Partial Automorphisms and Level of Symmetry of Asymmetric Graphs

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Abstract

Most graphs are asymmetric, i.e. they lack any nontrivial automorphisms. Even in the case of highly symmetric graphs, removing just a single vertex from a graphical regular representation may result in a graph with a trivial automorphism group. Nevertheless, asymmetric graphs can still contain relatively large induced subgraphs which do admit nontrivial automorphisms or relatively large distinct but isomorphic induced subgraphs. Such symmetric local structures play a crucial role in Babai's quasipolynomial algorithm for solving the Graph Isomorphism Problem. These observations called for the use of the concept of a partial automorphism of a graph Γ, which is either an isomorphism between two distinct induced subgraphs of Γ or an automorphism of one of its induced subgraphs. Based on the concept of a partial automorphism, the *symmetry level of a graph* Γ is defined as the ratio between the largest order of the domain of a nontrivial partial automorphism of Γ and the graph's order $|V(\Gamma)|$.

In our paper, we address several open questions concerning the symmetry levels of graphs posted by Cingel, Gál & Jajcayová (2023), and derive additional results using both computer aided and theoretical methods. We improve the best previously known lower bound for the symmetry levels of general graphs by proving that the symmetry level of any finite simple graph is at least $\frac{1}{2}$. In case of disconnected graphs without a unique isolated vertex, we prove that the symmetry level of such graphs is at least $\frac{3}{4}$. Furthermore, we present graphs that provide an answer to Question 3 posted by Cingel, Gál & Jajcayová by showing that higher symmetry level does not necessarily imply a larger number of partial automorphisms. We take the initial steps toward answering the main question of Cingel, Gál & Jajcayová. Finally, we discuss the relation between a measure of asymmetry introduced by Erdős & Rényi (1963) and the level of symmetry of graphs considered in our paper.

Keywords
symmetry of graphs, asymmetric graphs, partial automorphisms, graphs with small symmetry levels

1. Introduction

Almost all graphs are known to be *asymmetric* [4], i.e., having no nontrivial 'global' automorphisms. At the same time, all of them contain local symmetries. These observations remain true even if one restricts the graphs considered to the class of regular graphs, which is the class containing some of the most symmetric graphs - the vertex-transitive graphs [8]. Taking the opposite point of view, deleting a single vertex from a (vertex-transitive) graphical regular representation of odd order leads to an asymmetric graph. It is important to note, however, that the distorted graph obtained this way still has many induced subgraphs with nontrivial partial automorphisms [7], which suggests that the line between symmetric and asymmetric objects is surprisingly thin. Furthermore, local structures exhibiting high levels of symmetry are of significant importance in Babai's quasipolynomial algorithm for solving the Graph Isomorphism Problem [1]. These observations provide the motivation behind the study of partial automorphisms of graphs [6] and the related concept of the level of symmetry of graphs defined using the order of the largest domain of a nontrivial partial automorphism of a graph [3]. In the absence of a universally accepted name, we call a graph that possesses at least one nontrivial automorphism as *non-asymmetric*. The following definitions introduce two fundamental concepts used throughout our paper.

Definition 1.1 (*Partial automorphism*)**.** *Let Γ be a graph, and let D and R be non-empty subsets of the vertex set $V(\Gamma)$ of equal cardinalities. A partial automorphism $\varphi : D \to R$ is an isomorphism between the induced subgraphs $\Gamma[D]$ and $\Gamma[R]$; where the rank of φ is the cardinality $|D| = |R|$. We say that a partial automorphism is nontrivial if it maps at least one vertex in D to another (distinct) vertex in R.*

Definition 1.2 (*Symmetry level*)**.** *Let Γ be a graph of order $n \geq 2$, and let k be the largest postive integer for which Γ admits a nontrivial partial automorphism φ of rank k. The symmetry level of Γ is the ratio $S(\Gamma) := \frac{k}{n}$.*

The set of all partial automorphisms, denoted $\mathrm{PAut}(\Gamma)$, along with the operations of partial composition and partial inverse of partial maps, forms an inverse monoid, which was fully characterized for graphs by Jajcay et al. in [6]. Some of the basic results concerning

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partial automorphisms can be found in [5, 3, 2, 9].

2. Addressing open questions of Cingel, Gál & Jajcayová

In the article [3], the authors posed several questions we will address (and possibly answer) in this section.

2.1. Minimal level of symmetry

Question 1 ([3]). *What is the minimal level of symmetry of a graph Γ of order n as a function of n?*

Although we will not settle Question 1, in what follows, we improve the lower bound $S(\Gamma) \geq \frac{\log_{\sqrt{2}} n}{n}$ stated in [3] for graphs of order n. Let Γ be a finite simple graph of order $n \geq 2$, $V(\Gamma) = \{u_1, u_2, \ldots, u_n\}$, let u_i and u_j be two distinct vertices of Γ, and let φ be the partial map mapping the subset of vertices of $V(\Gamma)$ not contained in the symmetric difference of the neighborhoods $N_\Gamma(u_i) \triangle N_\Gamma(u_j)$ of u_i and u_j in Γ onto itself by swapping u_i and u_j, $\varphi(u_i) = u_j$ and $\varphi(u_j) = u_i$, and fixing all vertices of Γ distinct from u_i and u_j and not contained in the symmetric difference $N_\Gamma(u_i) \triangle N_\Gamma(u_j)$, $\varphi(u_l) = u_l$, for all $u_l \in V(\Gamma) - (N_\Gamma(u_i) \triangle N_\Gamma(u_j)) - \{u_i, u_j\}$. It is easy to see that φ is a partial automorphism of Γ of rank $|V(\Gamma) - (N_\Gamma(u_i) \triangle N_\Gamma(u_j))|$. Denoting the cardinality of the symmetric difference of the neighbourhoods of u_i and u_j by Δ_{jk} allows us now to derive the first (and in some sense, the most general) lower bound on the symmetry level of Γ that will serve as the basis of our forthcoming arguments:

$$S(\Gamma) \geq \frac{n - \min\{\Delta_{ij}\}}{n} = 1 - \frac{\min\{\Delta_{ij}\}}{n}, \quad (1)$$

where the minimum is taken over all pairs of distinct vertices $u_i, u_j \in V(\Gamma)$.

Using the notation introduced above and mimicking a proof from [4] yields now the following.

Theorem 2.1. *For any simple graph Γ of order $n \geq 2$,*

$$S(\Gamma) \geq 1 - \frac{n-1}{2n} > \frac{1}{2}.$$

Proof. Let Γ be a simple graph of order n. Using inequality (1) together with the obvious fact that

$$\min_{1 \leq i < j \leq n} \{\Delta_{ij}\} \leq \operatorname*{avg}_{1 \leq i < j \leq n} \{\Delta_{ij}\} = \frac{\sum_{1 \leq i < j \leq n} \Delta_{ij}}{\frac{n(n-1)}{2}},$$

yields

$$S(\Gamma) \geq 1 - \frac{2 \sum_{1 \leq i < j \leq n} \Delta_{ij}}{n^2(n-1)}. \quad (2)$$

Next, we use the following fact observed already in Erdős & Rényi [4]. The sum $\sum_{1 \leq i < j \leq n} \Delta_{ij}$ is equal to the number of ordered triples (v_i, v_j, v_l) of vertices in Γ such that Γ contains the edge $\{v_i, v_l\}$ but does not contain $\{v_j, v_l\}$, $1 \leq i \neq j \leq n$. Note that each of such triples consists of a vertex v_l together with one of its neighbors and one of its 'non-neighbors'. This implies that the number of such triples containing v_l is equal to $n_l(n - 1 - n_l)$. Hence,

$$\sum_{1 \leq i < j \leq n} \Delta_{ij} = \sum_{l=1}^{n} n_l(n - 1 - n_l), \quad (3)$$

and therefore

$$S(\Gamma) \geq 1 - \frac{2 \sum_{l=1}^{n} n_l(n - 1 - n_l)}{n^2(n-1)}. \quad (4)$$

Taking advantage of another simple algebraic identity used in Erdős & Rényi:

$$n_l(n - 1 - n_l) = (\frac{n-1}{2})^2 - \left(n_l - \frac{n-1}{2}\right)^2,$$

we obtain

$$S(\Gamma) \geq 1 - \frac{2 \sum_{l=1}^{n} (\frac{n-1}{2})^2 - \left(n_l - \frac{n-1}{2}\right)^2}{n^2(n-1)} \geq$$

$$1 - \frac{2 \sum_{l=1}^{n} (\frac{n-1}{2})^2}{n^2(n-1)} = 1 - \frac{n \frac{(n-1)^2}{2}}{n^2(n-1)} = 1 - \frac{n-1}{2n} \geq$$

$$\frac{1}{2}.$$

□

In order to obtain another formulation of Theorem 2.1, let n_i denote the degree of the i-th vertex of Γ, and let n_{ij} denote the cardinality of $N_\Gamma(u_i) \cap N_\Gamma(u_j)$, the set of shared neighbors of u_i and u_j. Using this notation, it is easy to see that

$$\Delta_{ij} = n_i + n_j - 2n_{ij} - 2\delta_{ij}, \quad (5)$$

for all $1 \leq i \neq j \leq n$, where $\delta_{ij} = 1$ if u_i and u_j are distinct and adjacent, and $\delta_{ij} = 0$ otherwise. Then,

$$\sum_{1 \leq i < j \leq n} \Delta_{ij} =$$

$$\sum_{1 \leq i < j \leq n} (n_i + n_j) - 2 \cdot \sum_{1 \leq i < j \leq n} n_{ij} - 2 \cdot \sum_{1 \leq i < j \leq n} \delta_{ij} \leq$$

$$4ns - 2 \sum_{l=1}^{n} \binom{n_l}{2} - 2s = (4n-2)s - \sum_{l=1}^{n} n_l(n_l - 1) =$$

$$(4n-2)s - \sum_{l=1}^{n}(n_l^2 - n_l) = 4ns - \sum_{l=1}^{n} n_l^2, \quad (6)$$

where s is the size (the number of edges) of Γ. Based on the above, we obtain the following corollary.

Corollary 2.2. *For any simple graph Γ of order $n \geq 2$ and size s,*

$$S(\Gamma) \geq 1 - \frac{8ns - 2\sum_{l=1}^{n} n_l^2}{n^2(n-1)}.$$

Proof. Returning to the inequality (2) and substituting the identity (6) yields the desired result. □

Revisiting the more precise (1), we note that computer searches we have conducted yielded many asymmetric graphs of order n and symmetry level $S(\Gamma) = 1 - \frac{\min\{\Delta_{ij}\}}{n}$, i.e., graphs whose symmetry level matches the lower bound in (1). On the other hand, despite considerable computational effort, we have not found a graph with symmetry level close to $\frac{1}{2}$. In his 2023 master thesis [5], Gál found graphs of order 14 and symmetry level equal to $\frac{5}{7}$. In our own investigation, we have found distinct graphs of the same order and the same symmetry level. We present one such graph Γ in Figure 1, for which $S(\Gamma) = \frac{5}{7}$ matches the lower bound from (1).

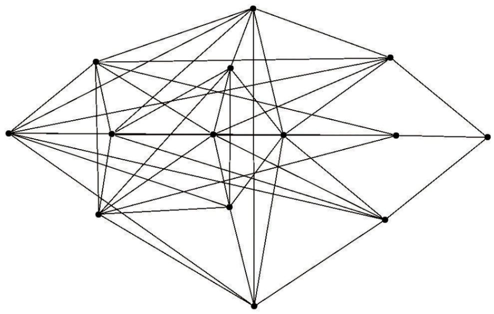


Figure 1: Graph Γ with 14 vertices and symmetry level $S(\Gamma) = \frac{5}{7}$.

Note that the lower bound in (1) can be computed in $O(n^2K)$ time, where K is the maximum degree in the graph. Computationally testing the relation between the lower bound in (1) and the true symmetry level of all graphs of order at most 10, we have learned that only $\frac{1210694}{8110708} \approx 15\%$ of all asymmetric graphs up to the order $n = 10$ have symmetry levels exceeding the lower bound $1 - \frac{\min\{\Delta_{ij}\}}{n}$. This means that for most asymmetric graphs of order not exceeding 10, a partial automorphism of maximal rank can be obtained by swapping two vertices and fixing all vertices not belonging to the symmetric difference of their neighborhoods. In the remaining 15% of the asymmetric graphs, all graphs were of order 10. In this subset of graphs, almost all graphs have the largest nontrivial partial automorphism of rank $(n - \min\{\Delta_{jk}\}) + 1 = n - 1$, which exceeds the rank predicted by the lower bound by 1, i.e., it is the closest possible to the considered lower bound. Moreover, the majority of these graphs have only few nontrivial partial automorphisms of the maximal rank $n - 1$, most of which swap only two vertices. Finally, there were also tens of graphs in which the largest rank of a nontrivial automorphism exceeds the lower bound by 2 and is equal to $n - \min\{\Delta_{jk}\} + 2 = n - 1$. For one such example, see Figure 2.

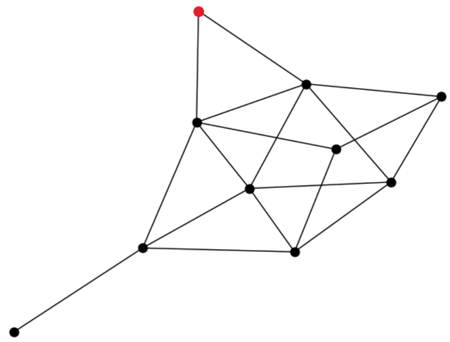


Figure 2: Graph in which $\min_{j \neq k}\{\Delta_{jk}\} = 3$ but there is a nontrivial partial automorphism of rank $n-1$ which can be obtained by deleting the red vertex and acts on the remaining vertices.

To complete the section, let us consider the special case of k-regular graphs. Inequality (4) applied to k-regular graphs of order n yields

$$S(\Gamma) \geq 1 - \frac{2\sum_{l=1}^{n} k(n-1-k)}{n^2(n-1)} = 1 - \frac{2nk(n-1-k)}{n^2(n-1)}.$$

Thus, for any fixed $k \geq 1$, and any infinite family of k-regular graphs Γ:

$$\lim_{|V(\Gamma)| \to \infty} S(\Gamma) = 1.$$

In case of unbounded k proportional to the order n of the k-regular graph, $k = \frac{n}{p}$, we obtain for any infinite family of $\frac{n}{p}$-regular graphs

$$\lim_{n \to \infty} \left(1 - \frac{2\sum_{l=1}^{n} k(n-1-k)}{n^2(n-1)}\right) =$$

$$= \lim_{n\to\infty}\left(1-\frac{2\sum_{l=1}^{n}\frac{n}{p}(n-1-\frac{n}{p})}{n^2(n-1)}\right) =$$

$$= \lim_{n\to\infty}\left(1-\frac{2(n(p-1)-p)}{(n-1)p^2}\right) =$$

$$= 1-\frac{2}{p^2}\lim_{n\to\infty}\frac{(np-n-p)}{(n-1)} =$$

$$= 1-\frac{2}{p^2}\lim_{n\to\infty}\frac{\frac{1}{n}(np-n-p)}{\frac{1}{n}(n-1)} =$$

$$= 1-\frac{2}{p^2}\lim_{n\to\infty}\frac{(p-1-\frac{p}{n})}{1-\frac{1}{n}} = 1-\frac{2\cdot(p-1)}{p^2}$$

which gives a better lower bound than the $\frac{1}{2}$ from Theorem 2.1 for all $p > 2$.

2.2. Relations between the symmetry level of a graph and the size of its inverse monoid of partial automorphisms

Question 2 ([3]). *When given two graphs of the same order, does a higher symmetry level of one of them necessarily mean that it will also have a larger monoid of partial automorphisms?*

Relying on the combination of trial and error attempts and exhaustive search of graphs of order at most 10 again, we found a pair of graphs of order 8 that provides a *negative answer to Question 2*, and is shown in Figure 3. While $S(\Gamma_1) = 1$ and $S(\Gamma_2) = \frac{7}{8}$, and thus $S(\Gamma_2) < S(\Gamma_1)$, the graph Γ_1 has fewer partial automorphisms than Γ_2; the exact numbers being 11033 vs. 13871 partial automorphisms.

This pair of graphs appears to be the first member of an infinite family of examples where every new pair is constructed from the previous pair by attaching two new vertices (one on each side) as shown in Figure 4. Each new pair consists of a graph of symmetry level 1 and an asymmetric graph of strictly smaller symmetry level $\frac{n-1}{n}$, with the second graph apparently having a larger monoid of partial automorphisms.

By determining the orders of the corresponding monoids, we have verified this observation for the first three pairs of graphs in the family. Specifically, the order of the monoid of partial automorphisms of the more symmetric graph of order 10 consists of 195779 vs. 255414 partial automorphisms in favor of the less symmetric graph. The pattern repeats for the next pair with 3859889 versus 5327803 partial automorphisms for graphs of order 12. As the difference between the orders of inverse monoids of partial automorphisms of the corresponding graphs keeps increasing, we formulate our observation in the form of a conjecture. Its proof would require a determination of the orders of the two corresponding monoids for all pairs of members of the infinite family.

Conjecture 2.3. *The order of the inverse monoid of partial automorphisms of the graph with a smaller symmetry level is larger than that of the graph of the symmetry level* 1 *for each pair of graphs constructed in the above described way from the pair in Figure 3.*

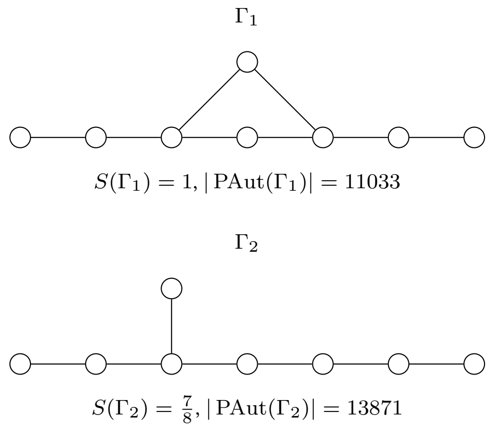


Figure 3: Graphs providing an answer to Question 2.

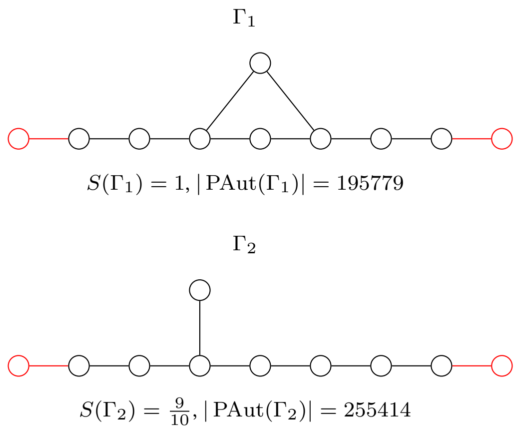
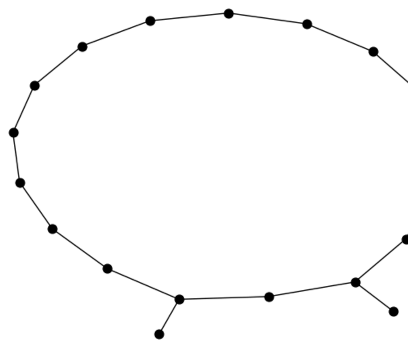


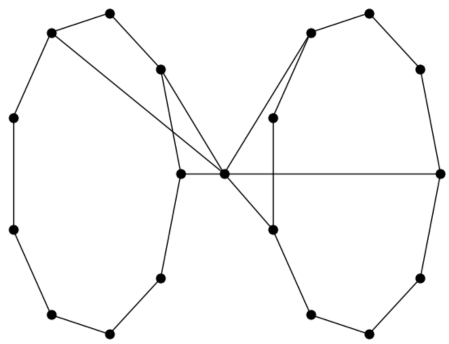
Figure 4: Extension of graphs providing an answer to Question 2.

Based on our negative answer to Question 2, we know that a higher symmetry level does not necessarily imply a larger number of partial automorphisms. This may be the consequence of the fact that the absence of nontrivial (global) automorphism does not negatively affect the existence of a large number of small partial automorphisms. However, it might be the case that the absence of nontrivial automorphisms in a graph of order n could have a negative impact on the number of partial automorphisms of rank $n-1$. That is why we propose a new question.

Question 3. *When given two graphs of the same order n, does one of them being asymmetric and another non-asymmetric necessarily mean that the asymmetric one will have fewer partial automorphisms of rank $n-1$?*



Graph Γ_1 with $|\operatorname{Aut}(\Gamma_1)| = 2$ and 50 partial automorphisms of rank $n-1$



Graph Γ_2 with $|\operatorname{Aut}(\Gamma_2)| = 1$ and 666 partial automorphisms of rank $n-1$

Figure 5: Graphs Γ_1 and Γ_2 of order $n = 19$ which provide a negative answer to Question 3.

We again answer this question in negative by constructing two graphs of order 19 shown in Figure 5. The first one, Γ_1, is non-asymmetric with $|\operatorname{Aut}(\Gamma_1)| = 2$ and has 50 partial automorphisms of rank $n-1$. The second graph, Γ_2, is asymmetric, $|\operatorname{Aut}(\Gamma_2)| = 1$, but it has 666 partial automorphisms of rank $n-1$. Thus, we have shown that if a graph is asymmetric, we cannot expect it to have necessarily fewer partial automorphisms of rank $n-1$ than more symmetric graphs, which further emphasises the need to improve our understanding of the correspondence between the symmetry levels and the corresponding orders of monoids of partial automorphisms.

We note that the non-asymmetric graph from Figure 5 has only a small automorphism group. Despite both our computational effort and trial and error attempts, we were unable to find graphs with larger automorphism groups such that some asymmetric graph of the same order n has more partial automorphisms of rank $n-1$.

2.3. Asymmetric depth of almost all graphs

Next, we address the following question.

Question 4 ([3]). *Does there exist a graph Γ of order n and level of symmetry equal to $\frac{n-d}{n}$ for arbitrarily large $d \geq 2$?*

To simplify our discussion, we shall refer to d as the *asymmetric depth* of Γ. By Theorem 2.1, we know that asymmetric depth cannot be greater than $\frac{1}{2}n$. We propose to use a probabilistic approach to build more intuition with regard to Question 4.

Recall that a graph Γ of order n is asymmetric if and only if $S(\Gamma) \leq \frac{n-1}{n}$. That means that the result asserting that almost all graphs are asymmetric proven in [4] can be reformulated using the language of partial automorphisms as follows.

Theorem 2.4. *The limit of probabilities*

$$\lim_{n\to\infty} P\left(S(\Gamma) \leq \frac{n-1}{n}\right) = 1,$$

where $P\left(S(\Gamma) \leq \frac{n-1}{n}\right)$ represents the probability that the symmetry level of a randomly chosen graph Γ of order n does not exceed $\frac{n-1}{n}$.

This formulation raises the question of whether the constant asymmetric depth 1 in the above formula can be replaced with any fixed integral asymmetric depth $d \geq 1$. We feel that the following conjecture stated originally in [2] is very likely true as well.

Conjecture 2.5. *Let $d \geq 1$ be an integer. The limit of probabilities*

$$\lim_{n\to\infty} P\left(S(\Gamma) \leq \frac{n-d}{n}\right) = 1,$$

where $P\left(S(\Gamma) \leq \frac{n-d}{n}\right)$ represents the probability that the symmetry level of a randomly chosen graph Γ of order n does not exceed $\frac{n-d}{n}$.

So far, we were unable to prove this conjecture. However, it might be easier to address a closely related question which focuses exclusively on partial automorphisms which are also automorphisms of some induced subgraph of a graph Γ. In order to formalize our reformulation, we define the *adjusted level of symmetry of a graph* Γ, $S'(\Gamma)$, to be the ratio of the order of a largest non-asymmetric induced subgraph of Γ and the order of Γ.

Clearly, for any graph Γ, $S'(\Gamma) \leq S(\Gamma)$, since Γ can also have partial automorphisms that do not have the same domain and range (are not nontrivial automorphisms of an induced subgraph).

We believe the following conjecture might be shown to be true if one could show that if we remove a constant number of randomly chosen vertices from a large random graph, we will again have a large random graph of a smaller order, which is also likely to be asymmetric. A formal proof of this statement would however require very careful manipulation of the concepts from the theory of random graphs.

Conjecture 2.6. *Let $d \geq 1$ be an integer. The limit of probabilities*

$$\lim_{n\to\infty} P\left(S'(\Gamma) \leq \frac{n-d}{n}\right) = 1,$$

where $P\left(S'(\Gamma) \leq \frac{n-d}{n}\right)$ represents the probability that the adjusted symmetry level of a randomly chosen graph Γ of order n does not exceed $\frac{n-d}{n}$.

2.4. Computer assisted searches for graphs of increasing asymmetric depth

Since the above probabilistic arguments appear to suggest a positive answer to Question 4, we set to find an explicit construction of an infinite family of graphs with increasing asymmetric depth. For this reason, we set to investigate the record graphs that increase d in $S(\Gamma) \geq \frac{n-d}{n}$. Previously, the record d for the graph symmetry level found in [3] was $S(\Gamma) \geq \frac{n-d}{n}; d = 5$; which means that one can delete any subset of up to 4 vertices without obtaining a graph admitting nontrivial partial automorphisms.

Some graphs of order 9 can already have millions of partial automorphisms. To determine the asymmetry depth d, we have to look at isomorphisms between all induced subgraphs of decreasing order until we find some nontrivial partial automorphism. In a graph of order n, there are $\binom{n}{n-d}$ induced subgraphs of order $n-d$. For each of these induced subgraphs, we have to compute its automorphism group, and then determine isomorphisms between all combinations of induced subgraphs of order $n-d$. Thus, computing the whole inverse monoid for a graph is already intractable for rather small graphs. However, taking advantage of the inequality $S(\Gamma) \geq \frac{n-\min\{\Delta_{jk}\}}{n}$, determining the parameter $S(\Gamma)$ can be done without computing the entire monoid of partial automorphisms of a given graph. Therefore, before computing partial automorphisms, we minimise the branching factor by introducing the following ad hoc heuristics.

1. Eliminating complements since $S(\Gamma) = S(\tilde{\Gamma})$ [3].
2. Utilising the result of Theorem 2.1 and by computing the bound using $\min\{\Delta_{jk}\}$ for some vertices v_j, v_k, omitting the graphs which have small $\min\{\Delta_{jk}\}$.
3. Omitting the graphs with many small or large degrees since for increasing the d, most of the vertices of the asymmetric graphs should have a degree roughly equal to $\frac{n}{2}$ [5].

With these ideas, we parallelised the search through the space of asymmetric graphs that are possible to be reached by extending the previous database of record graphs from [5]. After roughly three days of computations, we found a new record graph with $n = 27, d = 7$. It turns out that while most of the graphs are asymmetric, with our computations we find that most have only rather small d. We analysed the record graphs, but due to lack of any clear global structure of these graphs, we were unable to extend our example to larger graphs.

To further our experiments with regard to Question 4, we changed the approach by considering very "symmetric" graphs and making them asymmetric. Specifically, we considered n-dimensional hypercubes which have the property that $\min\{\Delta_{jk}\}$ increases with growing n, while they remain arc-transitive. Instead of necessarily removing vertices from these graphs, we noticed that multiple n-dimensional hypercubes can be joined asymmetrically. For example, one can glue hypercubes together and construct two columns of n-dimensional hypercubes, the first column with two n-dimensional hypercubes and the second column with three n-dimensional hypercubes; with the adjacent hypercubes sharing an $(n-1)$-dimensional hyperface. For $n = 2$, see Figure 6. As n increases, we add some diagonals to each hypercube to prevent the early occurrence of mirror symmetry. This allowed us to computationally increase d starting with $d = 1$ for 2-dimensional-hypercubes, to $d = 5$ for 5-dimensional-hypercubes. For this construction, we have not yet proved that increasing the dimension n of the hypercubes, necessarily makes d grow indefinitely and cover all $d \geq 2$. In fact, our construction skipped $d = 4$.

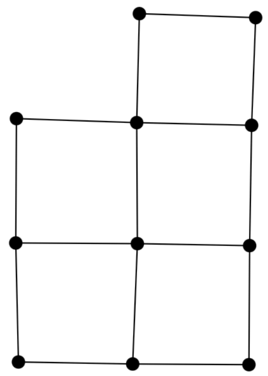


Figure 6: Asymmetric graph constructed of 2-dimensional hypercubes.

3. Further results on symmetry levels of graphs

3.1. Disconnected graphs and graphs with small cut sets

As we have seen at the end of Section 2.1, certain classes of graphs allow further improvements on the lower bounds on their symmetry levels. A slightly different version of the following theorem was proven for disconnected graphs in [2].

Theorem 3.1. *Let Γ be a disconnected graph of order n with c components $C_1, C_2, ..., C_c$.*

(i) For every C_j, $1 \leq j \leq c$, of cardinality bigger than one, we obtain the following lower bound for the symmetry level of Γ:

$$S(\Gamma) \geq \frac{|V(C_j)| \cdot S(C_j)}{n} + \frac{1}{n} \sum_{i \in \{1,\ldots,c\} \setminus \{j\}} |V(C_i)|.$$

(ii) If at least two of the components $C_1, C_2, ..., C_c$ are of cardinality 1, $S(\Gamma) = 1$.

(iii) If exactly one of the components $C_1, C_2, ..., C_c$ is of cardinality 1, say, $C_1 = \{u_1\}$, $S(\Gamma) = \frac{1+(n-1)S(\Gamma \setminus \{u_1\})}{n}$.

Proof. We only provide a quick sketch of the proof that is based on ideas introduced in Section 2.1. If $|V(C_j)| > 1$, a partial automorphism acting on a subset of C_j the same way as one of the partial automorphisms of C_j of the rank $|V(C_j)|S(C_j)$ and fixing vertices in all components distinct from C_j has the rank appearing in part (i) of the theorem. If at least two components, $C_i = \{u_i\}$ and $C_j = \{u_j\}$, are of cardinality one, the map swapping u_i and u_j and fixing all other vertices of Γ is an automorphism of Γ. If $C_1 = \{u_1\}$ is the unique component of cardinality 1, any non-trivial partial automorphism φ of $\Gamma \setminus \{u_1\}$ (Γ with u_1 removed) can be extended by adding $\varphi(u_1) = u_1$. □

Using the above theorem and Theorem 2.1, it is possible to prove the following corollary.

Corollary 3.2. *Let Γ be a disconnected graph without a unique isolated vertex. Then $S(\Gamma) \geq \frac{3}{4}$.*

Proof. Let Γ be a disconnected graph of order n without a unique isolated vertex with components $C_1, C_2, ..., C_c$. If there are at least two isolated vertices in Γ, then $S(\Gamma) = 1$. If Γ contains no isolated vertices, let C_j denote a smallest component of Γ. Then, $|V(C_j)| \leq \frac{n}{2}$. Theorem 2.1 yields that $S(C_j) \geq \frac{1}{2}$. Therefore, by Theorem 3.1,

$$nS(\Gamma) \geq \frac{1}{2}|V(C_j)| + \sum_{i \in \{1,\ldots,c\} \setminus \{j\}} |V(C_i)| =$$
$$= \frac{1}{2}|V(C_j)| + (n - |V(C_j)|) = n - \frac{1}{2}|V(C_j)| \geq$$
$$\geq n - \frac{1}{2} \cdot \frac{1}{2}n = \frac{3}{4}n.$$

□

Furthermore, if a graph Γ has a relatively small vertex cut set, it is usually possible to remove a small number of vertices from Γ and obtain a disconnected graph, which will have a level of symmetry at least $\frac{3}{4}$.

Corollary 3.3. *Let Γ be a graph of order n with a vertex separator S. Then $S(\Gamma) \geq \frac{\frac{3}{4}(n-|S|-1)}{n}$. Moreover, if $\Gamma \setminus S$ does not contain a unique isolated vertex, then $S(\Gamma) \geq \frac{\frac{3}{4}(n-|S|)}{n}$.*

Proof. Let Γ be a graph of order n with a vertex separator S. If $\Gamma \setminus S$ contains a unique isolated vertex v, then by Corollary 3.2 $S(\Gamma \setminus S \setminus \{v\}) \geq \frac{3}{4}$. Therefore, by removing S and v from Γ, we obtain $S(\Gamma) \geq \frac{\frac{3}{4}(n-|S|-1)}{n}$. If $\Gamma \setminus S$ does not contain a unique isolated vertex, then by Corollary 3.2 $S(\Gamma) \geq \frac{\frac{3}{4}(n-|S|)}{n}$. □

Corollary 3.4. *Let $s \in \mathbb{N}$. Let Γ_k be an infinite family of graphs such that every graph in this infinite family has a vertex separator of size at most s. Then*

$$\lim_{k \to \infty} S(\Gamma_k) \geq \frac{3}{4}.$$

3.2. Relation to measure of asymmetry introduced by Erdős & Rényi

In their 1963 paper [4], Erdős & Rényi considered a measure of asymmetry that might at first appear quite different from the one we consider in our paper. Their idea is based on deleting edges to obtain a non-asymmetric graph.

Definition 3.5 (Degree of asymmetry)**.** *The degree of asymmetry of a graph* Γ, $A^-(\Gamma)$, *is the minimum number of edges that need to be deleted so that* Γ *becomes non-asymmetric.*

As we have already pointed out in Section 2.1, both the general bounds for the symmetry levels proven therein and the degree of asymmetry results of Erdős & Rényi rely on the same idea of considering the minimum of the symmetric difference of the neighbourhoods over all pairs of vertices. This often results in asymmetric depth and degree of asymmetry of a graph being equal. This is further confirmed by the corresponding results concerning forests as listed in Table 1.

	$A^-(\Gamma)$	d	$S(\Gamma)$
general lower bound	$\frac{1}{2}n$	$\frac{1}{2}n$	$\frac{1}{2}$
lower bound for forests	1	1	$\frac{n-1}{n}$ [3]

Table 1
Comparison of lower bounds for the parameters $A^-(\Gamma)$, d and $S(\Gamma)$ where n is the order of the considered graph. The results in the first column come from [4].

It is therefore natural to ask how big can the difference between these two parameters be in general graphs. When addressing this question, we constructed graphs for which $3 = A^-(\Gamma) > d = 1$, where d corresponds to the number of vertices that have to be removed from Γ to obtain a non-asymmetric subgraph. One such graph can be seen in Figure 7.

On the other hand, there are also graphs where $A^-(\Gamma) < d$; e.g., the graph in Figure 8 for which $A^-(\Gamma) = 1$ and $d = 3$. We also entertained the question of what is the maximum possible difference between $A^-(\Gamma)$ and d. We have empirically discovered that finding graphs for which the difference would be greater than two is rather hard. We formulate questions inspired by our results in the most generalized form in Question 5. Despite the generality of this question, we believe that the answer is positive for any pair a and d. Some kind of a generalization of either the graph in Figure 7 or the graph in Figure 8 might be a good starting point in the construction of such graphs.

Question 5. *Does there exist for any pair of positive integers* a *and* d *a graph* Γ *such that* $A^-(\Gamma) = a$, *and the asymmetry depth of* Γ *is equal to* d*?*

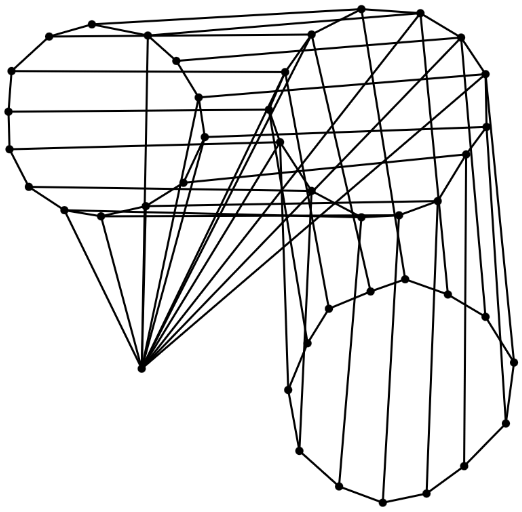


Figure 7: A graph where $d = 1$ and $A^-(\Gamma) = 3$. The graph was constructed by trial and error from three disjoint cycles of length 14 and one isolated vertex. We connected the isolated vertex to the first cycle in such a way that the vertex and the first cycle resulted in an asymmetric graph. This was done as follows. We connected the isolated vertex to some starting vertex from the cycle. We continued in the clockwise direction where we omitted the possible edge to the next neighbouring vertex. We made two subsequent edges and again omitted the next vertex. Lastly, we connected three subsequent vertices to the isolated vertex and omitted the remaining ones. We made the same kind of connections to the second cycle. We rotated the second cycle counterclockwise with the step of 5. We then linked the cycles as shown. Thus, all of the asymmetry is linked to just the one special vertex.

4. Final remarks

In this paper, we established that the level of symmetry of any simple graph is at least $\frac{1}{2}$, and for disconnected graphs without a unique isolated vertex, it is at least $\frac{3}{4}$ (and is even more for many other classes of graphs). By answering Question 2, we exhibited computationally that a higher level of symmetry does not imply a larger number of partial automorphisms; further emphasising the importance of studying local symmetries. We have also applied probabilistic and computational methods to build more intuition in understanding Question 4.

For the implementation and testing of our algorithms we used the mathematical software system - SageMath, version 10.1 [11] taking advantage of the existence of extensive and useful libraries in the Sage ecosystem. In determining the inverse monoids of the considered graphs, we used the interface to the system for discrete computational algebra - GAP, version 4.12.2 [10]. We also used GAP to represent the inverse monoids of partial automor-

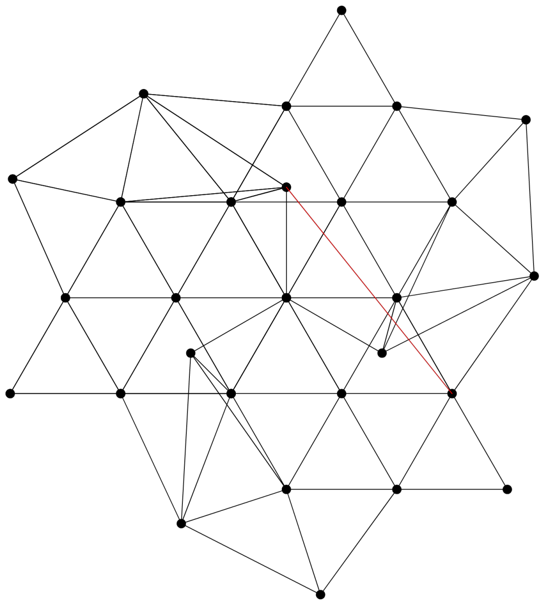


Figure 8: Graph where $d = 3$ and $A^{-}(\Gamma) = 1$. All of the asymmetry is linked to the red edge.

phisms of graphs. All our computations were utilised on a computer with a 12th Gen Intel Core i5-12450H processor, 4400 MHz, 8 cores, 12 logical processors, and 32GB of RAM.

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