Modeling and simulating of Duffing pendulum in the moved coordinate system

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Abstract

The paper deals with developing a mathematical framework to design novel discrete-time chaotic systems based on the known ones. Our development is based on applying coordinate transformation to the domain where the initial system dynamic is defined. We study the shift of 2D system coordinate origin and use it to define novel system state variables, which take into account this shift. The dynamical system obtained in such a way is considered the interval one with piecewise linear interval boundaries. This fact gives us the possibility to consider possible uncertainty caused by changes in system parameters and the presence of nonlinear functions and rewrite the system into a linear-like form. Unlike the initial nonlinear ones, performing all coordinate transformations for such type systems is easy. Our approach is based on transforming the continuous-time system dynamic into a discrete-time domain due to the possibility of its implementation in modern digital devices. Transformation into a discrete-time domain allows us to define system dynamics using its previous states to define the piecewise constant factors in the system equations. The system equation is designed in such a way that it is created on a solid background and uses information about previous system motions as well as its motion in the moved coordinate system and motions of the considered moved coordinate system. To make the system dynamic more complex, we offer to consider its perturbed motions as the difference between motions in the moved and stationary coordinate system.

Keywords

chaotic system, coordinate transformation, moved coordinate system, Duffing pendulum,

1. Introduction

Nowadays, data transmission using chaotic systems [1] refers to the practical usage of chaos theory [2] and chaotic signals [3] to secure information transmission. Chaotic systems are susceptible to initial conditions and exhibit complex, unpredictable behavior over time [4]. These facts about chaotic systems make them useful in secure communication [5] because chaotic signals can be difficult to predict, intercept, or reproduce without knowing the exact system parameters [6].

Such unique chaotic systems' features cause several key concepts in chaotic communication:

- Chaotic modulation involves embedding information into a chaotic signal [7]. The chaotic signal acts as a carrier wave, which is then modulated by the data. Only receivers knowledgeable about the chaotic system's parameters can demodulate and recover the original message.
- Synchronization of chaotic systems requires the transmitter and receiver must use identical or synchronized chaotic systems [8]. These systems must be synchronized so the receiver can extract the embedded message from the chaotic signal.
- Chaotic masking assumes the data signal is added to a chaotic carrier signal at the transmitter end [9]. The chaotic signal masks the data, making it indistinguishable from noise to an eavesdropper. The receiver, knowing the chaotic system, can subtract the chaotic carrier and retrieve the original data.

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- The noise-like signals concept assumes that chaotic signals appear similar to noise, making them hard to distinguish from random background noise in the communication channel [10]. This property provides inherent security, as an eavesdropper without the system parameters will find it challenging to extract meaningful data.

The above-shown concepts find their practical implementation in designing various chaotic modulation schemes. The main ones are chaos shift keying (CSK) [11] and chaotic phase modulation (CPM) [12]. These modulation schemes find their applications in establishing wireless communications. In this case, chaotic communication can be applied in wireless systems where robustness to interference is crucial. Since chaotic signals are noise-like and spread across a wide bandwidth, they can be used in environments with high electromagnetic interference. Also, various optical fiber communication systems use laser signals to transmit data securely. Optical chaos can be generated using semiconductor lasers, and synchronization between transmitter and receiver can be achieved with optical feedback.

In summary, chaotic systems offer a promising approach to secure data transmission by leveraging the unpredictable and noise-like nature of chaos, making it difficult for unauthorized parties to intercept or decode the communication.

The main drawback of known chaotic systems, which are used to implement chaotic generators and produce chaotic signals, is some subjectivism in the design of these systems. Since authors do not explain the influence of terms and factors in their equations, modifying and improving them is tough. We offer to avoid this drawback by designing novel chaotic systems with applying some transformations to known ones. Thus, our paper's goal is to design a novel chaotic system by combining motion equations of known ones and motions of the origin of the coordinate system where the above-mentioned chaotic system is considered. We believe that the goal achieving makes a systematic basis in chaotic system design.

Our paper is organized as follows: at first, we consider the generalized chaotic system and transform its equations into interval matrix form to represent it in a piecewise linear form. Then, we consider the system in the discrete-time domain to avoid solving any differential equations. Such a discrete-time dynamical system is viewed as a system in some coordinate system in which the origin changes its position relatively to a stationary one. We define the chaotic system position in the stationary coordinates as the sum of the chaotic system and the origin position. At last, we show the use of our approach by considering Duffing pendulum equations.

2. Method

2.1. Interval discrete-time model of the generalized second order dynamical system

Let us consider the generalized second order dynamical system

$$\dot{\mathbf{Y}} = \mathbf{F}\left(\mathbf{Y}\right) + \mathbf{U},\ \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix};\ \mathbf{F}\left(Y\right) = \begin{pmatrix} f_1\left(y_1, y_2\right) \\ f_2\left(y_1, y_2\right) \end{pmatrix};\ \mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1}$$

where y_i are system state variables and $f_i(.)$ are some nonlinear functions, and u_i are some input signals.

System nonlinearities make analysis of its motions, their transformations, or synthesis a quite hard problem which should be solved for each particular case in the separate way. That is why we offer to use interval methods and replace the system nonlinear functions $f_i(.)$ with domains $\mathbf{\Omega_i}$ where these functions are defined and do not exceed them on whole system operation range.

The boundaries for these domains can be defined in a different way. Thus, one can approximate boundaries for function $f_i(.)$ by nonlinear function $g_i(.)$ which are more simple than system nonlinearities and which use during system study of design does not cause any difficulties. One of such functions is a piecewise linear function which for the case of system with two arguments can be written down as follows

$$g_i(y_1, y_2) = \begin{cases} a_{i11}y_1 + a_{i21}y_2 + a_{i01} & \text{if} \quad (y_1 \in \mathbf{\Omega_{i1}}) \text{ and } (y_2 \in \mathbf{\Omega_{i1}})\,; \\ & \vdots \\ a_{i1n}y_1 + a_{i2n}y_2 + a_{i0n} & \text{if} \quad (y_1 \in \mathbf{\Omega_{in}}) \text{ and } (y_2 \in \mathbf{\Omega_{in}})\,, \end{cases} \tag{2}$$

here a_{ij} are factors of piecewise linear approximation which are defined in i-th subdomain $\mathbf{\Omega_{ij}}$ of system state variables' values.

It is clear that one can use different mashes to define subdomains $\mathbf{\Omega_{ij}}$. We believe that the most accurate one is a triangular mesh which we offer to use to define the subdomains $\mathbf{\Omega_{ij}}$.

Such an approach allows us to redefine i-th component of vector $\mathbf{F}$ in (1) as follows

$$\begin{aligned} & f_i\left(y_1, y_2\right) \in \mathbf{f_i}\left(\mathbf{y_1}, \mathbf{y_2}\right), \\ & \mathbf{f_i}\left(\mathbf{y_1}, \mathbf{y_2}\right) = \begin{cases} \begin{array}{l} [a_{i11\min}, a_{i11\max}]\,\mathbf{y_1} + \\ + [a_{i21\min}, a_{i21\max}]\,\mathbf{y_2} + \\ + [a_{i01\min}, a_{i01\max}] \end{array} & \text{if} \quad (\mathbf{y_1} \in \mathbf{\Omega_{i1}}) \text{ and } (\mathbf{y_2} \in \mathbf{\Omega_{i1}})\,; \\ & \vdots \\ \begin{array}{l} [a_{i1n\min}, a_{in1\max}]\,\mathbf{y_1} + \\ + [a_{i2n\min}, a_{i2n\max}]\,\mathbf{y_2} + \\ + [a_{i0n\min}, a_{i0n\max}] \end{array} & \text{if} \quad (\mathbf{y_1} \in \mathbf{\Omega_{in}}) \text{ and } (\mathbf{y_2} \in \mathbf{\Omega_{ij}})\,, \end{cases} \\ & \mathbf{y_i} = [y_{i\min}, y_{i\max}]. \end{aligned} \tag{3}$$

Here we consider the case when numbers of intervals in upper and lower boundaries equal each other and equal to n. In the most general case when upper and lower boundaries are defined with different numbers of intervals and/or these boundaries are defined for various subdomains Ω_{ij}, one should split intervals in (3) and check conditions for each boundary in a separate way.

Let us use (3) to rewrite (1) into linear-like interval form

$$\begin{aligned} & \dot{\mathbf{Y}} = \mathbf{AY} + \mathbf{A_0} + \mathbf{U}, \\ & \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix};\ \mathbf{A_0} = \begin{pmatrix} a_{01} \\ a_{02} \end{pmatrix};\ \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \\ & \mathbf{a_{ij}} = \begin{cases} [a_{ij1\min}, a_{ij1\max}] & \text{if} \quad (y_1 \in \mathbf{\Omega_{i1}}) \text{ and } (y_2 \in \mathbf{\Omega_{i1}})\,; \\ & \vdots \\ [a_{ijn\min}, a_{ij1\max}] & \text{if} \quad (y_1 \in \mathbf{\Omega_{in}}) \text{ and } (y_2 \in \mathbf{\Omega_{in}})\,. \end{cases} \end{aligned} \tag{4}$$

We call (4) as continuous-time interval model for dynamical system (1). Contrary to a solution of initial system the solution of interval one produce boundary motion trajectories $y_{i\min}$ and $y_{i\max}$ which are bound motion of initial system and which this system does not exceed.

The main feature of this system is a piecewise constant elements of matrices $\mathbf{A}$ and $\mathbf{A}_0$. Equations with piecewise constant factors can be implemented in the simplest way by using digital devices like MCU. That is why we rewrite them into discrete-time domain by using known approximations of derivative operator.

$$\frac{d}{dt} \approx h\left(1, z^{-1}\right), \tag{5}$$

here z^{-1} means backward signal shift in one sample time period T, $h(.)$ is a some approximation function.

We consider the simplest finite difference approximation in our paper

$$\frac{d}{dt} \approx \frac{1 - z^{-1}}{z^{-1}T}. \tag{6}$$

If one substitute (6) into (4), the discrete-time interval model of the considered dynamical system can be written down

$$\mathbf{Y} = z^{-1}\left(\mathbf{A1Y}\right) + z^{-1}\mathbf{A1_0} + z^{-1}T\mathbf{U},$$
$$\mathbf{A1} = \begin{pmatrix} a_{11}T+1 & a_{12}T \\ a_{21}T & a_{22}T+1 \end{pmatrix},\ \mathbf{A1_0} = \begin{pmatrix} a_{01}T \\ a_{02}T \end{pmatrix}. \tag{7}$$

Contrary to (4) the discrete-time state variables model $\mathbf{y}_i$ which are defined by using (7) depend on system previous state that is considered in time moment

$$\tau = T\left[\frac{t}{T}\right] - T, \tag{8}$$

here operator [.] means taking integer part from the number. It is clear that solution of (7) requires to save system previous state which can be easy implemented in all MCU programming languages. Also, it should be mentioned that shift operator in (7) applies to system state variables as components of $\mathbf{Y}$ matrix as well as the previous values of system coordinates are used to define piecewise constant factors $\mathbf{a_{ij}}$ in the matrices $\mathbf{A1}$ and $\mathbf{A1_0}$. Similar to (4) expression (7) allows us to define boundaries for all possible system motions. Systems (4) and (7) we call as the core of chaotic system and we use it to design some novel systems.

2.2. Interval model with moving origin

The above-given models are designed for the case of immovable coordinate system in which system phase portraits and motions are defined.

Nevertheless, sometimes system motions should be considered in some coordinate system which origin moves relatively some stationary base. To design the model which describe such motions let us determine the system position in stationary coordinates $\mathbf{Y0}$ as linear combination of its position in moved coordinates $\mathbf{Y}$ and origin position of moved coordinate system $\mathbf{Y1}$.

$$\mathbf{Y0} = \mathbf{Y1} + \mathbf{Y}, \tag{9}$$

here we think that vectors $\mathbf{Y0}$, $\mathbf{Y1}$, and $\mathbf{Y}$ have the same size and contain components which define system position in some phase plane and vector $\mathbf{Y1}$ are defined similar to $\mathbf{Y}$

$$\mathbf{Y1} = z^{-1}\left(\mathbf{B1Y1}\right) + z^{-1}\mathbf{B1_0} + z^{-1}\mathbf{U1}, \tag{10}$$

here $\mathbf{B1}$ and $\mathbf{B1_0}$ are some matrices which components are defined similar to components of $\mathbf{A1}$ and $\mathbf{A1_0}$ matrices.

If one substitutes (7) into (9), he can write down following expression

$$\mathbf{Y0} = z^{-1}\left(\mathbf{B1Y1}\right) + z^{-1}\left(\mathbf{A1Y}\right) + z^{-1}\mathbf{A1_0} + z^{-1}\mathbf{B1_0} + z^{-1}\mathbf{U} + z^{-1}\mathbf{U1}. \tag{11}$$

Expression (11) interrelate system motions in moved and stationary coordinate systems. It is clear that to define motion in stationary coordinate system one should to know system position in moved coordinate system and position of this coordinate system's origin. Since the both of positions in the general case are defined as solution of some equations, it is necessary to solve both of them to define system position. This fact can cause some computational issues. Also, the knowing of vectors $\mathbf{Y1}$ and $\mathbf{Y}$ can cause the necessity to transmit the components of these vectors from the moving systems to stationary base. That is why we offer to rewrite (11) in terms of components $\mathbf{Y0}$ vector only.

To perform such a transformation for (11) at first we solve (10) and(7) for $\mathbf{Y1}$ and $\mathbf{Y}$ vectors

$$\mathbf{Y} = \left(\mathbf{E} - z^{-1}\mathbf{A1}\right)^{-1} z^{-1}\left(\mathbf{A1_0} + \mathbf{U}\right);\ \mathbf{Y1} = \left(\mathbf{E} - z^{-1}\mathbf{B1}\right)^{-1} z^{-1}\left(\mathbf{B1_0} + \mathbf{U1}\right), \tag{12}$$

where $\mathbf{E}$ is the 2×2 identity matrix.

Then, we substitute (12) into (9)

$$\mathbf{Y0} = \left(\mathbf{E} - z^{-1}\mathbf{A1}\right)^{-1} z^{-1}\left(\mathbf{A1_0} + \mathbf{U}\right) + \left(\mathbf{E} - z^{-1}\mathbf{B1}\right)^{-1} z^{-1}\left(\mathbf{B1_0} + \mathbf{U1}\right) \tag{13}$$

and rewrite it as follows

$$\begin{aligned} &\det\left(\mathbf{E} - z^{-1}\mathbf{A1}\right)\det\left(\mathbf{E} - z^{-1}\mathbf{B1}\right)\mathbf{Y0} = \\ &= \operatorname{adj}\left(\mathbf{E} - z^{-1}\mathbf{A1}\right) z^{-1}\left(\mathbf{A1_0} + \mathbf{U}\right) + \operatorname{adj}\left(\mathbf{E} - z^{-1}\mathbf{B1}\right) z^{-1}\left(\mathbf{B1_0} + \mathbf{U1}\right), \end{aligned} \tag{14}$$

where adj(.) means adjugate matrix and $\det(.)$ means matrix determinant.

If one expand multiplication of determinants in left-hand expression of (14), he can rewrite this formula as follows

$$\begin{aligned} &\mathbf{Y0} = -\mathbf{q_1} z^{-1}\mathbf{Y0} - \mathbf{q_2} z^{-2}\mathbf{Y0} - \mathbf{q_3} z^{-3}\mathbf{Y0} - \mathbf{q_4} z^{-4}\mathbf{Y0} + \\ &+ \operatorname{adj}\left(\mathbf{E} - z^{-1}\mathbf{A1}\right) z^{-1}\left(\mathbf{A1_0} + \mathbf{U}\right) + \operatorname{adj}\left(\mathbf{E} - z^{-1}\mathbf{B1}\right) z^{-1}\left(\mathbf{B1_0} + \mathbf{U1}\right), \end{aligned} \tag{15}$$

where $\mathbf{q}_i$ are interval piecewise constant coefficients of the system characteristic polynomial.

Since the system (15) has piecewise constant factors which depend on motion systems (7) and (10) one should define expressions which interrelate the components of **Y0** vector with vectors **Y** and **Y1**.

We offer to find these expressions by considering (11) and expression which is obtained from (9) by shifting it for one sample time

$$z^{-1}\mathbf{Y0} = z^{-1}\mathbf{Y1} + z^{-1}\mathbf{Y}, \tag{16}$$

Solution of (16) and (11) allows us to write down following expressions

$$\begin{aligned} &z^{-1}\mathbf{Y} = \left(\mathbf{A1} - \mathbf{B1}\right)^{-1}\left(\mathbf{Y0} - z^{-1}\mathbf{B1}\mathbf{Y0} + z^{-1}\mathbf{U1} + z^{-1}\mathbf{U}\right); \\ &z^{-1}\mathbf{Y1} = \left(\mathbf{A1} - \mathbf{B1}\right)^{-1}\left(z^{-1}\mathbf{A1}\mathbf{Y0} - \mathbf{Y0} + z^{-1}\mathbf{U1} + z^{-1}\mathbf{U}\right). \end{aligned} \tag{17}$$

We call (15) as an interval model of the dynamical system in stationary coordinate system. This model consists of three parts: the first one use system's previous coordinates in the stationary coordinate system to define its current position. The second one uses the system position in the moved coordinate system and the third one uses information about moving of coordinate system origin.

3. Results and discussion

3.1. Duffing pendulum modeling and simulating in the stationary coordinate system

Let us consider the use of the proposed approach to design the system with chaotic dynamic. We use the well-known Duffing pendulum [13]

$$\dot{y}_1 = y_2;\ \dot{y}_2 = -a_1 y_1 - a_2 y_2 - a_3 y_1^3 + c_1 \cos c_2 t \tag{18}$$

as the basis for our system.

Under some pendulum parameters system (18) has a chaotic dynamic. We think that the base system parameters are a_2=0.02, a_1=1, a_3=5, c_1=8, c_2=0.5. Also, the parametric uncertainty is assumed in the relative interval ε=[0,9,1.1], which means possibility to 10% parameters drift. This interval allows us to define intervals of possible pendulum parameters in such a way

$$\mathbf{a_i} = a_i\varepsilon;\ \mathbf{c_i} = c_i\varepsilon. \tag{19}$$

The pendulum nonlinearity is approximated by piecewise linear domain which is shown in figure 1. The filled area in this figure shows a domain where pendulum nonlinearity is defined.

This domain is defined by the following intervals on horizontal

$$\mathbf{y_1} = \bigcup_{i=1}^{n} \mathbf{y_{1i}} \tag{20}$$

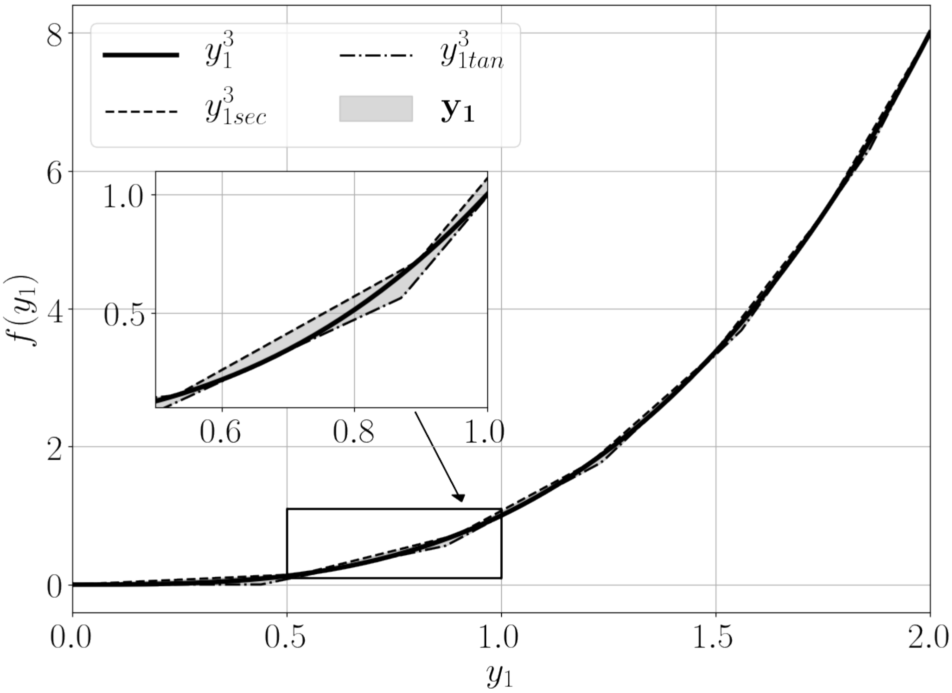

Figure 1: Piecewise linear interval approximation of pendulum's nonlinearity.

and vertical

$$\mathbf{f}(y_1) = \bigcup_{i=1}^{n} \mathbf{f_i}(y_{1i}) \tag{21}$$

axes.

In formulas (20) and (20) n is a number of piecewise linear intervals which for the considered case equals to 12, $\mathbf{f_i}(y_{1i})$ is an i-th interval of possible values of system nonlinearity and $\mathbf{y_{1i}}$ is an i-th interval of nonlinear function's argument

$$\begin{gathered} \mathbf{f}(y_1) = \mathbf{k_i} y_1 + \mathbf{y_{0i}},\ \mathbf{y_{1i}} = [y_{1i\min}, y_{1i\max}]\,,\ y_{1i} \in \mathbf{y_{1i}}, \\ \mathbf{k_i} = [k_{i\min}, k_{i\max}]\,; \mathbf{y_{0i}} = [y_{0i\min}, y_{0i\max}]\,, \end{gathered} \tag{22}$$

where k_{jimin}, k_{jimax} and y_{jimin}, y$_{jimax}$ are piecewise linear boundaries' factors.

If one substitutes (22) into (21) following expression can be written down

$$\begin{gathered} \mathbf{f}(y_1) = \mathbf{k}(y_1) y_1 + \mathbf{y_0}(y_1), \\ \mathbf{k}(y_1) = \bigcup_{i=1}^{n} \mathbf{k_i},\ \mathbf{y_0}(y_1) = \bigcup_{i=1}^{n} \mathbf{y_{0i}}. \end{gathered} \tag{23}$$

Intervals in (23) for the considered cubic nonlinearity are shown in table 1. These intervals as well as parameters of piecewise linear function, which replace pendulum cubic nonlinearity, obtained by using Nelder-Mead method from routine *minimize()* that is included in SciPy 1.11.0 library. We also use routine *scipy.integrate()* which solves differential equations from the above-mentioned Python library. All calculated data is stored in csv-files and used to visualize calculation results in package PGFplot which is a part of TEXLive-2023.

The use of intervals (19) and (22) gives us the possibility to rewrite (18) in the interval piecewise linear form

$$\dot{\mathbf{y}}_\mathbf{1} = \mathbf{y_2};\ \dot{\mathbf{y}}_\mathbf{2} = -\left(\mathbf{a_1} + \mathbf{a_3}\mathbf{k}\right)\mathbf{y_1} - \mathbf{a_2}\mathbf{y_2} - \mathbf{a_3}\mathbf{y_0} + c_1 \cos c_2 t \tag{24}$$

We call (24) as interval piecewise linear model of Duffing pendulum. One can use this model to define the boundary motions which shows the maximal and minimal possible amplitude of pendulum oscillations.

Table 1
Parameters of interval approximation.

No	Approximation parameters $\mathbf{k_i}$	$\mathbf{y_{0i}}$	$\mathbf{y_{1i}}$
1	[0.000003,0.283]	[0,0]	[0,0.000667]
2	[0.057,0.283]	[-0.00167,0]	[0.000667,0.439]
3	[1.304,0.283]	[-0.573,0]	[0.439,0.531]
4	[1.304,1.561]	[-0.573, -0.679]	[0.531,0.871]
5	[3.329,1.561]	[-2.338, -0.679]	[0.871,0.896]
6	[3.329,3.342]	[-2.338, -2.275]	[0.896,1.208]
7	[3.329,5.478]	[-2.338, -4.854]	[1.208,1.234]
8	[5.872,5.478]	[-5.477, -4.854]	[1.234,1.490]
9	[5.872,7.902]	[-5.477, -8.467]	[1.490,1.560]
10	[8.785,7.902]	[-10,021, -8.467]	[1.560,1.752]
11	[8.785,10.576]	[-10.021, -13.15]	[1.752,1.859]
12	[12,10.576]	[-16,13.15]	[1.859,2]

If one applies (6) to (24), he can rewrite interval pendulum motions equations in the discrete-time domain

$$\begin{aligned}&\mathbf{y_1} = z^{-1}\mathbf{y_1} + z^{-1}T\mathbf{y_2};\\&\mathbf{y_2} = z^{-1}\mathbf{y_2}\left(1-\mathbf{a_2}T\right) - z^{-1}T\left(\mathbf{a_1}+\mathbf{a_3}\mathbf{k}\right)\mathbf{y_1} - z^{-1}T\mathbf{a_3}\mathbf{y_0} + z^{-1}Tc_1\cos c_2 t.\end{aligned} \tag{25}$$

In the extended form (25) can be given as follows

$$\begin{aligned}y_{1\min} =& z^{-1}y_{1\min} + z^{-1}Ty_{2\min};\\y_{2\min} =& z^{-1}y_{2\min}\left(1-Ta_{2\max}\right) - z^{-1}Ta_{3\max}y_{0\max} -\\&-z^{-1}T\left(a_{1\max}+a_{3\max}k_{\max}\right)y_{1\min} + z^{-1}Tc_{1\min}\cos c_{2\max}t.\\y_{1\max} =& z^{-1}y_{1\max} + z^{-1}Ty_{2\max};\\y_{2\max} =& z^{-1}y_{2\max}\left(1-Ta_{2\min}\right) - z^{-1}Ta_{3\min}y_{0\min} -\\&-z^{-1}T\left(a_{1\min}+a_{3\min}k_{\min}\right)y_{1\max} + z^{-1}Tc_{1\max}\cos c_{2\min}t.\end{aligned} \tag{26}$$

Let us rewrite (26) into matrix form (4)

$$\begin{aligned}&\mathbf{Y} = z^{-1}\mathbf{A1Y} + z^{-1}\mathbf{A1_0} + z^{-1}\mathbf{U},\\&\mathbf{Y} = \begin{pmatrix}[y_{1\min}, y_{1\max}]\\ [y_{2\min}, y_{2\max}]\end{pmatrix};\ \mathbf{A1_0} = \begin{pmatrix}0\\ [-a_{3\max}y_{0\max}, -a_{3\min}y_{0\min}]\end{pmatrix};\\&\mathbf{A1} = \begin{pmatrix}1 & T\\ \begin{bmatrix}-T\left(a_{1\max}+a_{3\max}k_{\max}\right),\\ -T\left(a_{1\min}+a_{3\min}k_{\min}\right)\end{bmatrix} & \begin{bmatrix}1-Ta_{2\max},\\ 1-Ta_{2\min}\end{bmatrix}\end{pmatrix};\\&\mathbf{U} = \begin{pmatrix}0\\ [c_{1\min}\cos c_{2\max}t, c_{1\max}\cos c_{2\min}t]\end{pmatrix}.\end{aligned} \tag{27}$$

Piecewise constant elements of matrix $\mathbf{A1}$ can be found by known interval system output $\mathbf{y_1}$ which can be defined by (12)

$$\mathbf{y_1} = -\frac{z^{-1}T\mathbf{a_3}\mathbf{y_0} + z^{-1}Tc_1 cos(c_2 t)}{1 + z^{-1}\left(T\mathbf{a_2}-2\right) + z^{-2}\left(T^2\left(\mathbf{a_3}\mathbf{k}+\mathbf{a_1}\right) - T\mathbf{a_2}+1\right)}. \tag{28}$$

Simulation results which are obtained for interval system (26) are shown in figures 2, 3.

Analysis of given in figures 2 and 3 results shows that Duffing pendulum has the chaotic dynamic for all parameters combinations from the intervals (19) and (22). Chaotic nature of the considered interval

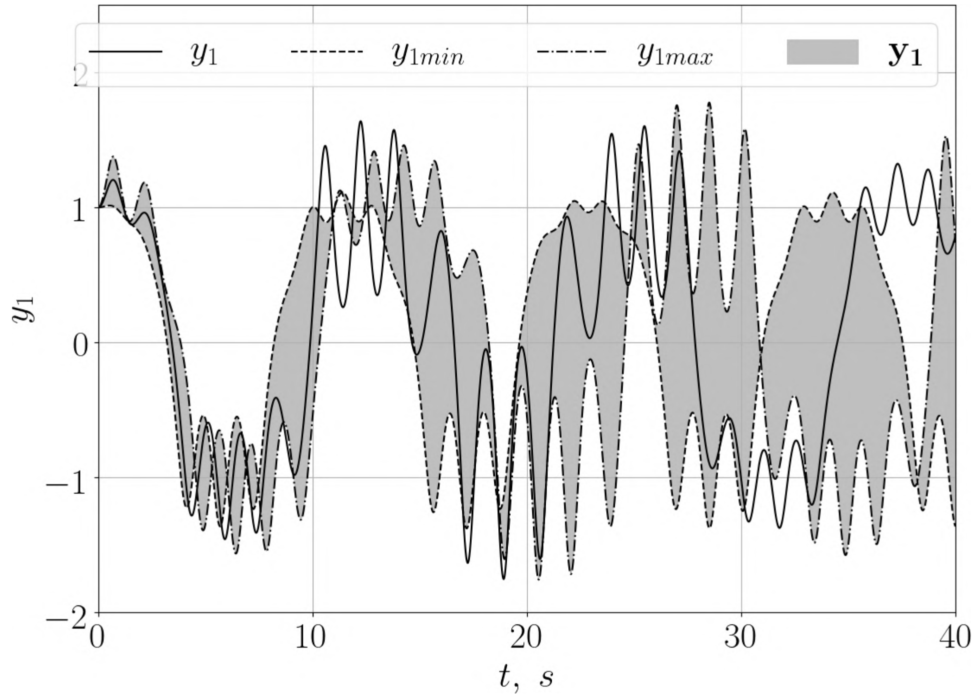

Figure 2: Chaotic oscillations in the studied dynamical system.

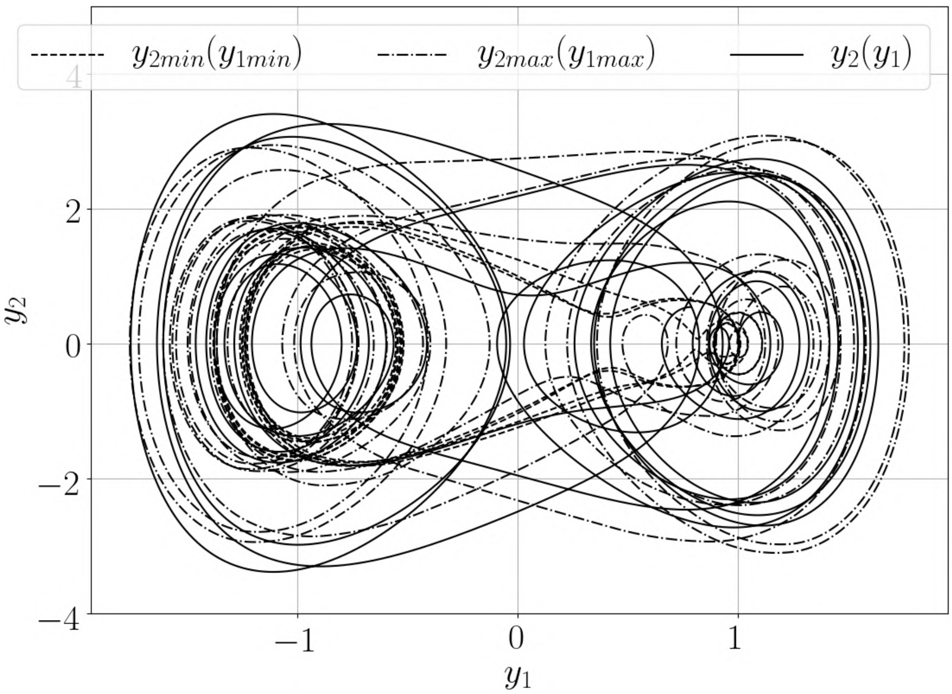

Figure 3: Chaotic attractors of (25).

system is proved by the fact that the motion of pendulum with exactly-known above-given parameters starts as motion which is bounded by motions in the upper and lower boundaries but after a quite short time which is near 10s it leaves the interval of pendulum boundary motions. If one analyzes these boundary motions, he finds that quite small variation of pendulum parameters dramatically changes its dynamic.

3.2. Duffing pendulum modeling and simulating in the moved coordinate system

Let us assume that origin of coordinate system, where the pendulum dynamic is defined, moves and this motion can be defined by using (18) with parameters a_2=0.01, a_1=-1, a_3=3, d_1=3, d_2=1 and the same relative interval. This assumption allows us to write down equation similar to (27)

$$\mathbf{Y1} = z^{-1}\mathbf{B1Y1} + z^{-1}\mathbf{B1_0} + z^{-1}\mathbf{U1}. \tag{29}$$

Matrices $\mathbf{B1}$, $\mathbf{B1_0}$ and $\mathbf{U1}$ are similar to $\mathbf{A1}$, $\mathbf{A1_0}$ and $\mathbf{U}$ but their elements are defined with replacing $\mathbf{a_i}$ to $\mathbf{b_i}$ and c_i to d_i.

Equations (29) and (27) allows us to define system dynamic in the stationary coordinates by rewriting (15) as follows

$$\begin{aligned}
&\mathbf{y_1} = -\mathbf{q_1}z^{-1}\mathbf{y_1} - \mathbf{q_2}z^{-2}\mathbf{y_1} - \mathbf{q_3}z^{-3}\mathbf{y_1} - \mathbf{q_4}z^{-4}\mathbf{y_1} + \left(\mathbf{w_4}z^{-4} + \mathbf{w_3}z^{-3} + \mathbf{w_2}z^{-2}\right)\mathbf{y_0} + \\
&+ \left(\mathbf{v_{14}}z^{-4} + \mathbf{v_{13}}z^{-3} + \mathbf{v_{12}}z^{-2}\right) c_1 \cos c_2 t + \left(\mathbf{v_{24}}z^{-4} + \mathbf{v_{23}}z^{-3} + \mathbf{v_{22}}z^{-2}\right) d_1 \cos d_2 t, \\
&\mathbf{q_1} = T\left(\mathbf{a_2} + \mathbf{b_2}\right) - 4;\ \mathbf{q_2} = T^2\left(\mathbf{a_2b_2} + \left(\mathbf{a_3} + \mathbf{b_3}\right)\mathbf{k} + \mathbf{a_1} + \mathbf{b_1}\right) - 3T\left(\mathbf{a_2} - \mathbf{b_2}\right) + 6; \\
&\mathbf{q_3} = (\mathbf{a_2b_3k} + \mathbf{a_3b_2k} + \mathbf{a_1b_2} + \mathbf{a_2b_1})T^3 - 2(\mathbf{a_2b_2} + \left(\mathbf{a_3} + \mathbf{b_3}\right)\mathbf{k} + \mathbf{a_1} + \mathbf{b_1})T^2 + \\
&+ 3(\mathbf{a_2} + \mathbf{b_2})T - 4;\ \mathbf{q_4} = 1 + (\mathbf{a_3b_3k}^2 + \left(\mathbf{a_1b_3} + \mathbf{a_3b_1}\right)\mathbf{k} + \mathbf{a_1b_1})T^4 - \\
&- \left(\mathbf{k}\left(\mathbf{a_2b_3} - \mathbf{a_3b_2}\right) - \mathbf{a_1b_2} - \mathbf{a_2b_1}\right)T^3 - (\mathbf{a_2} + \mathbf{b_2})T + (\mathbf{a_2b_2} + \left(\mathbf{a_3} + \mathbf{b_3}\right)\mathbf{k} + \mathbf{a_1} + \mathbf{b_1})T^2; \\
&\mathbf{w_4} = -(2\mathbf{a_3b_3k} - \mathbf{a_1b_3} - \mathbf{a_3b_1})T^3 + (\mathbf{a_2b_3} + \mathbf{a_3b_2})T^2 - T(\mathbf{a_3} + \mathbf{b_3}); \\
&\mathbf{w_3} = -(\mathbf{a_2b_3} + \mathbf{a_3b_2})T^2 + 2(\mathbf{a_3} + \mathbf{b_3})T;\ \mathbf{w_2} = -T(\mathbf{a_3} + \mathbf{b_3});\ \mathbf{v_{12}} = Tc_1;\ \mathbf{v_{22}} = Tc_1; \\
&\mathbf{v_{23}} = T^2\mathbf{a_2}c_1 - 2Tc_1;\ \mathbf{v_{14}} = T^3\left(\mathbf{b_3}c_1\mathbf{k} - \mathbf{b_1}c_1\right) + T^2\mathbf{b_2}c_1 - Tc_1;\ \mathbf{v_{13}} = T^2\mathbf{b_2}c_1 - 2Tc_1; \\
&\mathbf{v_{24}} = T^3\left(\mathbf{a_3}c_1\mathbf{k} - \mathbf{a_1}c_1\right) + T^2\mathbf{a_2}c_1 - Tc_1;
\end{aligned} \tag{30}$$

Comparison of (30) and (27) shows that taking into account motion of coordinate system increases order of the studied dynamical system and it is necessary to have information about four previous system positions instead of two ones. Iteration of (30) gives results which are shown in figures 4, 5.

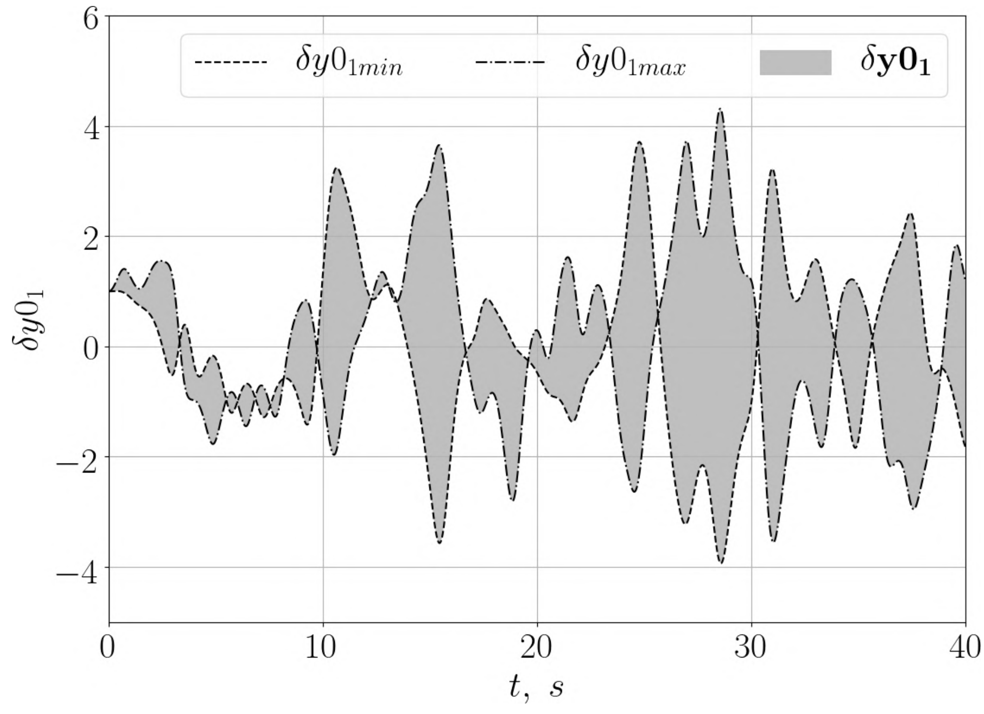

Figure 4: Chaotic oscillations in the moved dynamical system.

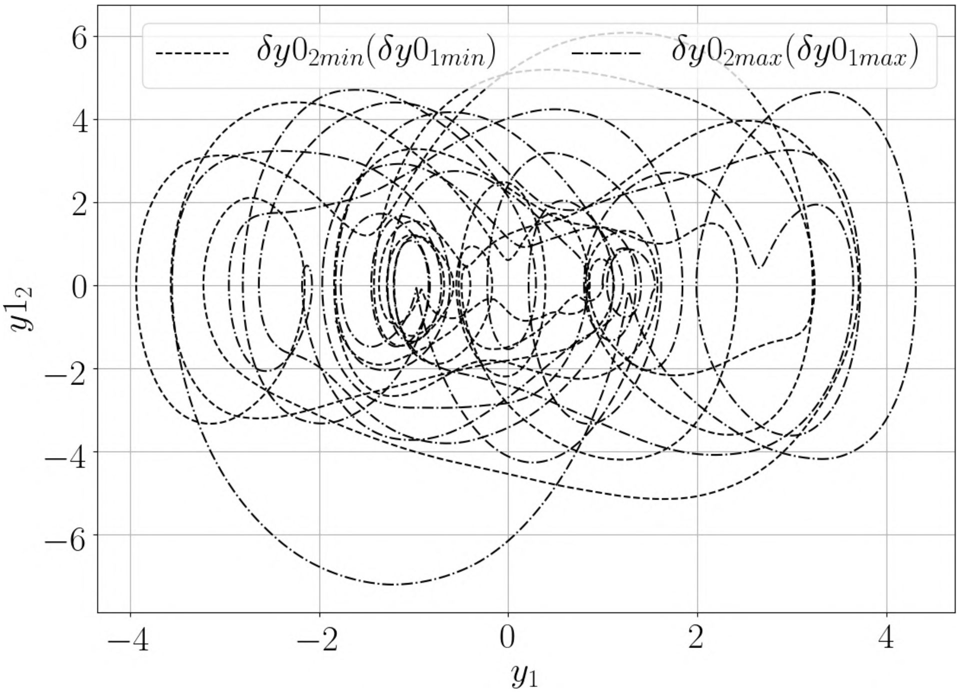

Figure 5: Chaotic attractors of (30).

Analysis of curves in figures 4 and 5 and comparison them with figures 2 and 3 allows us to claim that system motion in the moved coordinate system is more complex than stationary one. One can see three stationary points (-2.0), (0,0), and (2,0) in the system (30) instead of two (-1,0) and (1,0) for system (25). This fact makes backgrounds to improving system secured features.

4. Conclusion

The considering of chaotic system as dynamical system in moving coordinates gives us the possibility to produce novel chaotic oscillations by using well-known chaotic systems. This fact allows us to claim that novel chaotic system can be designed by changing one or both core system and system, which define motion of coordinate system's origin. In both cases system dynamic differs the core dynamic very much. The order of designed in such a way system equals to core system order and order of dynamical system which describe motion of coordinate system. The increasing system order requires to use more information about previous system states in case of discrete-time system implementation. Analysis of the obtained discrete-time models shows that chaotic system can be defined in class of discrete-time dynamical systems with piecewise constant parameters. It becomes possible due to the use of interval methods to describe system motions. Defining these parameters in some arrays, lists, tables and so on gives us the possibility to implement the considered systems by using wide range MCU and FPGA.

Declaration on Generative AI: We declare the no use of any Generative AI tools while preparing data and the paper's writing.

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