Temporal (Non-)Paradox:
Yablo's Sequences in LTL over Finite Traces

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Abstract
In this paper we investigate the relation of temporal logics and properties of Yablo sentences. We provide an (first in the literature) analysis of Yablo sentences (as formalized in temporal logic vocabulary) in LTL_f, i.e., the Linear Temporal Logic over finite traces.

Keywords
LTL on finite traces, Yablo's Paradox

1. Introduction

Yablo's Paradox is a semantic phenomenon discovered by Stephen Yablo in 1993. In his paper [2] the author provided a by now famous example of a semantic paradox which, according to the author, does not involve self-reference. Recall the paradox arises when one considers the following sequence of sentences:

$$
\begin{array}{ll}
Y_0 & \text{For any } k > 0,\ Y_k \text{ is false.} \\
Y_1 & \text{For any } k > 1,\ Y_k \text{ is false.} \\
Y_2 & \text{For any } k > 2,\ Y_k \text{ is false.} \\
 & \vdots \\
Y_n & \text{For any } k > n,\ Y_k \text{ is false.} \\
 & \vdots
\end{array}
$$

Take any sentence Y_n from the sequence and ask what would happen if it was true. Suppose it is. Then, things are as it says, and for any $j > n$ Y_j is false. In particular Y_{n+1} is false and also for any $j > n + 1$ Y_j is false.

But the second conjunct is exactly what Y_{n+1} states, so it turns that Y_{n+1} is true after all. The assumption that Y_n is true led therefore to a contradiction. So it is false. This means that not all sentences following Y_n are false, and so one of them, say Y_k, is true. But then, we can again obtain a contradiction by repeating for Y_k the same reasoning that we have just given for Y_n. So, whether Y_n is true, or false, a contradiction follows. Hence the paradox.

Yablo's Paradox has already been demonstrated relevant to the field of multiagent systems. In particular, it has been investigated from the perspective of epistemic game theory, starting from the assumption that other people's beliefs about our beliefs shape our decision-making strategies. Epistemic game theory formalizes how players think about each other's beliefs, examining their reasoning processes before making final decisions in a game. A paper [3] introduced a non-self-referential paradox, termed the 'Yablo-like Brandenburger-Keisler paradox', within epistemic game theory. This paradox demonstrated that it is impossible to comprehensively model players' epistemic beliefs and assumptions. Additionally, the authors proposed an interactive temporal belief and assumption logic to appropriately formalize this paradox, transforming it into a theorem within this logic framework.

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Specifically, the so-called Assumption Logic is a multi-modal logic designed to formalize the beliefs and assumptions of agents within a multi-agent system. Initially introduced by Bonanno for belief revision, it was later simplified by Brandenburger and Keisler to articulate their paradox in epistemic game theory. To reframe their two-player impossibility result within a modal logic framework, Brandenburger and Keisler developed an interactive version of Assumption Logic, featuring two operators. Temporal Assumption Logic extends this framework, allowing for the study of how agents' beliefs change over time.

Ensuring the reliability of both software and hardware systems is a complex task, particularly when these systems are distributed. Over the past fifty years, researchers have proposed various solutions to this challenge. One notable success is the application of formal methods techniques [4]. These methods allow for the verification of a system's correctness by formally assessing whether a mathematical model of the system meets a formal representation of its intended behavior. Linear Time Temporal Logic (LTL) [5] is a formalism used to describe sequences of events in a linear, chronological order. It allows for the specification and verification of temporal properties in systems, such as safety and liveness conditions and it is extensively used as a specification language in formal methods. LTL over finite traces (LTL_f) [6] adapts Linear Time Temporal Logic to handle finite sequences of events, making it ideal for systems that eventually stop. Unlike standard LTL, which assumes events continue indefinitely, LTL_f focuses on properties within a limited timeframe. This is especially useful for scenarios like workflows or test cases where the process has a clear endpoint. This is particularly relevant in areas such as Planning and Business Process Management. A key computational advantage of finite-trace interpretation is the ability to use standard finite-state automata for modeling and reasoning, instead of the more complex omega-automata needed for infinite traces. In this paper, we give a formalization of Yablo's Paradox within the framework of LTL_f.

1.1. Related Work

Yablo's paradox is one of major phenomena in formal semantics and has been investigated from many logical, mathematical, computer-scientific and philosophical points of view. A body of works that is particuarly interesting and relevant from the point of view of this paper, concerns topics such as the metalogical properties of Yablo sequences, as formalized in arithmetic and axiomatic theories of truth, metalogical properties of Yablo sequences when interpreted over potentially infinite domains with the semantics formalizing the notion of truth in the limit, and last, but not least, properties of Yablo sentences, as formalized in the temporal logic LTL. S. Salehi and A. Karimi in [7] have initiated the study of Yablo's sentences in LTL, obtaining some preliminary results on the semantics of Yablo's sentences in LTL. The study was further extended by A. Karimi in [8], where syntactic proofs using an appropriate axiomatization of LTL were given for Yablo's paradox in its various variants.

A fruitful study of Yablo's paradox formalized over arithmetic performed e.g. in [9] and in [10] has revealed that the reasoning has the following interesting feature. If we formalize the Yablo sentences over arithmetic, then in order to derive the contradiction, one needs to use a strong assumption concerning the notion of truth: namely one has to assume "for all n, Y_n if and only if $\ulcorner Y_n \urcorner$ is true." $\forall n\ (Y_n \equiv Tr(Y_n))$. If we wanted to replace this *uniform disquotation* with an infinity of *local disquotation* instances, contradiction could be obtained only if we used some infinitary inference rule (requiring an infinite number of premises) such as the ω-rule.

So far, the story is rather well-known. What is somewhat less known, is that there is a way of handling the paradox which relies on finitistic assumptions. After all, if the world is finite, there aren't enough things in the world to interpret all sentences from the Yablo sequence, and the last interpreted one is vacuously true without any threat of paradox. Yablo's paradox can be thought of as an infinitary version of the Liar paradox, so perhaps thinking it can be dealt with by tackling the notion of infinity isn't extremely implausible.

1.2. Our contributions

In this paper we fill a certain gap in the literature concerning the relation of temporal logics and properties of Yablo sentences and for the first time in the literature, we perform the analysis of Yablo sentences (as formalized in temporal logic vocabulary) in LTL_f, i.e., the Linear Temporal Logic over finite traces.

2. Yablo's Paradox in LTL

In this section, we give a formalization of Yablo's Paradox within the framework of LTL. Before doing so let us recall the syntax and semantics of LTL.

Definition 1 (Syntax of LTL). *Let V be a set of propositional variables. The alphabet of a basic propositional language $\mathscr{L}_{LTL}$ is given by:*

- *the set V,*
- *logical connectives: $\rightarrow$, and $\neg$,*
- *the bracket symbols: $($, and $)$,*
- *logical operators: $\circ$, and $\mathscr{U}$.*

We use the following recursive definition of the set of formulas of $\mathscr{L}_{LTL}$:

1. *every propositional variable $p \in V$ is a formula,*
2. *if φ is a formula, then $\neg\varphi$, $\circ\varphi$ are formulas*
3. *if φ, and ψ are formulas, then $(\varphi \rightarrow \psi)$ and $\varphi \mathscr{U} \psi$ are formulas,*

and nothing else is a formula.

Semantical interpretations in classical propositional logic are given by boolean valuations. For LTL we have to extend this concept according to our informal idea that formulas are evaluated over sequences of states ('time scales').

Definition 2 (Semantics of LTL). *A temporal (Kripke) structure for V is an infinite sequence $K = (\eta_0, \eta_1, \eta_2, \ldots)$ of mappings $\eta_i : V \rightarrow \{0, 1\}$ called states. The mapping η_0 is called initial state of K. Observe that states are just valuations in the classical logic sense. For K and $i \in \mathbb{N}$ we define $K, i \vDash \varphi$ (in another formalism denoted by $K_i(\varphi) = 1$), informally meaning the 'truth value of φ in the ith state of K' for every formula φ inductively as follows:*

1. *$K, i \vDash v$ iff $\eta_i(v) = 1$ for each $v \in V$,*
2. *$K, i \vDash \neg\varphi$ iff $K, i \nvDash \varphi$,*
3. *$K, i \vDash \varphi \rightarrow \psi$ iff $K, i \vDash \neg\varphi$ or $K, i \vDash \psi$,*
4. *$K, i \vDash \circ\varphi$ iff $K, i + 1 \vDash \varphi$,*
5. *$K, i \vDash \varphi \mathscr{U} \psi$ iff there exists $j \geq i$ s.t. $K, j \vDash \psi$ and for each $i \leq l < j$ it holds that $K, l \vDash \varphi$*

The Boolean connectives $\vee$, $\wedge$ and $\top$ (and their semantics) can be defined as usual. We write $\Diamond\varphi$ as a shortcut for $\top \mathscr{U} \varphi$ and $\Box\varphi$ as a shortcut for $\neg\Diamond\neg\varphi$.

Definition 3 (Validities of LTL). *A formula φ of $\mathscr{L}_{LTL}$ is called **valid** in the temporal structure K for V (or K satisfies φ), denoted by $K \vDash \varphi$, if $K, i \vDash \varphi$ for every $i \in \mathbb{N}$. A formula φ is called a **consequence** of a set Δ of formulas $\Delta \vDash \varphi$ if $K \vDash \varphi$ holds for every K such that $K \vDash \psi$ for all $\psi \in \Delta$. A formula φ is called (universally) **valid** $\vDash \varphi$ if $\emptyset \vDash \varphi$. Then we also say that the formula φ is a law of LTL or that LTL entails it, denoted as $LTL \vDash \varphi$. A formula φ is called (locally) satisfiable if there is a temporal structure K and $i \in \mathbb{N}$ such that $K, i \vDash \varphi$.*

It can be easily seen that the following are examples of formulas that are universally valid in LTL:

$$\circ\Box\varphi \leftrightarrow \Box \circ \varphi,$$

and

$$\neg \circ \varphi \leftrightarrow \circ\neg\varphi.$$

The latter, called the duality law for *Next* operator of LTL, will be of particular importance when analyzing the main differences between behavior of Yablo sentences in LTL and LTL_f.

It immediately follow from the equivalences below that the following equivalences hold universally as well:

$$\circ\Box\neg\varphi \leftrightarrow \Box \circ \neg\varphi \leftrightarrow \Box\neg \circ \varphi.$$

The Yablo formula in LTL is defined as the equivalence:

$$\varphi \leftrightarrow \circ\Box\neg\varphi.$$

The Yablo Paradox means that the following theorem holds:

Theorem 1 (A. Karimi, S. Salehi [7]). *The following is a theorem of LTL:*

$$\neg\Box\,(\varphi \leftrightarrow \circ\Box\neg\varphi.)$$

The proof goes by an argument demonstrating that the formula

$$\Box\,(\varphi \leftrightarrow \circ\Box\neg\varphi.)$$

is unsatisfiable in LTL. We will show below that this is in contrast with what happens in LTL_f.

As Yablo's paradox comes in several varieties, in the paper [7] by S. Salehi and A. Karimi it has also been demosntrated that other variants of Yablo formulas are paradoxical in the same sense as obve, when formalized in the framework of temporal logic rather than arithmetic.

The original version of Yablo's sequence consists of the so-called **Always-Y-sentences**: $Y_n \leftrightarrow \forall k > n\, Y_k$ is not true, which when formalized in $\mathscr{L}_{LTL}$ gives the formula mentioned above, i.e., $\varphi \leftrightarrow \circ\Box\neg\varphi$.

Other natural variants considered in the literature are:

1. **Sometimes-Y-sentences**: $Y_n \leftrightarrow \exists k > n\, Y_k$ is not true, which has the following form when formalized in $\mathscr{L}_{LTL}$:
$$\varphi \leftrightarrow \circ\Diamond\neg\varphi,$$
2. **Almost-Always-Y-sentences**: $Y_n \leftrightarrow \exists k > n \forall j > k\, Y_j$ is not true, which has the following form when formalized in $\mathscr{L}_{LTL}$:
$$\varphi \leftrightarrow \circ\Diamond\Box\neg\varphi,$$
3. **Unboundedly-Often-Y-sentences**: $Y_n \leftrightarrow \forall k > n \exists j > k\, Y_j$ is not true, which has the following form when formalized in $\mathscr{L}_{LTL}$:
$$\varphi \leftrightarrow \circ\Box\Diamond\neg\varphi.$$

A. Karimi and S. Salehi show in [7] that all the formulas above they are paradoxical in the sense that for each equivalence ψ of the above (Sometimes, Almost-Always, and Unboundedly-Often) the following holds:

$$LTL \models \neg\Box\psi,$$

meaning that the formula $\Box\psi$ is unsatisfiable.

Below, we demonstrate that this is in contrast with the situation in LTL_f.

3. LTL_f

The syntax of LTL_f is identical to the syntax of LTL. The semantics of LTL_f is given in terms of LT_f-interpretations, i.e., interpretations over finite traces denoting a finite sequence of consecutive instants of time. LT_f-interpretations are represented here as finite words π over the alphabet of $\{0,1\}^V$, i.e., as alphabet we have all the possible propositional interpretations of the propositional symbols in V.

We use the following notation. We denote the length of a trace π as $|\pi|$. We denote the positions, i.e., instants, on the trace as $\pi(i)$ with $0 \leq i \leq \max$, where $\max = |\pi| - 1$ is the last element of the trace. We denote by $\pi[i,j]$ the segment (i.e., the subword) obtained from π starting from position i and terminating in positon j, $0 \leq i \leq j \leq \max$.

Definition 4 (Semantics of LTL_f). *A temporal (Kripke) structure for V is a finite sequence* $\pi = (\eta_0, \eta_1, \eta_2, \dots, \eta_{\max})$ *of mappings* $\eta_i : V \to \{0,1\}$ *called states. The mapping* η_0 *is called initial state of* π*. Observe that states are just valuations in the classical logic sense. For* π *and* $i \in \mathbb{N}$ *we define* $\pi, i \vDash \varphi$ *(in another formalism denoted by* $\pi_i(\varphi) = 1$*), informally meaning the 'truth value of* φ *in the ith state of K' for every formula* φ *inductively as follows:*

1. $\pi, i \vDash v$ *iff* $\eta_i(v) = 1$ *for each* $v \in V$,
2. $\pi, i \vDash \neg\varphi$ *iff* $\pi, i \nvDash \varphi$,
3. $\pi, i \vDash \varphi \to \psi$ *iff* $\pi, i \vDash \neg\varphi$ *or* $\pi, i \vDash \psi$,
4. $\pi, i \vDash \circ\varphi$ *iff* $i < \max$ *and* $\pi, i+1 \vDash \varphi$,
5. $\pi, i \vDash \Diamond\varphi$ *iff there exists* $i \leq j \leq \max$ *s.t.* $\pi, j \vDash \varphi$.

Thus, by the semantics above, for each formula φ of the language $\mathscr{L}_{LTL}$ we have that for any trace π the following holds:

$$\pi, \max \vDash \neg \circ \varphi,$$

but simultaneously

$$\pi, \max \nvDash \circ \neg\varphi,$$

which proves that the law of duality of *Next* of LTL does not hold universally for LTL_f, i.e., it is not a validity of the logic.

4. Yablo's sentences in LTL_f

The Yablo sentences employ a very interesting feature in models of LTL_f, and allow us to recover the counterpart of the arithmetical result on non-paradoxicality of the Yablo sequence when considered under *sl*-semantics over *FM*-domains.

We now present a result that shall e interpreted as one stating that there is no Yablo Paradox in linear temporal logic over finite traces.

Theorem 2. *The formula*

$$\Box(\varphi \leftrightarrow \circ\Box\neg\varphi)$$

is satisfiable in LTL_f

Proof. We demonstrate a stronger result, namely that for each natural number $n \geq 2$ there exists an LT_f-interpretation (i.e., a finite trace) of size n that satisfies the Yablo formula. Recall that $\max = n - 1$.

Let $n \geq 2$ be arbitrary natural number. Consider a finite trace π with $|\pi| = n$. Let φ be formula mentionted in the Yablo equivalence, i.e., defined as equivalent to $\circ\Box\neg\varphi$.

We define the valuation of φ in subsequent states of the trace π.

For each $i = 0 \dots, n-3$ (observe that if $n = 2$, then there are no such i's, but it does not result in any problems for the valuation) put:

$$\pi, i \vDash \neg\varphi.$$

For $i = n-2 = \max -1$ put:

$$\pi, i \models \varphi.$$

Finally, for $i = \max = n-1$ define:

$$\pi, i \models \neg\varphi.$$

As it can be easily seen, it holds that

$$\pi, 0 \models \Box(\varphi \leftrightarrow \circ\Box\neg\varphi).$$

This implies that

$$LTL_f \not\models \neg\Box(\varphi \leftrightarrow \circ\Box\neg\varphi)$$

in contrast to the status of the formula in LTL. □

Observe the valuation above is the only consistent one that can be defined on any given finite trace of size at least 2.

We also obtain similar results concerning the avoidance of paradoxicality of other Yablo sentences in LTL_f.

Theorem 3. *The temporal Sometimes-Y-formula formula*

$$\Box(\varphi \leftrightarrow \circ\Diamond\neg\varphi)$$

is satisfiable in LTL_f

Proof. We demonstrate a stronger result, namely that for each natural number $n \geq 2$ there exists an LT_f- interpretation (i.e., a finite trace) of size n that satisfies the Sometimes-Y formula. Recall that $\max = n-1$.

Let $n \geq 2$ be arbitrary natural number. Consider a finite trace π with $|\pi| = n$. Let φ be formula mentionted in the Yablo equivalence, i.e., defined as equivalent to $\circ\Box\neg\varphi$.

We define the valuation of φ in subsequent states of the trace π.

For each $i = 0 \ldots, n-2$ put:

$$K, i \models \varphi.$$

Finally, for $i = \max = n-1$ define:

$$K, i \models \neg\varphi.$$

As it can be easily seen, it holds that

$$\pi, 0 \models \Box(\varphi \leftrightarrow \circ\Diamond\neg\varphi).$$

This implies that

$$LTL_f \not\models \neg\Box(\varphi \leftrightarrow \circ\Diamond\neg\varphi)$$

in contrast to the status of the formula in LTL. □

The valutaion constructed in the proof of satisfiability of the Sometimes-Y-formulas in LTL_f gives us also proofs of satisfiability of the Almost-Always-Y-formula and the Unboundedly-Often-Y-formula:

Theorem 4. *The temporal Almost-Always-Y-formula*

$$\Box(\varphi \leftrightarrow \circ\Diamond\Box\neg\varphi)$$

is satisfiable in LTL_f

Finally, we have:

Theorem 5. *The temporal Unboundedly-Often-Y-formula*

$$\Box(\varphi \leftrightarrow \circ\Box\Diamond\neg\varphi)$$

is satisfiable in LTL_f

Theorem 6. *Let φ be the Unboundedly-Often-Y-formula, i.e., $\Box(\varphi \leftrightarrow \circ\Box\Diamond\neg\varphi)$. Then: $\varphi \in sl(\Pi)$, where Π is the class of all finite traces over the set of propositional variables V.*

5. Conclusions

In this paper we have investigated the relation of temporal logics and properties of Yablo sentences, providing the analysis of Yablo sentences (as formalized in temporal logic vocabulary) in LTL_f, i.e., the Linear Temporal Logic over finite traces.

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