemma 3.3, $\ell$ intersects $\mathcal{P}_{-\frac{k^2}{4}}$ and the difference of the first coordinates of the two intersections is $(2j+1)k$. Hence one of the two points belongs to $G_k$ and the other one belongs to $H_k$. The statement follows from Lemma 3.10. ☐

The main result of this section is the following theorem, which derives from Propositions 3.9, 3.11 and 3.13.

**Theorem 3.14.** *Let $p$ be an odd prime. If an almost parabolic one-factorization $\mathcal{F}$ of $K_{p+1}$ contains a one-factor which is represented by a vertical line, then the one-factorization is parabolic.*

*Proof.* For $p < 11$ the statement follows from an exhaustive computer search, see [6].

If $p \geq 11$, then $\lceil \frac{p+1}{4} \rceil \geq 3$. Hence, by Lemma 3.9, $\mathcal{F}$ contains either one, or at least three one-factors that are represented by lines. In the former case, we are done. The latter case leads to a contradiction, since, by Lemma 3.11, the number of vertical lines is at most two, and by Lemma 3.13, there is no non-vertical line among the lines representing the one-factors. □

## 4. Linear one-factorizations

In this section, we consider one-factorizations whose all one-factors are represented by lines.

Let $\Omega$ be the set of points of an irreducible conic in $PG(2, q)$ with $q \geq 5$ odd.

**Theorem 4.1.** *Let $K_{q+1}$ be a one-factorization on $\Omega$ whose one-factors are represented by lines. Then some of those lines are a chord of $\Omega$.*

*Proof.* Let $K_{q+1}$ be represented by the lines $\ell_1, \ldots, \ell_q$. Assume on the contrary that each those lines is an external line to $\Omega$. Let $L_1, \ldots, L_q$ be the poles of $\ell_1, \ldots, \ell_q$ with respect to the orthogonal polarity $\pi$ associated with $\Omega$. For $i = 1, \ldots, q$, let $\varphi_i$ be the involutory perspectivity with center $L_i$ and axis $\ell_i$ which preserves $\Omega$. Let $G \cong PGL(2, q)$ be the orthogonal group of $\Omega$, i.e. the subgroup of $PGL(3, q)$ which commutes with $\pi$. Then $G$ preserves $\Omega$ and acts on its points as $PGL(2, q)$ on the projective line over $\mathbb{F}_q$. Furthermore, $\varphi_i \in G$. Let $\mathcal{F}$ be the set consisting of $\varphi_i$, $i = 1, \ldots, q$ together with the identity of $G$. Since $K_{q+1}$ is a one-factorization, for any two points $P, Q \in \Omega$ there exists a unique $\varphi \in \mathcal{F}$ such that $\varphi(P) = q$. Therefore, $\mathcal{F}$ is a sharply transitive permutation set on $\Omega$ containing the identity. From the classification of sharply transitive subsets of $PGL(2, q)$ [15], it turns out $\mathcal{F}$ is a subgroup of $PGL(2, q)$ of order $q + 1$. On the other hand, from Dickson's classification of subgroups of $PGL(2, q)$, the subgroups of $PGL(2, q)$ entirely consisting of involutions together with the identity, have order either 2 or 4. But then $q \leq 3$, a contradiction. □

We conclude by reporting a conjecture that is supported by computer-aided searches. With the aid of Magma [16] we verified that the conjecture holds for $p \leq 23$.

**Conjecture 4.2.** *[6, Conjecture 3.6] Let $p > 7$ be an odd prime, $\mathcal{F}$ be a one-factorization of the complete graph $K_{p+1}$ such that each one-factor of $\mathcal{F}$ is either represented by a line or a parabola. Then $\mathcal{F}$ is either a parabolic one-factorization or each one-factor of $\mathcal{F}$ is represented by a line.*


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## A. Tables

Corollary 3.14 states that almost parabolic one-factorization of $K_{p+1}$, $p$ odd prime, containing a vertical line are parabolic. In this Appendix, we report examples of such parabolic one-factorizations of the complete graphs $K_{14}$, $K_{18}$ and $K_{20}$, found by computations on Magma, [16]. $K_4$ admits only 1 one-factorization, and are well-known the 6 examples of 1-factorizations of $K_6$. We refer to [6] for the cases $K_8$ and $K_{12}$.

### A.1. $p = 13$

$F_0 : \{P_{0,\infty} : (0:1:0), P_{1,6} : (1:0:3), P_{2,12} : (1:0:12), P_{3,5} : (1:0:1), P_{4,11} : (1:0:9), P_{5,4} : (1:0:10), P_{6,10} : (1:0:4)\}$

$G_1 : \{P_{6,\infty} : (0:1:12), P_{1,7} : (1:1:4), P_{1,9} : (1:3:12), P_{1,11} : (1:5:2), P_{1,0} : (1:7:0), P_{1,2} : (1:9:6), P_{1,4} : (1:11:7)\}$

$H_1 : \{P_{7,\infty} : (0:1:1), P_{1,8} : (1:2:7), P_{1,10} : (1:4:6), P_{1,12} : (1:6:0), P_{1,1} : (1:8:2), P_{1,3} : (1:10:12), P_{1,5} : (1:12:4)\}$

$G_2 : \{P_{12,\infty} : (0:1:11), P_{2,1} : (1:2:3), P_{2,5} : (1:6:9), P_{2,9} : (1:10:8), P_{2,0} : (1:1:0), P_{2,4} : (1:5:11), P_{2,8} : (1:9:2)\}$

$H_2 : \{P_{1,\infty} : (0:1:2), P_{2,3} : (1:4:2), P_{2,7} : (1:8:11), P_{2,11} : (1:12:0), P_{2,2} : (1:3:8), P_{2,6} : (1:7:9), P_{2,10} : (1:11:3)\}$

$G_3 : \{P_{5,\infty} : (0:1:10), P_{3,8} : (1:3:10), P_{3,1} : (1:9:4), P_{3,7} : (1:2:5), P_{3,0} : (1:8:0), P_{3,6} : (1:1:2), P_{3,12} : (1:7:11)\}$

$H_3 : \{P_{8,\infty} : (0:1:3), P_{3,11} : (1:6:11), P_{3,4} : (1:12:2), P_{3,10} : (1:5:0), P_{3,3} : (1:11:5), P_{3,9} : (1:4:4), P_{3,2} : (1:10:10)\}$

$G_4 : \{P_{11,\infty} : (0:1:9), P_{4,2} : (1:4:12), P_{4,10} : (1:12:10), P_{4,5} : (1:7:6), P_{4,0} : (1:2:0), P_{4,8} : (1:10:5), P_{4,3} : (1:5:8)\}$

$H_4 : \{P_{2,\infty} : (0:1:4), P_{4,6} : (1:8:8), P_{4,1} : (1:3:5), P_{4,9} : (1:11:0), P_{4,4} : (1:6:6), P_{4,12} : (1:1:10), P_{4,7} : (1:9:12)\}$

$G_5 : \{P_{4,\infty} : (0:1:8), P_{5,9} : (1:5:9), P_{5,6} : (1:2:1), P_{5,3} : (1:12:11), P_{5,0} : (1:9:0), P_{5,10} : (1:6:7), P_{5,7} : (1:3:6)\}$

$H_5 : \{P_{9,\infty} : (0:1:5), P_{5,1} : (1:10:6), P_{5,11} : (1:7:7), P_{5,8} : (1:4:0), P_{5,5} : (1:1:11), P_{5,2} : (1:11:1), P_{5,12} : (1:8:9)\}$

$G_6 : \{P_{10,\infty} : (0:1:7), P_{6,3} : (1:6:1), P_{6,2} : (1:5:3), P_{6,1} : (1:4:7), P_{6,0} : (1:3:0), P_{6,12} : (1:2:8), P_{6,11} : (1:1:5)\}$

$H_6 : \{P_{3,\infty} : (0:1:6), P_{6,9} : (1:12:5), P_{6,8} : (1:11:8), P_{6,7} : (1:10:0), P_{6,6} : (1:9:7), P_{6,5} : (1:8:3), P_{6,4} : (1:7:1)\}$

### A.2. $p = 17$

$F_0 : \{P_{0,\infty} : (0:1:0), P_{1,8} : (1:0:4), P_{2,16} : (1:0:16), P_{3,7} : (1:0:2), P_{4,15} : (1:0:13), P_{5,6} : (1:0:15), P_{6,14} : (1:0:8), P_{7,5} : (1:0:9), P_{8,13} : (1:0:1)\}$

$G_1 : \{P_{8,\infty} : (0:1:16), P_{1,9} : (1:1:5), P_{1,11} : (1:3:13), P_{1,13} : (1:5:12), P_{1,15} : (1:7:2), P_{1,0} : (1:9:0), P_{1,2} : (1:11:6), P_{1,4} : (1:13:3), P_{1,6} : (1:15:8)\}$

$H_1 : \{P_{9,\infty} : (0:1:1), P_{1,10} : (1:2:8), P_{1,12} : (1:4:3), P_{1,14} : (1:6:6), P_{1,16} : (1:8:0), P_{1,1} : (1:10:2), P_{1,3} : (1:12:12), P_{1,5} : (1:14:13), P_{1,7} : (1:16:5)\}$

$G_2 : \{P_{16,\infty} : (0:1:15), P_{2,1} : (1:2:3), P_{2,5} : (1:6:1), P_{2,9} : (1:10:14), P_{2,13} : (1:14:8), P_{2,0} : (1:1:0), P_{2,4} : (1:5:7), P_{2,8} : (1:9:12), P_{2,12} : (1:13:15)\}$

$H_2 : \{P_{1,\infty} : (0:1:2), P_{2,3} : (1:4:15), P_{2,7} : (1:8:12), P_{2,11} : (1:12:7), P_{2,15} : (1:16:0), P_{2,2} : (1:3:8), P_{2,6} : (1:7:14), P_{2,10} : (1:11:1), P_{2,14} : (1:15:3)\}$

$G_3 : \{P_{7,\infty} : (0:1:14), P_{3,10} : (1:3:11), P_{3,16} : (1:9:15), P_{3,5} : (1:15:6), P_{3,11} : (1:4:1), P_{3,0} : (1:10:0), P_{3,6} : (1:16:3), P_{3,12} : (1:5:10), P_{3,1} : (1:11:4)\}$

$H_3 : \{P_{10,\infty} : (0:1:3), P_{3,13} : (1:6:4), P_{3,2} : (1:12:10), P_{3,8} : (1:1:3), P_{3,14} : (1:7:0), P_{3,3} : (1:13:1), P_{3,9} : (1:2:6), P_{3,15} : (1:8:15), P_{3,4} : (1:14:11)\}$

$G_4 : \{P_{15,\infty} : (0:1:13), P_{4,2} : (1:4:12), P_{4,10} : (1:12:4), P_{4,1} : (1:3:5), P_{4,9} : (1:11:15), P_{4,0} : (1:2:0), P_{4,8} : (1:10:11), P_{4,16} : (1:1:14), P_{4,7} : (1:9:9)\}$

$H_4 : \{P_{2,\infty} : (0:1:4), P_{4,6} : (1:8:9), P_{4,14} : (1:16:14), P_{4,5} : (1:7:11), P_{4,13} : (1:15:0), P_{4,4} : (1:6:15), P_{4,12} : (1:14:5), P_{4,3} : (1:5:4), P_{4,11} : (1:13:12)\}$

$G_5 : \{P_{6,\infty} : (0:1:12), P_{5,11} : (1:5:6), P_{5,4} : (1:15:2), P_{5,14} : (1:8:11), P_{5,7} : (1:1:16), P_{5,0} : (1:11:0), P_{5,10} : (1:4:14), P_{5,3} : (1:14:7), P_{5,13} : (1:7:13)\}$

$H_5 : \{P_{11,\infty} : (0:1:5), P_{5,16} : (1:10:13), P_{5,9} : (1:3:7), P_{5,2} : (1:13:14), P_{5,12} : (1:6:0), P_{5,5} : (1:16:16),$

$P_{5,15}:(1:9:11), P_{5,8}:(1:2:2), P_{5,1}:(1:12:6)\}$

$G_6:\{P_{14,\infty}:(0:1:11), P_{6,3}:(1:6:10), P_{6,15}:(1:1:9),$
$P_{6,10}:(1:13:7), P_{6,5}:(1:8:4), P_{6,0}:(1:3:0),$
$P_{6,12}:(1:15:12), P_{6,7}:(1:10:6), P_{6,2}:(1:5:16)\}$

$H_6:\{P_{3,\infty}:(0:1:6), P_{6,9}:(1:12:16), P_{6,4}:(1:7:6),$
$P_{6,16}:(1:2:12), P_{6,11}:(1:14:0), P_{6,6}:(1:9:4),$
$P_{6,1}:(1:4:7), P_{6,13}:(1:16:9), P_{6,8}:(1:11:10)\}$

$G_7:\{P_{5,\infty}:(0:1:1PZ0), P_{7,12}:(1:7:7), P_{7,9}:(1:4:8),$
$P_{7,6}:(1:1:10), P_{7,3}:(1:15:13), P_{7,0}:(1:12:0),$
$P_{7,14}:(1:9:5), P_{7,11}:(1:6:11), P_{7,8}:(1:3:1)\}$

$H_7:\{P_{12,\infty}:(0:1:7), P_{7,2}:(1:14:1), P_{7,16}:(1:11:11),$
$P_{7,13}:(1:8:5), P_{7,10}:(1:5:0), P_{7,7}:(1:2:13),$
$P_{7,4}:(1:16:10), P_{7,1}:(1:13:8), P_{7,15}:(1:10:7)\}$

$G_8:\{P_{13,\infty}:(0:1:9), P_{8,4}:(1:8:14), P_{8,3}:(1:7:16),$
$P_{8,2}:(1:6:3), P_{8,1}:(1:5:9), P_{8,0}:(1:4:0),$
$P_{8,16}:(1:3:10), P_{8,15}:(1:2:5), P_{8,14}:(1:1:2)\}$

$H_8:\{P_{4,\infty}:(0:1:8), P_{8,12}:(1:16:2), P_{8,11}:(1:15:5),$
$P_{8,10}:(1:14:10), P_{8,9}:(1:13:0), P_{8,8}:(1:12:9),$
$P_{8,7}:(1:11:3), P_{8,6}:(1:10:16), P_{8,5}:(1:9:14)\}$

## A.3. $p = 19$

$F_0:\{P_{0,\infty}:(0:1:0), P_{1,9}:(1:0:14), P_{2,18}:(1:0:18),$
$P_{3,8}:(1:0:12), P_{4,17}:(1:0:15), P_{5,7}:(1:0:8),$
$P_{6,16}:(1:0:10), P_{7,6}:(1:0:2), P_{8,15}:(1:0:3),$
$P_{9,5}:(1:0:13)\}$

$G_1:\{P_{9,\infty}:(0:1:18), P_{1,10}:(1:1:15), P_{1,12}:(1:3:4),$
$P_{1,14}:(1:5:1), P_{1,16}:(1:7:6), P_{1,18}:(1:9:0),$
$P_{1,1}:(1:11:2), P_{1,3}:(1:13:12), P_{1,5}:(1:15:11),$
$P_{1,7}:(1:17:18)\}$

$H_1:\{P_{10,\infty}:(0:1:1), P_{1,11}:(1:2:18), P_{1,13}:(1:4:11),$
$P_{1,15}:(1:6:12), P_{1,17}:(1:8:2), P_{1,0}:(1:10:0),$
$P_{1,2}:(1:12:6), P_{1,4}:(1:14:1), P_{1,6}:(1:16:4),$
$P_{1,8}:(1:18:15)\}$

$G_2:\{P_{18,\infty}:(0:1:17), P_{2,1}:(1:2:3), P_{2,5}:(1:6:16),$
$P_{2,9}:(1:10:4), P_{2,13}:(1:14:5), P_{2,17}:(1:18:0),$
$P_{2,2}:(1:3:8), P_{2,6}:(1:7:10), P_{2,10}:(1:11:6),$
$P_{2,14}:(1:15:15)\}$

$H_2:\{P_{1,\infty}:(0:1:2), P_{2,3}:(1:4:15), P_{2,7}:(1:8:6),$
$P_{2,11}:(1:12:10), P_{2,15}:(1:16:8), P_{2,0}:(1:1:0),$
$P_{2,4}:(1:5:5), P_{2,8}:(1:9:4), P_{2,12}:(1:13:16),$
$P_{2,16}:(1:17:3)\}$

$G_3:\{P_{8,\infty}:(0:1:16), P_{3,11}:(1:3:2), P_{3,17}:(1:9:17),$
$P_{3,4}:(1:15:9), P_{3,10}:(1:2:16), P_{3,16}:(1:8:0),$
$P_{3,3}:(1:14:18), P_{3,9}:(1:1:13), P_{3,15}:(1:7:4),$
$P_{3,2}:(1:13:10)\}$

$H_3:\{P_{11,\infty}:(0:1:3), P_{3,14}:(1:6:10), P_{3,1}:(1:12:4),$
$P_{3,7}:(1:18:13), P_{3,13}:(1:5:18), P_{3,0}:(1:11:0),$
$P_{3,6}:(1:17:16), P_{3,12}:(1:4:9), P_{3,18}:(1:10:17),$
$P_{3,5}:(1:16:2)\}$

$G_4:\{P_{17,\infty}:(0:1:15), P_{4,2}:(1:4:12), P_{4,10}:(1:12:7),$
$P_{4,18}:(1:1:16), P_{4,7}:(1:9:1), P_{4,15}:(1:17:0),$
$P_{4,4}:(1:6:13), P_{4,12}:(1:14:2), P_{4,1}:(1:3:5),$
$P_{4,9}:(1:11:3)\}$

$H_4:\{P_{2,\infty}:(0:1:4), P_{4,6}:(1:8:3), P_{4,14}:(1:16:5),$
$P_{4,3}:(1:5:2), P_{4,11}:(1:13:13), P_{4,0}:(1:2:0),$
$P_{4,8}:(1:10:1), P_{4,16}:(1:18:16), P_{4,5}:(1:7:7),$
$P_{4,13}:(1:15:12)\}$

$G_5:\{P_{7,\infty}:(0:1:14), P_{5,12}:(1:5:14), P_{5,3}:(1:15:5),$
$P_{5,13}:(1:6:6), P_{5,4}:(1:16:17), P_{5,14}:(1:7:0),$
$P_{5,5}:(1:17:12), P_{5,15}:(1:8:15), P_{5,6}:(1:18:9),$
$P_{5,16}:(1:9:13)\}$

$H_5:\{P_{12,\infty}:(0:1:5), P_{5,17}:(1:10:13), P_{5,8}:(1:1:9),$
$P_{5,18}:(1:11:15), P_{5,9}:(1:2:12), P_{5,0}:(1:12:0),$
$P_{5,10}:(1:3:17), P_{5,1}:(1:13:6), P_{5,11}:(1:4:5),$
$P_{5,2}:(1:14:14)\}$

$G_6:\{P_{16,\infty}:(0:1:13), P_{6,3}:(1:6:8), P_{6,15}:(1:18:11),$
$P_{6,8}:(1:11:17), P_{6,1}:(1:4:7), P_{6,13}:(1:16:0),$
$P_{6,6}:(1:9:15), P_{6,18}:(1:2:14), P_{6,11}:(1:14:16),$
$P_{6,4}:(1:7:2)\}$

$H_6:\{P_{3,\infty}:(0:1:6), P_{6,9}:(1:12:2), P_{6,2}:(1:5:16),$
$P_{6,14}:(1:17:14), P_{6,7}:(1:10:15), P_{6,0}:(1:3:0),$
$P_{6,12}:(1:15:7), P_{6,5}:(1:8:17), P_{6,17}:(1:1:11),$
$P_{6,10}:(1:13:8)\}$

$G_7:\{P_{6,\infty}:(0:1:12), P_{7,13}:(1:7:13), P_{7,8}:(1:2:6),$
$P_{7,3}:(1:16:11), P_{7,17}:(1:11:9), P_{7,12}:(1:6:0),$
$P_{7,7}:(1:1:3), P_{7,2}:(1:15:18), P_{7,16}:(1:10:7),$
$P_{7,11}:(1:5:8)\}$

$H_7:\{P_{13,\infty}:(0:1:7), P_{7,1}:(1:14:8), P_{7,15}:(1:9:7),$
$P_{7,10}:(1:4:18), P_{7,5}:(1:18:3), P_{7,0}:(1:13:0),$
$P_{7,14}:(1:8:9), P_{7,9}:(1:3:11), P_{7,4}:(1:17:6),$
$P_{7,18}:(1:12:13)\}$

$G_8:\{P_{15,\infty}:(0:1:11), P_{8,4}:(1:8:10), P_{8,1}:(1:5:9),$
$P_{8,17}:(1:2:7), P_{8,14}:(1:18:4), P_{8,11}:(1:15:0),$
$P_{8,8}:(1:12:14), P_{8,5}:(1:9:8), P_{8,2}:(1:6:1),$
$P_{8,18}:(1:3:12)\}$

$H_8:\{P_{4,\infty}:(0:1:8), P_{8,12}:(1:16:12), P_{8,9}:(1:13:1),$
$P_{8,6}:(1:10:8), P_{8,3}:(1:7:14), P_{8,0}:(1:4:0),$
$P_{8,16}:(1:1:4), P_{8,13}:(1:17:7), P_{8,10}:(1:14:9),$
$P_{8,7}:(1:11:10)\}$

$G_9:\{P_{5,\infty}:(0:1:10), P_{9,14}:(1:9:18), P_{9,13}:(1:8:1),$
$P_{9,12}:(1:7:5), P_{9,11}:(1:6:11), P_{9,10}:(1:5:0),$
$P_{9,9}:(1:4:10), P_{9,8}:(1:3:3), P_{9,7}:(1:2:17),$
$P_{9,6}:(1:1:14)\}$

$H_9:\{P_{14,\infty}:(0:1:9), P_{9,4}:(1:18:14), P_{9,3}:(1:17:17),$
$P_{9,2}:(1:16:3), P_{9,1}:(1:15:10), P_{9,0}:(1:14:0),$
$P_{9,18}:(1:13:11), P_{9,17}:(1:12:5), P_{9,16}:(1:11:1),$
$P_{9,15}:(1:10:18)\}$

## B. Magma code

Below, we report the code used to find the parabolic one-factorizations.

```
p:=19;
F:=GF(p);
AG:=AffineSpace(F,3);
r:=(p-1)/2;
rr:=(p-3)/2;
F0:={AG![0,1,0]};
for k in [1..r] do
kk:=F!k;
l:=AG![1,0,(-kk/2)^2+(-kk/2)*kk];
F0:=F0 join {l};
end for;
printf "F 0 : %o\n" ,F0;
for k in [1..r] do
kk:=F!k;
G:={AG![0,1,-kk]};
H:={AG![0,1,kk]};
for j in [0..rr] do
jj:=F!j;
i:=F!(kk/2+2*jj*kk);
ii:=F!(kk/2+(2*jj+1)*kk);
m:=AG![1,i+kk/2,i^2+i*kk];
n:=AG![1,ii+kk/2,ii^2+ii*kk];
G:=G join {m};
H:=H join {n};
end for;
printf "G %o : %o\n" ,kk,G;
printf "H %o : %o\n" ,kk,H;
end for;
```