Many-valued Temporal Description Logics with Typicality: an Abridged Report

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Abstract

In this paper, we develop a many-valued semantics for the description logic $LTL_{\mathcal{ALC}}$, a temporal extension of description logic $\mathcal{ALC}$, based on Linear-time Temporal Logic (LTL). We add a typicality operator to represent defeasible properties, and discuss the use of the (many-valued) temporal conditional logic and of weighted KBs for explaining the dynamic behaviour of a network.

Keywords

Preferential Logics, Temporal Logics, Many-valued Description Logics, Explainability

1. Introduction

Preferential extensions of Description Logics (DLs) allow for reasoning with exceptions through the identification of *prototypical properties* of individuals or classes of individuals. *Defeasible inclusions* are allowed in the knowledge base, to model typical, defeasible, non-strict properties of individuals. Their semantics extends DL semantics with a preference relation among domain individuals, along the lines of the preferential semantics introduced by Kraus, Lehmann and Magidor [2, 3] (KLM for short). *Multi-preferential* extensions of DLs have been developed, to provide a semantics for ranked and weighted knowledge bases with typicality [4, 5, 6].

Temporal extensions of Description Logics are very well-studied in DLs literature [7, 8]. Preferential extensions of Linear Time Temporal Logic (LTL) with defeasible temporal operators have been recently studied [9, 10] to enrich temporal formalisms with non-monotonic reasoning features. On a different route, a preferential extension of the temporal description logic $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ has been proposed in [11], extending $LTL_{\mathcal{ALC}}$ [7] with a typicality operator $\mathbf{T}$, which selects the most typical instances of a concept, to represent defeasible temporal properties of concepts, i.e., temporal properties which admit exceptions.

It is proven that the preferential extension of $LTL^{\mathbf{T}}_{\mathcal{ALC}}$ can be polynomially encoded into $LTL_{\mathcal{ALC}}$, and this approach allows borrowing decidability and complexity results from $LTL_{\mathcal{ALC}}$. A similar encoding can be given for a multi-preferential extension of $LTL^{\mathbf{T}}_{\mathcal{ALC}}$, by allowing a concept-wise preferential semantics, where different preferences are associated to different concepts.

In this short paper, an abridged version of [12], we describe a many-valued extension of $LTL_{\mathcal{ALC}}$ with typicality, making it possible to represent concept inclusions such as

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$$\exists lives\_in.Town \sqcap Young \sqsubseteq \mathbf{T}(\Diamond Granted\_Loan),$$

(meaning that people living in town and being young, normally are eventually granted a loan), where the interpretation of some concepts (such as, $Young$) may be non-crisp.

This many-valued temporal extension of $\mathcal{ALC}$ builds on many-valued DLs, which are widely studied in the literature, both for the fuzzy case [13, 14, 15] and for the finitely-valued case [16, 17]. We then add a typicality operator, to get a many-valued temporal extension of $\mathcal{ALC}$ with typicality.

We briefly discuss the definition of a closure construction for weighted knowledge bases with typicality [5, 18, 19] in the temporal case. The formalism allows for a finer grained representation of the prototypical properties of a concept, including temporal properties, by assigning weights to typicality properties. It is also discussed how the many-valued preferential temporal logic can be used to provide a logical interpretation of the transient behavior of recurrent neural networks.

2. Fuzzy $\mathcal{ALC}$

Fuzzy description logics have been widely studied in the literature for representing vagueness in DLs [13, 14, 15] based on the idea that concepts and roles can be interpreted as fuzzy sets. Formulas in Mathematical Fuzzy Logic [20] have a degree of truth in an interpretation rather than being true or false; similarly, axioms in a fuzzy DL have a degree of truth, usually in the interval $[0, 1]$. The finitely many-valued case is also well studied for DLs [16, 17]. We breifly recall the semantics of a fuzzy extension of $\mathcal{ALC}$, following [15]; then we consider the finitely-valued case.

Let N_C be a set of concept names, N_R a set of role names and N_I a set of individual names. The set of $\mathcal{ALC}$ *concepts* (or, simply, concepts) is defined inductively from concept names and the $\top$ and $\bot$ concepts, using intersection $C \sqcap D$, union $C \sqcup D$, negation $\neg C$, as well as universal and existential restrictions $\forall r.C$, $\exists r.C$.

A *fuzzy interpretation* I, given a non-empty domain Δ, assigns to each individual name $a \in N_I$ an element $a^I \in \Delta$; to each concept name $A \in N_C$ a function $A^I : \Delta \to [0, 1]$; and to each role name $r \in N_R$ a function $r^I : \Delta \times \Delta \to [0, 1]$. That is, an element $x \in \Delta$ belongs to the extension of A to some degree in $[0, 1]$, i.e., A^I is a fuzzy set; and similarly for roles. The interpretation function $\cdot^I$ is extended to other concepts as follows:

$$\top^I(x) = 1 \qquad \bot^I(x) = 0 \qquad (\neg C)^I(x) = \ominus C^I(x)$$
$$(C \sqcap D)^I(x) = C^I(x) \otimes D^I(x) \qquad (C \sqcup D)^I(x) = C^I(x) \oplus D^I(x)$$
$$(\exists r.C)^I(x) = \sup_{y\in\Delta}\ r^I(x, y) \otimes C^I(y) \qquad (\forall r.C)^I(x) = \inf_{y\in\Delta}\ r^I(x, y) \rhd C^I(y)$$

where $x \in \Delta$, and $\otimes$, $\oplus$, $\rhd$ and $\ominus$ are arbitrary but fixed *t-norm*, *s-norm*, implication function, and negation function, chosen among the combination functions of some fuzzy logic. In particular, in Gödel logic $a \otimes b = min\{a, b\}$, $a \oplus b = max\{a, b\}$, $a \rhd b = 1$ *if* $a \leq b$ *and* b *otherwise*; $\ominus a = 1$ *if* $a = 0$ *and* 0 *otherwise*. In Łukasiewicz logic, $a \otimes b = max\{a + b - 1, 0\}$, $a \oplus b = min\{a + b, 1\}$, $a \rhd b = min\{1 - a + b, 1\}$ and $\ominus a = 1 - a$. Following [15], we do not commit to a specific choice of combination functions,

A *fuzzy* $\mathcal{ALC}$ *knowledge base* K is a pair $(\mathcal{T}, \mathcal{A})$ where $\mathcal{T}$ is a fuzzy TBox and $\mathcal{A}$ is a fuzzy ABox. A fuzzy TBox is a set of *fuzzy concept inclusions* of the form $C \sqsubseteq D\ \theta\ n$, where $C \sqsubseteq D$ is an $\mathcal{ALC}$ concept inclusion axiom, $\theta \in \{\geq, \leq, >, <\}$ and $n \in [0, 1]$. A fuzzy ABox $\mathcal{A}$ is a set of *fuzzy assertions* of the form $C(a)\theta n$ or $r(a, b)\theta n$, where C is an $\mathcal{ALC}$ concept, $r \in N_R$, $a, b \in N_I$, $\theta \in \{\geq, \leq, >, <\}$ and $n \in [0, 1]$. Following Bobillo and Straccia [21], we assume that fuzzy interpretations are *witnessed*, i.e., the sup and inf are attained at some point of the involved domain. The interpretation function $\cdot^I$ is also extended to axioms as follows:

$$(C \sqsubseteq D)^I = inf_{x\in\Delta^I} C^I(x) \rhd D^I(x) \qquad (C(a))^I = C^I(a^I)$$

This allows defining the satisfiability of fuzzy concept inclusions, $I \models C \sqsubseteq D\ \theta\alpha$ if $(C \sqsubseteq D)^I\ \theta\alpha$; while, for fuzzy assertions, $I \models C(a)\ \theta\alpha$ if $C^I(a^I)\ \theta\alpha$, and $I \models r(a, b)\ \theta\ n$ if $r^I(a^I, b^I)\theta\ n$. If $I \models \Gamma$, we say that I *satisfies* Γ or that I is a *model* of Γ (for Γ being an axiom, a set of axioms, or a KB), meaning that I satisfies all the axioms in Γ.

For the finitely many-valued case, we assume the *truth space* $\mathcal{S}$ to be equipped with a preorder relation $\leq^{\mathcal{S}}$, a bottom element $0^{\mathcal{S}}$, and a top element $1^{\mathcal{S}}$. We denote by $<^{\mathcal{S}}$ and $\sim^{\mathcal{S}}$ the related strict preference relation and equivalence relation. In the following we assume $\mathcal{S}$ to be the unit interval $[0,1]$ or the finite set $\mathcal{C}_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\}$ for an integer $n \geq 1$ [16, 17], and that $\otimes$, $\oplus$, $\rhd$ and $\ominus$ are a t-norm, an s-norm, an implication function, and a negation function in some well known system of many-valued logic. In particular, in the following we restrict to *continuous* t-norms.

3. A many-valued semantics for $LTL_{\mathcal{ALC}}$

Temporal extensions of DLs, their complexity and decidability are very well-studied in the literature (see, e.g., [7, 8]). The temporal Description Logic $LTL_{\mathcal{ALC}}$ extends $\mathcal{ALC}$ with LTL operators $\bigcirc$ (next), $\mathcal{U}$ (until), $\Diamond$ (eventually) and $\Box$ (always); the set of *temporally extended concepts* is the following:

$$C ::= A \mid \top \mid \bot \mid C \sqcap D \mid C \sqcup D \mid \neg C \mid \exists r.C \mid \forall r.C \mid \bigcirc C \mid C \mathcal{U} D \mid \Diamond C \mid \Box C$$

where $A \in N_C$, and C and D are temporally extended concepts.

While we refer to [7] for the two-valued semantics of $LTL_{\mathcal{ALC}}$, we develop a many-valued semantics for $LTL_{\mathcal{ALC}}$, by interpreting, at each time point, all concepts and role names over a *truth degree set* $\mathcal{S}$.

A *many-valued temporal interpretations for* $LTL_{\mathcal{ALC}}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty domain; $\cdot^{\mathcal{I}}$ is an interpretation function that maps each concept name $A \in N_C$ to a function $A^{\mathcal{I}} : \mathbb{N} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{S}$, each role name $r \in N_R$ to a function $r^{\mathcal{I}} : \mathbb{N} \times \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow \mathcal{S}$, and each individual name $a \in N_I$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. For simplicity, following [7], we assume individual names to be *rigid*, i.e., having the same interpretation at any time point n. Given a time point $n \in \mathbb{N}$ and a domain element $d \in \Delta^{\mathcal{I}}$, the interpretation $A^{\mathcal{I}}$ of a concept name A assigns to the pair (n, d) a value $A^{\mathcal{I}}(n, d) \in \mathcal{S}$ representing the *degree of membership of* d *in concept* A *at time point* n; and similarly for roles. By adapting the formulation of the semantics of temporal operators from [22], the interpretation function $\cdot^{I}$ is extended to complex concepts as follows:

$$\bot^{\mathcal{I}}(n,x) = 0 \qquad \top^{\mathcal{I}}(n,x) = 1 \qquad (\neg C)^{\mathcal{I}}(n,x) = \ominus C^{\mathcal{I}}(n,x)$$
$$(C \sqcap D)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n,x) \otimes D^{\mathcal{I}}(n,x) \qquad (C \sqcup D)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n,x) \oplus D^{\mathcal{I}}(n,x)$$
$$(\exists r.C)^{\mathcal{I}}(n,x) = sup_{y \in \Delta}\ r^{\mathcal{I}}(n,x,y) \otimes C^{\mathcal{I}}(n,y) \qquad (\bigcirc C)^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n+1,x)$$
$$(\forall r.C)^{\mathcal{I}}(n,x) = inf_{y \in \Delta}\ r^{\mathcal{I}}(n,x,y) \rhd C^{\mathcal{I}}(n,y) \qquad (\Diamond C)^{\mathcal{I}}(n,x) = \bigoplus_{m \geq n} C^{\mathcal{I}}(m,x)$$
$$(C\mathcal{U}D)^{\mathcal{I}}(n,x) = \bigoplus_{m \geq n} (D^{\mathcal{I}}(m,x) \otimes \bigotimes_{k=n}^{m-1} C^{\mathcal{I}}(k,x)) \qquad (\Box C)^{\mathcal{I}}(n,x) = \bigotimes_{m \geq n} C^{\mathcal{I}}(m,x)$$

The semantics of $\Diamond$, $\Box$ and $\mathcal{U}$ requires a passage to the limit. Following [22], bounded versions for $\Diamond$, $\Box$ and $\mathcal{U}$ can be introduced, using additional temporal operators $\Diamond_t$ (eventually in the next t time points), $\Box_t$ (always within t time points) and $\mathcal{U}_t$, with the interpretation:

$$(\Diamond_t C)^{\mathcal{I}}(n,x) = \bigoplus_{m=n}^{n+t} C^{\mathcal{I}}(m,x) \qquad (\Box_t C)^{\mathcal{I}}(n,x) = \bigotimes_{m=n}^{n+t} C^{\mathcal{I}}(m,x)$$
$$(C\mathcal{U}_t D)^{\mathcal{I}}(n,x) = \bigoplus_{m=n}^{n+t} (D^{\mathcal{I}}(m,x) \otimes \bigotimes_{k=n}^{m-1} C^{\mathcal{I}}(k,x))$$

so that $(\Diamond C)^{\mathcal{I}}(n,x) = lim_{t \rightarrow +\infty} (\Diamond_t C)^{\mathcal{I}}(n,x)$ and $(\Box C)^{\mathcal{I}}(n,x) = lim_{t \rightarrow +\infty} (\Box_t C)^{\mathcal{I}}(n,x)$ and $(C\mathcal{U}D)^{\mathcal{I}}(n,x) = lim_{t \rightarrow +\infty} (C\mathcal{U}_t D)^{\mathcal{I}}(n,x)$. The existence of the limits is ensured by the fact that $(\Diamond_t C)^{\mathcal{I}}(n,x)$ and $(C\mathcal{U}_t D)^{\mathcal{I}}(n,x)$ are increasing in t, while $(\Box_t C)^{\mathcal{I}}(n,x)$ is decreasing in t.

Here, we have not considered the additional temporal operators ("soon", "almost always", etc.) introduced by Frigeri et al. [22] for representing vagueness in the temporal dimension. As a consequence, for the case $\mathcal{S} = [0,1]$, the semantics above is an extension to $\mathcal{ALC}$ of the FLTL (Fuzzy Linear-time Temporal Logic) semantics by Lamine and Kabanza [23] and, for all concepts C and D, and time points n, the following properties hold:

$$(\Diamond C)^{\mathcal{I}}(n,x) = C^{I}(n,x) \oplus (\Diamond C)^{\mathcal{I}}(n+1,x) \qquad (\Box C)^{\mathcal{I}}(n,x) = C^{I}(n,x) \otimes (\Box C)^{\mathcal{I}}(n+1,x)$$
$$(C\mathcal{U}D)^{\mathcal{I}}(n,x) = D^{I}(n,x) \oplus (C^{I}(n,x) \otimes (C\mathcal{U}D)^{\mathcal{I}}(n+1,x))$$

Although we have considered a constant domain $\Delta^{\mathcal{I}}$, for a many-valued preferential temporal interpretation $\mathcal{I}$, expanding domains could be considered, as for $LTL_{\mathcal{ALC}}$ in the two-valued case [7].

For simplicity, we consider knowledge bases with non-temporal TBox and ABox, where a non-temporal TBox $\mathcal{T}$ is a set of concept inclusions $C \sqsubseteq D$, where C, D are temporally extended concepts, and no temporal operator is applied in front of concept inclusions themselves. The notions of satisfiability and model of a knowledge base can be easily generalized to a many-valued $LTL_{\mathcal{ALC}}$ knowledge base with non-temporal ABox and TBox. The assertions in a non-temporal ABox $\mathcal{A}$ are evaluated at time point 0. Concept inclusions in the non-temporal TBox $\mathcal{T}$ are evaluated by considering all time points n.

Given a many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, the interpretation function $\cdot^{I}$ is extended to inclusion axioms as follows:

$$(C \sqsubseteq D)^{I} = inf_{x \in \Delta^{I}, n \in \mathbb{N}}(C^{I}(n,x) \rhd D^{I}(n,x))$$

Let K be an $LTL_{\mathcal{ALC}}$ knowledge base $K = (\mathcal{T}, \mathcal{A})$ with non-temporal ABox and TBox. Given a many-valued temporal interpretation for $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, satisfiability of an axiom in $\mathcal{I}$ is defined as:

- $\mathcal{I} \models C \sqsubseteq D \; \theta\alpha$ if $(C \sqsubseteq D)^{\mathcal{I}} \; \theta\alpha$;
- $\mathcal{I} \models C(a) \; \theta\alpha$ if $C^{\mathcal{I}}(0, a^{\mathcal{I}}) \; \theta\alpha$;
- $\mathcal{I} \models r(a,b) \; \theta \; \alpha$ if $r^{\mathcal{I}}(0, a^{\mathcal{I}}, b^{\mathcal{I}})\theta \; \alpha$.

The interpretation $\mathcal{I}$ is a *model* of $K = (\mathcal{T}, \mathcal{A})$ if $\mathcal{I}$ satisfies all concept inclusions in $\mathcal{T}$ and all assertions in $\mathcal{A}$. A knowledge base $K = (\mathcal{T}, \mathcal{A})$ is *satisfiable* in the many-valued extension of $LTL_{\mathcal{ALC}}$ if a many-valued temporal model $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of K exists.

4. A many-valued $LTL_{\mathcal{ALC}}$ with Typicality

As in the two-valued case [11], the language of a many-valued $LTL_{\mathcal{ALC}}$ can be extended with *typicality concepts* of the form $\mathbf{T}(C)$ representing the set of typical instances of concept C. The typicality operator $\mathbf{T}$ may occur both in concepts of TBox and ABox, but it cannot be nested. *Extended concepts* can be built by combining the concept constructors in $LTL_{\mathcal{ALC}}$ with the typicality operator, by allowing $\mathbf{T}(C)$ as a concept. They can freely occur in concept inclusions as in:

$$\mathbf{T}(\mathit{Professor}) \sqsubseteq (\exists \mathit{teaches}.\mathit{Course}) \mathcal{U} \mathit{Retired}$$
$$\exists \mathit{lives\_in}.\mathit{Town} \sqcap \mathit{Young} \sqsubseteq \mathbf{T}(\Diamond \mathit{Granted\_Loan})$$

Inclusions of the form $\mathbf{T}(C) \sqsubseteq D$ correspond to conditionals $C \mathrel{|\!\sim} D$ in KLM preferential logics [2, 3]. While the semantics in [11] was two-valued, in this example, the interpretation of some concepts, e.g., Young and $\mathit{Granted\_Loan}$, may have a non-crisp value in $[0,1]$. Indeed, being young is a fuzzy concept and in place of $\mathit{Granted\_Loan}$ we could have $\mathit{Gets\_Positive\_Loan\_Evaluation}$, the non-binarized outcome of some classifier.

Given a temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ over a truth degree set $\mathcal{S}$, a preference relation $\prec_C^n$ on $\Delta^{\mathcal{I}}$ is induced by the many valued interpretation of C in $\mathcal{I}$, at time point n, as follows: for all $x, y \in \Delta^{\mathcal{I}}$,

$$x \prec_C^n y \text{ if and only if } C^{\mathcal{I}}(n,y) <^{\mathcal{S}} C^{\mathcal{I}}(n,x),$$

where $x \prec_C^n y$ means that x *is preferred to* y *with respect to* C *at time point* n.

The many-valued temporal semantics introduced in the previous section easily extends to the language with typicality (see below). We regard typical C-elements (at time point n) as the domain elements x which are preferred with respect to $\prec_C^n$ among all domain elements (and such that $C^{\mathcal{I}}(x) \neq 0^{\mathcal{S}}$). Note that this semantics is inherently multi-preferential. The interpretation of typicality concepts $\mathbf{T}(C)$ can be defined as follows:

Definition 1. *Given an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, for all $n \in \mathbb{N}$, $x \in \Delta^{\mathcal{I}}$, $(\mathbf{T}(C))^{\mathcal{I}}(n,x) = C^{\mathcal{I}}(n,x)$, if there is no $y \in \Delta^{\mathcal{I}}$ such that $y \prec_C^n x$; $(\mathbf{T}(C))^{\mathcal{I}}(n,x) = 0^{S}$, otherwise.*

When $(\mathbf{T}(C))^{\mathcal{I}}(x) > 0^{\mathcal{S}}$, x is said to be a *typical* C*-element* in $\mathcal{I}$. Note that, when $\leq^{\mathcal{S}}$ is a total preorder (as it is in the cases $\mathcal{S} = [0,1]$ and $\mathcal{S} = \mathcal{C}_n$), relation $\prec^n_C$ is an irreflexive, transitive and modular relation over $\Delta^{\mathcal{I}}$, like ranked preference relations in KLM-style rational interpretations by Lehmann and Magidor [3]. For finitely-many truth values, $\prec^n_C$ is also *well-founded*.

For $LTL_{\mathcal{ALC}}$ with typicality, the notion of satisfiability of an axiom in a multi-preferential temporal interpretation $\mathcal{I}$ and the notion of model of a KB, are as in Section 3.

In the following, we denote with $LTL_{\mathcal{ALC}}{}^n\mathbf{T}$ the many-valued extension of $LTL_{\mathcal{ALC}}$ with typicality with truth degree set $\mathcal{S} = \mathcal{C}_n$, for $n \geq 1$, and with $LTL_{\mathcal{ALC}}{}^{\mathbf{F}}\mathbf{T}$ the fuzzy extension of $LTL_{\mathcal{ALC}}$ with typicality (where $\mathcal{S} = [0,1]$).

4.1. Weighted temporal knowledge bases

Besides a set of *strict* concept inclusions in the TBox, weighted KBs also allow a set of *typicality inclusions* (or *defeasible inclusions*), each one with a weight. *Weighted typicality inclusions* for a concept C_i have the form $(\mathbf{T}(C_i) \sqsubseteq D_j, w_{ij})$, and describe the *prototypical properties of* C_i*-elements* (where D_j is a concept, and the weight w_{ij} is a real number). The concepts C_i for which weighted typicality inclusions are provided are called *distinguished concepts*.

A *weighted temporal knowledge base* is a tuple $\langle \mathcal{T}, \mathcal{D}, \mathcal{A} \rangle$, where the (strict) TBox $\mathcal{T}$ is a set of strict inclusions, the *defeasible TBox* $\mathcal{D}$ is a set of weighted typicality inclusions, and $\mathcal{A}$ is a set of assertions.

Consider the weighted $LTL_{\mathcal{ALC}}{}^n\mathbf{T}$ knowledge base $K = \langle \mathcal{T}, \mathcal{D}, \mathcal{A} \rangle$, over the set of distinguished concepts $\{\mathit{Student}, \mathit{Employee}, \mathit{Person}, \ldots\}$, with $\mathcal{T}$ containing, for instance, the inclusion $\mathit{Student} \sqsubseteq \mathit{Person} \geq \mathit{1}$. and $\mathcal{D}$ containing the following weighted typicality inclusions, describing the *prototypical properties* of concept *Student*:

$(\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Has\_Classes}$, +50), $(\mathbf{T}(\mathit{Student}) \sqsubseteq \mathit{Active}$,+35) ,
$(\mathbf{T}(\mathit{Student}) \sqsubseteq \exists \mathit{has\_Boss}.\top$, -70),

That is, a student normally has classes and is active, but she usually does not have a boss (negative weight). Accordingly, a student having classes, but not a boss, is more typical than an active student having classes and a boss. In the two valued case, one can evaluate how typical are two domain individuals mary and tom as students, by considering their weight with respect to concept $\mathit{Student}$, i.e., by summing the (positive or negative) weights of the defeasible inclusions satisfied by mary and tom, and comparing them. The higher the weight, the more typical is the individual. In the many-value case, in defining the weight of a domain element x with respect to a distinguished concept C_i, we have to consider that, in an interpretation $\mathcal{I}$, at time point n, element x may belong to other concepts to some degree (e.g., at time point n, mary may be active with degree 0.8, i.e., $\mathit{Active}^{\mathcal{I}}(n, \mathit{mary}) = 0.8$).

The many-valued temporal interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ with $\mathcal{S} = [0,1]$ or a subset of it, the *weight of* $x \in \Delta^{\mathcal{I}}$ *in* $\mathcal{I}$ *with respect to a distinguished concept* C_i *at time point* n is given by

$$W^{\mathcal{I}}_{i,n}(x) = \textstyle\sum_{(\mathbf{T}(C_i) \sqsubseteq D_j, w_{ij}) \in \mathcal{D}} \; w_{ij} \; D^{\mathcal{I}}_j(n, x).$$

Intuitively, the higher the value of $W^{\mathcal{I}}_{i,n}(x)$, the more typical is x as an instance of C_i), at time point n (considering the defeasible properties of C_i). Here, the membership degree $D^{\mathcal{I}}_j(n, x)$ of x in each concept D_j at time point n is considered.

For $LTL_{\mathcal{ALC}}{}^n\mathbf{T}$ and $LTL_{\mathcal{ALC}}{}^{\mathbf{F}}\mathbf{T}$, the notions of faithful, coherent and φ-coherent semantics introduced for many-valued weighted KBs in [5, 6, 19] can be smoothly extended to the temporal case. Generalizing from the non-temporal case, we expect the membership degree of a domain element x in a concept C_i at a time point n to be in agreement with the weight of x with respect to concept C_i, at the same time point n. Different *agreement conditions at different time points* n can also be considered (see [12]); one is φ*-coherence at* n, imposing that for all $x \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}_i(n, x) = \varphi_i(W^{\mathcal{I}}_{i,n}(x)))$.

A many-valued temporal interpretation $\mathcal{I}$ can be regarded as a sequence $J^0, J^1, J^2, \ldots$ of many-valued preferential interpretations (as those considered in [19]), for each time point. Different notions of *agreement* at different time points can then be combined to give rise to different semantics of a temporal weighted KB, and different notions of entailment (based on different closure constructions). In particular, a notion of *transient φ-coherence at* n (i.e., for all $x \in \Delta^\mathcal{I}$, $C_i^\mathcal{I}(n+1,x) = \varphi_i(W_{i,n}^\mathcal{I}(x)))$ is introduced in [12] to provide a logical characterization of the transient behavior of a recurrent multilayer network.

4.2. Temporal weighted KBs and the transient behaviour of a neural network

In [19] it has been shown that many-valued weighted KBs with typicality can provide a logical interpretation to some neural network model. Specifically, the φ-coherent semantics allows to capture the stationary states of multilayer networks as well as of networks with cyclic dependencies. In this subsection, we are interested in the transient behavior of a network.

Let us consider a trained network $\mathcal{N}$. We do not put restrictions on the topology the network. Following the approach in [19], $\mathcal{N}$ can be mapped into a (non-temporal) weighted conditional knowledge base $K^\mathcal{N}$ [5, 19], by regarding the units in the network as concept names and the synaptic connections between units as weighted inclusions. If C_k is the concept name associated to unit k and $C_{j_1}, \ldots, C_{j_m}$ are the concept names associated to units $j_1, \ldots, j_m$, whose output signals are the input signals for unit k, with synaptic weights $w_{k,j_1}, \ldots, w_{k,j_m}$, then unit k can be associated a set $\mathcal{T}_{C_k}$ of weighted typicality inclusions: $\mathbf{T}(C_k) \sqsubseteq C_{j_1}$ with $w_{k,j_1}, \ldots$, $\mathbf{T}(C_k) \sqsubseteq C_{j_m}$ with w_{k,j_m}.

It has been proven that the input-output behavior of a multilayer network $\mathcal{N}$ can be captured by a preferential interpretation $I_\mathcal{N}^\Delta$ built over a set of input stimuli Δ (e.g., the test set), through a simple construction, which exploits the activity level of units for the input stimuli.

This approach allows for the verification of conditional properties of the network (of the form $\mathbf{T}(C) \sqsubset D \geq \theta$) by *model checking* over the preferential interpretation $I_\mathcal{N}^\Delta$, or by using *entailment* from the conditional knowledge base $K^\mathcal{N}$ (e.g., in an ASP encoding for finitely-valued semantics [18]). Both the model checking and entailment approach have been used in the verification of properties of feedforward neural networks for the recognition of basic emotions [24, 19].

When we consider a temporal preferential model $\mathcal{I}$ of the weighted knowledge base $K^\mathcal{N}$, we can represent different states of the network at different time points. When $\mathcal{I}$ is φ-coherent at time point n, the coherence condition above imposes that the (non-temporal) interpretation J^n at time point n represents a stationary state of network $\mathcal{N}$. In such a case, φ_i plays the role of the activation function, and the sum $\sum_h w_{ih}\ D_h^\mathcal{I}(n,x)$ plays the role of the induced local field.

The temporal formalism also allows to capture the dynamic behavior of the network beyond stationary states. When the network $\mathcal{N}$ is recurrent, the knowledge base $K^\mathcal{N}$ contains cyclic dependencies in DBox. By imposing the condition that $\mathcal{I}$ is a *transient φ-coherent interpretation at all time points* n, one can enforce that the interpretations $J^0, J^1, J^2, \ldots$ at successive time points describe the dynamic evolution of the activity of units in the network (where the activity of each unit at time point $n+1$ depends on the activity of incoming units at time point n). The temporal formalism provides a semantics for capturing the trajectories of the network state, as well as time delayed feedback connections.

5. Conclusions

In this paper, we develop a many-valued, temporal description logic with typicality, extending $LTL_{\mathcal{ALC}}$ to deal with defeasible reasoning. Our extension of $LTL_{\mathcal{ALC}}$ builds, on the one hand, on fuzzy and many-valued DLs, and, on the other hand, on preferential DLs with typicality. We have first developed a many-valued semantics for $LTL_{\mathcal{ALC}}$, and then added to the language a typicality operator, based on a (multi-) preferential semantics. Finally, we have defined an extension of weighted knowledge bases with typicality to the temporal many-valued case, for representing prototypical properties of different concepts in the temporal case.

On a different route, preferential extensions of LTL with defeasible temporal operators have been recently studied by Chafik et al. [9, 10] to enrich temporal formalisms with non-monotonic reasoning features, by considering defeasible versions of the LTL operators.

Much work has been recently devoted to the combination of neural networks and symbolic reasoning [25, 26, 27]. While conditional weighted KBs have been shown to capture (in the many-valued case) the stationary states of a neural network (or its finite approximation) [5, 19], and allow for combining empirical knowledge with elicited knowledge for reasoning and for post-hoc verification, adding a temporal dimension opens to the possibility of verifying properties concerning the dynamic behaviour of the network, based on a model checking approach or an entailment based approach.

An interesting direction for future work, is an extension to the temporal case of the model-checking approach developed in Datalog [24, 19] for the verification of conditional properties of a network, for post-hoc verification.

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