When Contranominal Scales Give a Solution to the Zarankiewicz Problem?

Dmitry I. Ignatov

HSE University, Moscow, Russia

Abstract

The paper formulates the Zarankiewicz problem in terms of formal contexts as follows: What is $z(m, n; s, t)$, the largest size of the incidence relation of a formal context with m objects and n attributes such that there is no a formal concept with the given extent s and t intent sizes and larger? Exact formulas for the case $n = m$, and $s + t = n + 1 + k$ with valid ranges of s, t, and k using the contranominal scales of sizes $n - k$ and maximal symmetric contexts are obtained. Moreover, symmetric versions of $z_{\lceil n/2 \rceil}(n)$ function are studied and expected ansatz-based solutions as second-degree polynomials for $z_{\lfloor n/2 \rfloor}(n)$ are disproven with Formal Concept Analysis assisted tools and concrete lower bounds obtained for $z_5(11)$, $z_6(13)$, $z_7(15)$, $z_8(17)$, and $z_9(19)$.

Keywords

Zarankiewicz problem, maximal biclique, formal concepts, contranominal scale, extremal combinatorics

1. Introduction

Recently, such IT giants as DeepMind have set their eye on trying to exploit AI-related and machine learning techniques to solve fundamental mathematical problems like the capset problem in extremal combinatorics [1] or an alternative characterisation of algebraic structures in knot and group theory via machine learning attribution methods [2]. In this paper, we deal with Formal Concept Analysis not only as a mathematical language but as an AI-involved tool suitable for the purposes of extremal combinatorics and Experimental Mathematics [3] on the example of the Zarankiewicz problem.

The Zarankiewicz problem dates back to the 1950s and asks for the maximal number of edges in a bipartite graph of fixed size free of bicliques with given sizes of its parts [4]. This is an analogue of a famous problem studied by Turan on the maximal size of a graph free of p-clique. The corresponding function $z(m, n; s, t)$ counting the number of edges in a bipartite graph with parts of sizes m and n and no biclique with sizes of components s and t respectively is called the Zarankiewicz function or number. It is the subject of ongoing research, while the problem is still open in general.

It is interesting that the original problem was published first in terms of grids in French [4]. We take its translation from [5] except the term grid not lattice to avoid confusion with the French “trellis” normally used for lattice; moreover, in the original Zarankiewicz formulation “un réseau plan formé” was used):

> “Let R_n where $n > 3$ be an $n \times n$ square grid. Find the smallest natural number $k_2(n)$ for which every subset of R_n of size $k_2(n)$ contains 4 points that are all the intersections of 2 rows and 2 columns. More generally, find the smallest natural number $k_j(n)$ for which every subset of R_n of size $k_j(n)$ contains j^2 points that are all the intersections of j rows and j columns.”

Note that $k_j(n)$ is $z(n, n; j, j) + 1$.

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dignatov@hse.ru (D. I. Ignatov)
https://www.hse.ru/en/staff/dima (D. I. Ignatov)
0000-0002-6584-8534 (D. I. Ignatov)

Such incidence structures like grids are naturally represented by binary relations, Boolean matrices, and formal contexts, while the latter serve for object-attribute incidence representation in Formal Concept Analysis.

Formal Concept Analysis (FCA) is a branch of modern lattice theory, and it studies (formal) concepts and their hierarchies [6]. The adjective "formal" indicates a strict mathematical definition of a pair of sets (of objects and attributes, respectively), called the extent and the intent, and named the formal concept as a whole. This formalisation is possible because of the use of algebraic lattice theory and Galois connections.

So, our goal here is to consider the formulation of the Zarankiewicz problem in terms of FCA and to see what this approach and the existing tools can add to the state of the art. Thus, bipartite graphs can be considered as formal contexts, and their maximal bicliques as formal concepts of the context. This is not the first note on this problem; thus, the problem was properly summarised as part of an extensive survey on mathematical aspects of Formal Concept Analysis in the middle of the 1990s [7].

Moreover, recent results on extremal lattice theory and Boolean matrix factorisation with FCA show that formal contexts called contranominal scales are of high importance. For example, the work of Albano and Chornomaz [8] answers the question "How large is the size of a concept lattice?" when contranominal scales of a certain size are not contained in the input context of a fixed size, while our previous work shows that the state-of-the-art Boolean matrix factorisation algorithms are suboptimal on contranominal scales. This is also a basic fact that a contranominal scale of size $n \times n$ has the largest possible number of formal concepts, 2^n, for the given n, while the concept extent sizes run through all $\{1, 2, \ldots, n\} = [n]$. So, we answer the question "What is the role of contranominal scales in the Zarankiewicz problem?".

In addition, we include a summary of the relevant decision problems related to those formulated in [9] on the existence of the concepts with the fixed sizes (cf. non-strictly larger or smaller than that size) of their extent, intent and perimeter.

The paper is organised as follows. Section 2 gives the basics of FCA theory. Section 3, formulates the studied problem for $z(n, m; s, t)$ in FCA terms. Section 4 presents obtained theoretical results including the fully symmetric case for the Zarankiewicz function like $z(n, n; \lfloor n/2 \rfloor)$. Section 5 briefly overviews the most relevant works. Section 6 concludes the paper.

2. FCA Basics

We mainly follow the notation from [6].

Definition 1. Formal context $\mathbb{K}$ *is a triple* (G, M, I) *where* G *is a set of* objects*,* M *is a set of* attributes*, and* $I \subseteq G \times M$ *is an incidence binary relation.*

The binary relation I is interpreted as follows: for $g \in G$, $m \in M$ we write gIm if the object g has the attribute m.

For a formal context $\mathbb{K} = (G, M, I)$ and any $A \subseteq G$ and $B \subseteq M$, a pair of mappings is defined:

$$A^{\uparrow} = \{m \in M \mid gIm \text{ for all } g \in A\}, \quad B^{\downarrow} = \{g \in G \mid gIm \text{ for all } m \in B\},$$

these mappings define the Galois connection between partially ordered sets $(2^G, \subseteq)$ and $(2^M, \subseteq)$ on disjunctive union of G and M. The set A is called *closed*, if $A^{\uparrow\downarrow} = A$ [10].

Definition 2. *A* formal concept *of the formal context* $\mathbb{K} = (G, M, I)$ *is a pair* (A, B)*, where* $A \subseteq G$*,* $B \subseteq M$*,* $A^{\uparrow} = B$ *and* $B^{\downarrow} = A$*. The set* A *is called the* extent*, and* B *is the* intent *of the formal concept* (A, B)*.*

It is evident that the extent and intent of any formal concept are closed sets.

The set of all formal concepts of a context $\mathbb{K}$ is denoted by $\mathfrak{B}(G, M, I)$. This set forms an algebraic lattice called concept lattice where the concepts are ordered via set inclusion of their extents (dually intents).

Note that $(.)^{\uparrow}$ and $(.)^{\downarrow}$ derivation operators are usually unified by a single symbol like prime $(.)'$ or $(.)^{I}$, when formal contexts with different incidence relations, say I and J are used simultaneously.

Note also that similarly to the case of the definition of bipartite graph, the definition of formal context could include a requirement for sets G and M being disjunctive; however, since G and M are technically related with two different operators $(.)^{\uparrow}$ and $(.)^{\downarrow}$ we can safely operate with $G = \{1, \ldots, n\}$ and $M = \{1, \ldots, n\}$ in what follows without loss of generality.

For every set S the *contranominal scale* is defined as $\mathbb{N}^c_S = (S, S, \neq)$. In what follows, we consider $\mathbb{N}^c_n$ with $S = [n] = \{1, \ldots, n\}$ without loss of generality.

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The surveys on advances in FCA theory and its applications can be found in [11, 12].

3. Problem Statement

We propose the following most general formulation of the Zarankiewicz problem.

Problem 1. *What is $z(m, n; s, t)$, the largest size of the incidence relation I of a formal context $\mathbb{K} = (G, M, I)$ with $|G| = m$ and $|M| = n$, for which there is no any formal concept (A, B) with $|A| \geq s$ and $|B| \geq t$?*

We need these inequalities in the formulation, $|A| \geq s$ and $|B| \geq t$, since there might be a concept of size $(s + 1) \times t = |A||B|$ but not of $s \times t$ containing the subcontext of sizes $s \times t$ due to maximality of concepts in terms of the number of objects and attributes (cf. maximal bicliques).

In what follows, we use p,q when the extent and intent sizes of a concept (A, B) are involved with $p = |A|$ and $q = |B|$. If $m = n$, we write $z(n; s, t)$ for $z(n, n; s, t)$ and if in addition $s = t$, then we write $z(n; t)$ for $z(n; s, t)$.

For a given instance of Problem 1, we will call the formal contexts (similarly, for binary matrices or bipartite graphs [5, 13]) with the maximum number of pairs in I *admissible* if there is no any concept (A, B) with $|A| \geq s$ and $|B| \geq t$.

Here and in what follows we also assume that all the variables n, m, p, q, s, t and k are non-negative integers.

4. Results

4.1. Contranominal Scales and Maximal Symmetric Contexts

Lemma 1. *([14, 5]) Let $\mathbb{K} = (G, M, I \subseteq G \times M)$ with $G = \{g_1, \ldots, g_n\}$ and $M = \{m_1, \ldots, m_n\}$, then this context does not contain a concept with extent and intent sizes p and q or larger, respectively, if*

$$\sum_{i=1}^{n} \binom{|m_i'|}{p} \leq (q - 1)\binom{n}{p}. \tag{1}$$

Lemma 1 is the instantiation of the pigeonhole principle [15], where the pigeons are subsets of attributes' extents of size p, while the holes are subsets of objects of the same size.

Lemma 1 implies bounds on the values of z suitable for our purposes, while stronger results can be found, for example, in Roman [16]. Also, a stronger version of Lemma 1 exists for the sum over q attributes (see, [13]).

Lemma 2. *([6]) For each concept (A, B) of the contranominal scale of size n, $([n], [n], \neq)$, $|A| + |B| = n$.*

Property 1. $z(n; n + 1, q) = n^2$ *for any $q > 0$.*

Property 1 is needed if we would like to extend the values of z for the sizes of p, q(symmetrically) exceeding the size n of the input context.

Form Lemma 2 we infer Property 2.

Property 2. *If a formal context* $\mathbb{K} = (G, M, I)$ *contains as its subcontext a contranominal scale of size* k, $\mathbb{N}^c_k$, *then* $\mathbb{K}$ *should contain a formal concept* (A, B) *with* $|A| \geq p$ *and* $|B| \geq q$ *where* $p + q = k$.

In the next theorems by the solution to the Zarankiewicz problem for given n, p and q, we mean an admissible formal context with $|I| = z(n; p + 1, n) = z(n; p, q + 1)$.

Theorem 1. *A contranominal scale of size* n, $([n], [n], \neq)$, *gives a solution to the Zarankiewicz problem with* $m = n$, $p + q = n$ *for* $p, q > 0$, *and* $z(n; p + 1, q) = z(n; p, q + 1) = n(n - 1)$.

Proof. 1) Admissibility. By Lemma 1 we should have

$$\sum_{i=1}^{n} \binom{n-1}{p+1} \leq (q-1)\binom{n}{p+1}.$$

Or

$$(n - 1 - p)\binom{n}{p+1} \leq (q-1)\binom{n}{p+1}$$

(since we plug in n into the binomial coefficient $\binom{n-1}{p+1}$ and pull out $(n - 1 - p)$),

$$n - 1 - p \leq q - 1.$$

We substitute $n - p = q$ by the condition and get the identity $q - 1 \leq q - 1$.

One can also show that our context is free of any concept (A, B) with $|A| = p + 1$, $|B| = q$ and $|A| = p$, $|B| = q + 1$. By Lemma 2 $|A| + |B| = n$, which implies $p + q + 1 = n$, the contradiction.

2) Maximality. Then let us also show that the contranominal scale is the maximal context in terms of its number of incident object-attribute pairs.

Assume that we can add one more object-attribute pair, say (g_n, m_n) to the contranominal scale such that the resulting context is admissible. Then our context will contain one full row, full column and a contranominal scale of size $n - 1$ as subcontext disjoint from these full row and column. Since full rows and columns are both reducible, then the resulting context gives rise to the same number of concepts that the contranominal scale of size $n - 1$ has. For each concept (A, B) of the $\mathbb{N}^c_n$ with $|A| = p$ and $|B| = q$, either $g_n \in A$ ($m_n \notin B$) or $m_n \in B$ ($g_n \notin A$), so the concept of the new context I, (A^{II}, A^{I}) or (B^{I}, B^{II}), will have extent and intent sizes $p, q + 1$ or $p + 1, q$, respectively. Due to the context symmetry, the concepts of sizes $p, q + 1$ and $p + 1, q$ are both realised for each p, q pair. Then the new context is not admissible for given p and q, the contradiction.

Similarly, for Lemma 1, its inequality becomes false.

□
□

Can we also use contranominal scales for other types of solutions? The answer is yes. Thus one can place new object-attribute pair on the main diagonal of a contranominal scale.

Theorem 2. *A context* $\mathbb{K}$ *obtained from a contranominal scale of size* n *as* $\mathbb{K} = ([n], [n], \neq \cup (i, i))$ *for* $i \in [n]$ *gives a solution to the Zarankiewicz problem with* $m = n$, $p + q = n - 1$ *for* $p, q > 0$, *and* $z(n; p + 2, q + 1) = z(n; p + 1, q + 2) = n(n - 1) + 1$.

Proof. 1) Admissibility. By Lemma 1 the following inequality should hold

$$\binom{n}{p+2} + \sum_{i=1}^{n-1} \binom{n-1}{p+2} \leq q\binom{n}{p+2}.$$

After simplification we get the inequality $\frac{n-1}{n}(q-1) \leq q-1$.

2) Let us check maximality by adding a new pair, say (g_{n-1}, m_{n-1}). First, we should note that our context contains a contranominal scale isomorphic to $([n-1], [n-1], \neq)$. So, the sum of sizes of its concept's extent and intent is $n-1 = p+q$, but since we have an extra full row and column, each concept of the considered context will have extent and intent size $p+1$ and $q+1$. It implies that there is no any concept with sizes of its extent $p+2$ and intent $q+1$ (same for $p+2$ and $q+1$).

□

□

Can we generalise this solution up to k added pairs to the contranominal scale of size n? Again, the answer is yes.

Theorem 3. *A context $\mathbb{K}$ obtained from a contranominal scale of size n as*

$$\mathbb{K} = ([n], [n], \neq \cup \bigcup_{i \in S} (i, i))$$

for $S \in \binom{[n]}{k}$ gives a solution to the Zarankiewicz problem with $m = n$, $p+q = n-k$, $0 < k \leq n$ for 1) $p, q > 0$ (or $p = q = 0$), and

$$z(n; p+1+k, q+k) = z(n; p+k, q+1+k) = n(n-1)+k,$$

2) $q = 0, p > 0$,

$$z(n; p+1+k, q+k) = n^2$$

and

$$z(n; p+k, q+1+k) = n(n-1)+k,$$

3) $p = 0, q > 0$,

$$z(n; p+1+k, q+k) = n(n-1)+k$$

and

$$z(n; p+k, q+1+k) = n^2.$$

Proof. The proof of Admissibility is similar.

$$k\binom{n}{p+k+1} + (n-k)\binom{n-1}{p+k+1} \leq (q+k-1)\binom{n}{p+k+1}.$$

We obtain inequality $\frac{n-k}{n}(q-1) \leq q-1$, which is true for $q > 0$. For $q = 0$, the inequality is false.

The solution for 2) $q = 0$ falls into two basic cases (up to the symmetry p and q). When $p = 0, n = k$ and by Property 1 $z(n; n+1, n) = z(n; n, n+1) = n^2$, which coincides with $n(n-1)+k$. When $p > 0, p+k = n$, and

$$z(n; n+1, k) = n^2$$

(by Property 1) and

$$z(n; n, 1+k) = n(n-1)+k$$

(has to be proven).

The last subcase is admissible by noting that any new object-attribute pair on the diagonal will result in the full column and we get the concept of size $n \times (1+k)$. The contranominal scale of size $n-k$ cannot be replaced by any other subcontext within the context region of size $n-k \times n$ with the same number of incident pairs since placing at least two missing pairs in one row gives rise to a full column but placing all $n-k$ into distinct rows results in the presence of the contranominal scale of the same size. Case 3) is similar.

The maximality condition for 1) is proven similarly to Theorem 2.

□ □

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lceil n/2 \rceil$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
$z_{\lceil \frac{n}{2} \rceil}(n)$	0	0	6	9	20	26	42	51	72	84	110	125	156	174	210	231	272	296	342	369

Corollary 1. *A context* $([n], [n], = \setminus \bigcup\limits_{i \in S} (i, i))$ *with* $S \in \binom{[n]}{k}$ *has the maximal number of incident pairs being free from the concept with extent and intent sizes* n *and* $k+1$ *(symmetrically,* $k+1$ *and* n*), respectively.*

Remark 1. *Note that* $z(n; p+1+k, q+k)$ *and* $z(n; p+k, q+1+k)$ *can be recast as* $z(n; p+1+k, n-p)$ *and* $z(n; p+k, n-p-1)$.

Note that Theorems 1 and 2 look less general or less detailed than Theorem 3, though we keep these two theorems not as propositions similarly to the work of Balbuena et al. [17], which is discussed in Subsection 4.3. Summing it all, the obtained results allow us to compute all the values $z(n, n; s, t)$ where $s + t$ is greater than n (see also Subsection 4.3).

4.2. Single variable Zarankiewicz function

In earlier works, Zarankiewicz, Guy [14] and others paid special attention to the function $z_a(n) = z(n, n; a, a)$.

An interesting question would be what the obtained results can do for this case when the function is totally symmetrised. As it is mentioned in the related work section, [18] considered the so-called half-half case $z(2s, 2t; s, t)$ and one possible symmetrisation would be to deal with $z(2t, 2t; t, t)$ (even case).

For an odd n, i.e. $n = 2t - 1$ for $t > 0$, we get

$$z(2t-1; t, t) = 2(2t-1)(t-1),$$

but we cannot tackle the even case since $p + 1$ and q have different parity when $p + q = 2t$ and cannot be equal. However, by Theorem 1.3 from [17] we have

$$z(2t; t, t) = 4t^2 - 3t - 1 = 2(2t - 1/2)(t - 1).$$

We combine these two results into a single formula as follows:

$$z(n; \lceil n/2 \rceil, \lceil n/2 \rceil) = 2\Big(2t - \frac{1}{4} + (-1)^n \frac{3}{4}\Big)\Big(t - 1\Big),$$

here $t = \lceil n/2 \rceil$

Or

$$z_{\lceil \frac{n}{2} \rceil}(n) = 4\lceil n/2 \rceil^2 - \frac{9}{2}\lceil n/2 \rceil + \frac{1}{2} + (-1)^n \Big(\frac{3}{2}\lceil n/2 \rceil - \frac{3}{2}\Big).$$

This upper symmetrisation for odd $n = 2t - 1$ is possible with contranominal scales, however they do not work in the case of $z(n; \lfloor n/2 \rfloor, \lfloor n/2 \rfloor) = z(2t-1; t-1, t-1)$, since $\lfloor n/2 \rfloor = t - 1$ along with $p = t - 2$ and $q = t - 1$ violate $p + q = n$. Similarly, for $n = 2t + 1$ we deal with $z(2t + 1; t, t)$ and have $p + q = t - 1 + t = 2t - 1 \neq 2t + 1$, the violation. This case is also beyond reach for Theorem 3 (since $p + q = (t - 1 - k) + (t - k) \neq n - k$), Theorems 1.2 ($n = 2t + 1$ implies the contradiction $2t + 1 \leq 2t - 1$)) and 1.3 (valid for even cases) from [17].

We know from the literature [5] and OEIS that $z_2(5) = 12$, $z_3(7) = 33$, and $z_4(9) = 61$. This is enough to find coefficients of $z(2t + 1; t, t)$ as a quadratic polynomial $at^2 + bt + c$. The case $n = 3$ is omitted, resulting in zero pairs by the definition.

The system

$$\begin{cases} 4a + 2b + c = 12 \\ 9a + 3b + c = 33 \\ 16a + 4b + c = 61. \end{cases}$$

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lfloor n/2 \rfloor$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
$z_{\lfloor \frac{n}{2} \rfloor}(n)$	0	0	9	12	26	33	51	61	84	?	125	?	174	?	231	?	296	?	369

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\lfloor n/2 \rfloor$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10
$Q_{2t+1}(t)$		0		12		33		**63**		102		150		207		273		348	
$z_{\lfloor \frac{n}{2} \rfloor}(n)$	0	0	9	12	26	33	51	61	84	$\geq$97	125	$\geq$142	174	$\geq$ 192	231	$\geq$253	296	$\geq$320	369
$P_{2t+1}(t)$				12		33		61		**96**		138		187		243		306	

results in $P_{2t+1}(t) = \frac{7}{2}t^2 + \frac{7}{2}t - 9 = \frac{7}{2}t(t+1) - 9$ as a candidate for $z(2t+1; t, t)$.

If our ansatz[1] based on the fact that $z(n, n; p, q)$ is totally bounded by n^2 and all possible variables enter linearly or quadratically is correct, we should obtain the next value for $z_5(11)$ as 96.

In reality, the following contexts for $n = 11$ in Figure 1 and 2 were obtained ad hoc based on the usage of contranominal scales as building blocks and validated with our implementations of CbO [7] and NextClosure [6] and cross-checked with the concept generation algorithm In-Close [20], by adding extra crosses (pairs) to the context and checking the absence of concepts larger than or equal to 5×5 and 6×6 full subcontexts, respectively, in terms of extent times intent sizes.

$\mathbb{K}_{z_5(11) \geq 97}$	1	2	3	4	5	6	7	8	9	10	11
1	×	×				×	×	×	×	×	×
2		×			×	×	×	×	×	×	×
3		×	×	×	×	×	×	×		×	×
4	×		×	×	×	×	×	×	×	×	×
5	×	×		×	×	×		×	×	×	×
6	×	×	×		×		×	×	×	×	×
7	×	×	×	×		×	×	×	×	×	
8	×	×	×	×	×		×	×	×		
9	×	×	×	×	×	×		×	×		
10	×	×	×	×	×	×	×		×	×	×
11	×	×	×	×	×	×	×	×			×

Figure 1: A formal context for the obtained lower bound $z_5(11) \geq 97$

So, the knowledge base on the behaviour of $z_{2t+1}(t)$ is updated. At least, it is not that regular to be described by the same polynomial of degree 2, $P_{2t+1}(t)$, for the range $t \geq 2$.

But what if we still have doubts, especially, since $z(2t-1, t, t) = 2(2t-1)(t-1)$ and its even counterpart, $n = 2t$, have their roots at $t = 1$? We can consider another quadratic polynomial $Q_{2t+1}(t) = \frac{9}{2}t^2 - \frac{3}{2}t - 3 = \frac{9}{2}(t + \frac{2}{3})(t-1)$. Then for $n = 9$, $t = 4$ we have $Q_{2t+1}(4) = 63$ but it contradicts previous knowledge $z_9(4) = 61$ ([5], OEIS sequence for $k_n(4) = z_n(4) + 1$ is A006616[2]).

Q starts to overestimate z at t=4, while P underestimates z first at $t = 11$, the lowest upper bound for $z_4(9)$ from the best known ones is by Roman [16] (the bound by Nikiforov [21] gives higher values) is 64, while for $z_5(11)$ it is 102, and 148 for $z_6(13)$ (smaller than $Q_{2t+1}(13)$).

All the contexts and codes are available on GitHub https://github.com/dimachine/Zarankiewicz.

4.3. Reality Checks

Let us have a look at the function behaviour for some small n, for example, 5. The rows and columns of Table 1 with $s = 1$ or $t = 1$ are filled via the conditions that every object (or attribute) should have (be shared by) $t - 1$ attributes ($s - 1$ objects).

[1]an assumption about the form of an unknown function which is made in order to facilitate solution of an equation or other problem; e.g. [19]

[2]https://oeis.org/a006616

$\mathbb{K}_{z_6(13)\geq 142}$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	×	×				×	×	×	×	×	×	×	×
2		×			×	×	×	×	×	×	×	×	×
3		×	×	×	×	×	×	×	×	×		×	×
4	×		×	×	×	×	×	×	×	×	×	×	×
5	×	×		×	×	×	×	×		×	×	×	×
6	×	×	×		×	×	×		×	×	×	×	×
7	×	×	×	×		×		×	×	×	×	×	×
8	×	×	×	×	×		×	×	×	×	×	×	×
9	×	×	×	×	×	×		×	×	×	×	×	
10	×	×	×	×	×	×	×		×	×	×		
11	×	×	×	×	×	×	×	×		×	×		
12	×	×	×	×	×	×	×	×	×		×	×	×
13	×	×	×	×	×	×	×	×	×	×			×

Figure 2: A formal context for the obtained lower bound $z_6(13) \geq 142$

Table 1
$z(5,5,s,t)$; the numbers given in the literature are italic, while obtained by our formulas are below the stepwise line on the diagonal (and also in bold if present in the referenced literature, e.g., in [13])

s,t	1	2	3	4	5
1	0	5	10	15	20
2	5	*12*	*16*	***20***	21
3	10	*16*	***20***	***21***	22
4	15	***20***	***21***	***22***	23
5	20	21	22	23	24

s,t	1	2	3	4	5	6
1	0	6	12	18	24	30
2	6	*16*	*22*	*25*	30	31
3	12	*22*	*26*	***30***	31	32
4	18	*25*	***30***	***31***	32	33
5	24	30	31	32	33	34
6	30	31	32	33	34	35

Assembling Theorems 1, 2, 3 altogether, we have $p + q = n - k$ and $p + 1 + k + q + k = s + t$. For $t = s$ in the region $s + t = n + k + 1$, $0 \leq k \leq n - 1$, if we substitute k in $n(n-1) + k$, we get $z(n; t, t) = n^2 - 2n + 2t - 1$ for $n + 1 \leq 2t \leq 2n$. Actually, within that region (on and below the backward diagonal), in axes n, z when t is fixed, we deal with the parabola, while on the level $n = const$ we have the family of disjoint lines.

What if we would like to see a solution for a certain non-trivial value of z from those tables above the stepwise line? We can check a suitable context of a given size with $|I| = z(n; s, t)$.

For example, take the value $z(6; 2, 4) = z(6; 4, 2) = 25$ (see OEIS sequence A006614[3]).

$\mathbb{K}_{z(6;4,2)}$	a	b	c	d	e	f
1			×	×	×	×
2		×		×	×	×
3	×		×		×	×
4	×	×		×		×
5	×	×	×			×
6	×	×	×	×	×	

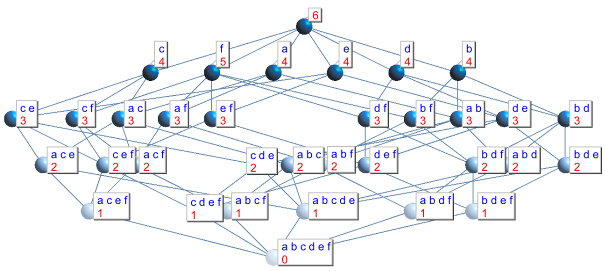

Figure 3: A formal context for $z(6; 4, 2) = 25$ and its concept lattice diagram with labeling by extent size and full intent

Thus, starting with a contranominal scale of size $5 = 4 - 1 + 2$, we have found the non-extensible (by adding new crosses) context $\mathbb{K}_{z(6;4,2)}$ shown in Figure 3. Its concept lattice diagram shows that there are no concepts of sizes 4×2 or (2×4).

Actually, if we are given s and t and an examined context, we should care that contranominal scales

[3] https://oeis.org/006614

of size $s+t$ are not contained in the given context, while scales of size $s+t-1$ can be used as building blocks, properly allocated and modified.

4.4. Complexity of Related Decision Problems

One of our peers posed the following question: What is the computational complexity of decision problems concerning the Zarankiewicz function, such as deciding whether $z(n, m; s, t)$ equals k, is at most k, or is at least k?

In general, these problems are presumably not in P since otherwise many more values of the Zarankiewicz function would be known.

We formulate these decision problems here along with an intermediate problem which should be used as a proxy routine for them.

Problem 2. *INSTANCE Given* n, m, s, t *and* k.
QUESTION Is it true that $z(n, m; s, t) = k$*?*

Problem 3. *INSTANCE Given* n, m, s, t *and* k.
QUESTION Is it true that for given n, m, s, t*,* $z(n, m; s, t) \leq k$*?*

Problem 4. *INSTANCE Given* n, m, s, t *and* k.
QUESTION Is it true that for given n, m, s, t*,* $z(n, m; s, t) \geq k$*?*

As the aforementioned intermediate problem, we consider the following one.

Problem 5. *INSTANCE Given* $\mathbb{K} = (G, M, I)$*, and* p *and* q.
QUESTION: Does there exist a formal concept $(A, B) \in \mathfrak{B}(G, M, I)$ *with* $|A| = p$ *and* $|B| = q$*?*

Note that Problem 5 when $p = q = k$ is also equivalent to the problem of the balanced biclique, which is NP-complete [22].

To answer the question of Problem 3 in a brute-force manner one needs to check every context with $|I| = k$ ($\binom{nm}{k}$ such contexts) with an algorithm solving an instance of Problem 4 with all concept sizes p, q such that $p \geq s$ and $q \geq t$ until at least one context without all these concepts is found (if the context is not found, the answer is no). Note that in general such a context trivially exists for all $st > k$ and $s \leq n, t \leq m, k \leq nm$. Then one needs to check all the contexts with $|I| = k + 1$ ($\binom{nm}{k+1}$ such contexts) for the absence of any concept of extent and intent sizes p, q, respectively, such that $p \geq s$ and $q \geq t$. Then the answer is yes.

Note that for the case when $n = m$ and $s + t \geq n$, we can use the König theorem to find the size of the formal concept with $p + q \geq n$ as in Theorem 3 from [9] in polynomial time via the size of the maximum matching for the bipartite graph B for the complement of I. By the König theorem, the number of vertices in the largest independent set is $2n - |T|$, where $2n$ is the number of vertices in the bipartite graph B given by the complement of I, and $|T|$ is the number of edges in the maximum matching of B. Since $|T| \leq n$, we cannot do that for the cases when $p + q < n$ since the maximum matching leaves us with n vertices at minimum for the size of the related maximal independent set in B and hence the concept is in $\mathfrak{B}(G, M, I)$. Or, in general, this can be done for appropriate cases when $s + t \geq mn - \min(n, m) + 1$. This explains why we can have exact formulas for the cases with $n = m$ and $s + t \geq n$.

We leave all the remaining cases with formal proofs for the extended publication.

5. Related Work

The most relevant for our studies are the works by Balbuena et al. [17] and Tan [5]. The work of Tan [5] demonstrates how to obtain not only values but also possible solutions to the first several dozen values of n for $z_a(n)$ and $a \in \{2, 3, 4\}$ with SAT solvers. In [17] more general cases for $z(m, n; s, t)$ are considered under $max\{m, n\} \leq s + t - 1$ and $z(m, n; t, t)$ if $2t \leq n \leq 3t - 1$; the exact formulas obtained. The authors used matchings to subtract them from the considered bipartite graphs and obtain the solutions and claimed formulas. Our theorems are in accordance with their results where the scopes of the theorems overlap for $m = n$. They also rely on [18], where the so-called half-half case was considered with $z(2s, 2t; s, t)$, which is not applicable for cases with odd n.

We partially reproduce Theorem 1.2 and Theorem 1.3 from [17] since we rely on them in Subsection 4.2.

Theorem 4. *(A part of Theorem 1.2 [17]) Let m, n, s, t be integers with $2 \leq s < m$, $2 \leq t < n$ and such that $\max\{m, n\} \leq s + t - 1$. Then*

$$z(m, n; s, t) = mn - (m + n - s - t + 1).$$

Theorem 5. *(A part of Theorem 1.3 [17]) Theorem 1.3. Let m, t be integers such that $2 \leq t \leq m \leq 2t$. Then*

$$z(m, 2t; t, t) = m \cdot 2t - (2m - t + 1).$$

There is also Theorem 1.4 but it forbids $t = n/2$.

A large amount of past and recent works devoted to various inequalities [16] and asymptotic studies [21, 23] whose estimates are usually overly high for rather small n like 11 or not general enough by considering special cases for $z_a(n)$ with small a like 2, 3 or with rather complex a being a polynomial, or cases where the ratio of m and n in $z(m, n; s, t)$ is rather high up to some binomial coefficient including m or n to sample out.

6. Conclusion

One can see that FCA as a theory and as an analytical tool can help to study combinatorial mathematical problems, which is in line with the work of B. Ganter [24] on integer partition lattices, the work of C. Jakel on the ninth Dedekind number [25] and on our previous works on (maximal) antichains enumeration in Boolean and partition lattices [26] and symmetric contexts and maximal independent sets for the cover graph of a Boolean cube [27]. We hope it can also do both in other cases when we deal with Boolean matrices or ordered structures, to provide theoretical keys to enumeration and counting problems and help to compute missing numbers, which may lead to interesting conjectures and theorems. Last but not least, it contributes to the inventory of Experimental Mathematics as an AI-assisted tool.

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Declaration on Generative AI

The author has not employed any Generative AI tools. However, during the preparation of this work, the author used the Overleaf built-in spell checker. Further, the author used FCA-based software tools for drawing the lattice diagram and for inputting the formal contexts in figures 1, 2, and 3. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the publication's content.

References

[1] B. Romera-Paredes, M. Barekatain, A. Novikov, M. Balog, M. P. Kumar, E. Dupont, F. J. R. Ruiz, J. S. Ellenberg, P. Wang, O. Fawzi, P. Kohli, A. Fawzi, Mathematical discoveries from program search with large language models, Nature 625 (2024) 468–475. URL: https://doi.org/10.1038/s41586-023-06924-6. doi:10.1038/s41586-023-06924-6.

[2] A. Davies, P. Veličković, L. Buesing, S. Blackwell, D. Zheng, N. Tomašev, R. Tanburn, P. Battaglia, C. Blundell, A. Juhász, M. Lackenby, G. Williamson, D. Hassabis, P. Kohli, Advancing mathematics by guiding human intuition with ai, Nature 600 (2021) 70–74. URL: https://doi.org/10.1038/s41586-021-04086-x. doi:10.1038/s41586-021-04086-x.

[3] D. Zeilberger, What is experimental mathematics?, 2006.

[4] K. Zarankiewicz, Problem p 101, in: Colloq. Math., 2, 1951, p. 301.

[5] J. Tan, An attack on Zarankiewicz's problem through SAT solving, arXiv e-prints (2022) arXiv–2203.

[6] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer, Berlin/Heidelberg, 1999.

[7] S. O. Kuznetsov, Mathematical aspects of concept analysis, Journal of Mathematical Sciences 80 (1996) 1654–1698.

[8] A. Albano, B. Chornomaz, Why concept lattices are large: extremal theory for generators, concepts, and VC-dimension, Int. J. Gen. Syst. 46 (2017) 440–457. URL: https://doi.org/10.1080/03081079.2017.1354798. doi:10.1080/03081079.2017.1354798.

[9] S. O. Kuznetsov, On computing the size of a lattice and related decision problems, Order 18 (2001) 313–321. URL: https://doi.org/10.1023/A:1013970520933. doi:10.1023/A:1013970520933.

[10] G. Birkhoff, Lattice Theory, eleventh printing ed., Harvard University, Cambridge, MA, 2011.

[11] J. Poelmans, D. I. Ignatov, S. O. Kuznetsov, G. Dedene, Formal concept analysis in knowledge processing: A survey on applications, Expert Syst. Appl. 40 (2013) 6538–6560.

[12] J. Poelmans, S. O. Kuznetsov, D. I. Ignatov, G. Dedene, Formal concept analysis in knowledge processing: A survey on models and techniques, Expert Syst. Appl. 40 (2013) 6601–6623.

[13] R. K. Guy, A many-facetted problem of zarankiewicz, in: G. Chartrand, S. F. Kapoor (Eds.), The Many Facets of Graph Theory, Springer Berlin Heidelberg, Berlin, Heidelberg, 1969, pp. 129–148.

[14] R. K. Guy, A problem of Zarankiewicz, 12, The University of Calgary, Department of Mathematics, 1967.

[15] M. Aigner, G. M. Ziegler, Pigeon-hole and double counting, Springer Berlin Heidelberg, Berlin, Heidelberg, 2018, pp. 195–205. URL: https://doi.org/10.1007/978-3-662-57265-8_28. doi:10.1007/978-3-662-57265-8_28.

[16] S. Roman, A problem of Zarankiewicz, Journal of Combinatorial Theory, Series A 18 (1975) 187–198. URL: https://www.sciencedirect.com/science/article/pii/0097316575900072. doi:https://doi.org/10.1016/0097-3165(75)90007-2.

[17] C. Balbuena, P. García-Vázquez, X. Marcote, J. Valenzuela, New results on the Zarankiewicz problem, Discrete Mathematics 307 (2007) 2322–2327. URL: https://www.sciencedirect.com/science/article/pii/S0012365X06008326. doi:https://doi.org/10.1016/j.disc.2006.11.002.

[18] J. R. Griggs, C.-C. Ho, On the half–half case of the Zarankiewicz problem, Discrete Mathematics 249 (2002) 95–104. URL: https://www.sciencedirect.com/science/article/pii/S0012365X01002370. doi:https://doi.org/10.1016/S0012-365X(01)00237-0, combinatorics, Graph Theory, and Computing.

[19] D. Zeilberger, The c-finite ansatz, The Ramanujan Journal 31 (2013) 23–32. URL: https://doi.org/10.1007/s11139-012-9406-6. doi:10.1007/s11139-012-9406-6.

[20] S. Andrews, In-close2, a high performance formal concept miner, in: Conceptual Structures for Discovering Knowledge - 19th International Conference on Conceptual Structures, ICCS 2011, Derby, UK, July 25-29, 2011. Proceedings, 2011, pp. 50–62. URL: https://doi.org/10.1007/978-3-642-22688-5_4. doi:10.1007/978-3-642-22688-5_4.

[21] V. Nikiforov, A contribution to the zarankiewicz problem, Linear Algebra and its Applications

432 (2010) 1405–1411. URL: https://www.sciencedirect.com/science/article/pii/S0024379509005540. doi:https://doi.org/10.1016/j.laa.2009.10.040.
[22] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman, 1979.
[23] G. Chen, D. Horsley, A. Mammoliti, Zarankiewicz numbers near the triple system threshold, Journal of Combinatorial Designs 32 (2024) 556–576.
[24] B. Ganter, Notes on integer partitions, International Journal of Approximate Reasoning 142 (2022) 31–40. URL: https://www.sciencedirect.com/science/article/pii/S0888613X21001894. doi:https://doi.org/10.1016/j.ijar.2021.11.004.
[25] C. Jäkel, A computation of the ninth Dedekind Number, 2023. arXiv:2304.00895.
[26] D. I. Ignatov, A note on the number of (maximal) antichains in the lattice of set partitions, in: M. Ojeda-Aciego, K. Sauerwald, R. Jäschke (Eds.), Graph-Based Representation and Reasoning, Springer Nature Switzerland, Cham, 2023, pp. 56–69.
[27] D. I. Ignatov, On the maximal independence polynomial of the covering graph of the hypercube up to n=6, in: D. Dürrschnabel, D. López Rodríguez (Eds.), Formal Concept Analysis, Springer Nature Switzerland, Cham, 2023, pp. 152–165.