Proof Search and Countermodel Construction for iCK4

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Abstract
We present a proof search procedure for the minimal coreflection logic iCK4, an intuitionistic modal logic with the normality axiom and the coreflection principle. The procedure is based on a sequent calculus Gbu-iCK4 that ensures strong termination of backward proof search. Gbu-iCK4 is shown to be complete via a dual refutation calculus that enables the extraction of countermodels when the proof search fails. To support practical experimentation, we provide an implementation of the proof search and the countermodel extraction procedures.

1. Introduction

Within the framework of intuitionistic modal logics, normal systems that consider only the $\Box$ modality and satisfy the *coreflection principle* $p \to \Box p$ have attracted significant attention due to their connections with provability and epistemic interpretations and applications in the formal verification; for a comprehensive discussion, see [1, 2, 3, 4, 5].

Here, we consider the *minimal coreflection logic* iCK4 which is the logic obtained by extending intuitionistic propositional logic IPL with the *normality axiom* $\Box(p \to q) \to (\Box p \to \Box q)$ and the *coreflection principle*. Adopting these principles has important consequences in terms of the usual Kripke birelational semantics for intuitionistic modal logics; indeed, in the resulting models, which are called *strong*, the modal accessibility relation R is constrained by the intuitionistic one ($R \subseteq \leq$) and, as a consequence, the persistence of forcing (characterizing the $\leq$ relation in intuitionistic Kripke models) also holds for the modal accessibility relation R. In this work, we investigate proof search in iCK4 by following the approach used in [6] for the Intuitionistic Strong Löb Logic iSL, which extends iCK4 with the Gödel-Löb axiom $\Box(\Box p \to p) \to \Box p$, and which in turn is based on the ideas behind the calculus Gbu for Intuitionistic propositional Logic presented in [7, 8]. Our focus is on designing a sequent calculus $\mathcal{C}$ in which backward proof search always terminates, that is: given any sequent of $\mathcal{C}$, repeated upward applications of the rules of $\mathcal{C}$ eventually halts, regardless of the strategy employed. A calculus with this property is called *(strongly) terminating*, and it can be characterized as follows: there exists a well-founded relation $\prec$ on the sequents of $\mathcal{C}$ such that, for every rule application ρ in $\mathcal{C}$, if σ is the conclusion of ρ and σ' is any of its premises, then $\sigma' \prec \sigma$.

Following [6, 7, 8], the crucial step in achieving our goal is to decorate sequents with one of the labels b (blocked) or u (unblocked). The calculus for iCK4, we call Gbu-iCK4, is strongly terminating and it is shown to be complete via a dual refutation system, Rbu-iCK4. This refutation calculus enables the extraction of a countermodel for an unblocked sequent σ when the proof search for σ in Gbu-iCK4 fails. We note that, compared to the calculi for iSL introduced in [6], the case of iCK4 is more challenging, due to the presence of reflexive worlds (i.e., worlds w such that wRw holds) that are not admitted in the iSL-models. This characteristic complicates both the design of inference rules and the process of countermodel construction. Moreover, as a consequence of this feature, unlike the case of iSL, the calculus for iCK4 does not satisfy the standard subformula property. Instead, it satisfies a weaker version, which we refer to as the *extended subformula property*: every formula occurring in an Gbu-iCK4 proof

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tree is either a subformula of the root sequent or a subformula of the sequent obtained by removing all occurrences of the $\Box$ modality from the root sequent.

Regarding the comparison with existing literature, to the best of our knowledge, the only available calculus for iCK4 is the natural deduction system introduced in [2], which is not designed to support proof search effectively.

To complement our theoretical results with an applied contribution, we have implemented both the proof search procedure and the countermodel extraction in the Java framework JTabWB [9]; the implementation is available online at [10].

2. The Logic iCK4

Formulas, denoted by lowercase Greek letters, are built from an enumerable set of propositional variables $\mathcal{V}$, the constant $\bot$ and the connectives $\vee, \wedge, \to$ and $\Box$; $\neg\alpha$ is an abbreviation for $\alpha \to \bot$ and $\alpha \leftrightarrow \beta$ an abbreviation for $(\alpha \to \beta) \wedge (\beta \to \alpha)$. We denote multisets (and sets) of formulas by uppercase Greek letters. Let α be a formula and Γ a multiset of formulas; by $\Box\Gamma$ we denote the multiset $\{\Box\alpha | \alpha \in \Gamma\}$. By α^-, we denote the formula obtained from α by erasing every occurrence of $\Box$; Γ^- is the multiset $\{\alpha^- | \alpha \in \Gamma\}$. We write $\mathrm{Sf}(\alpha)$ to denote the set of the subformulas of α, including α itself; $\mathrm{Sf}(\Gamma)$ is the union of the sets $\mathrm{Sf}(\alpha)$, for every α in Γ. The size of α, denoted by $|\alpha|$, is the number of symbols in α; the size of Γ, denoted by $|\Gamma|$, is the sum of the sizes of formulas α in Γ, taking into account their multiplicity.

We introduce the semantics of logic iCK4; to simplify the presentation, we assume that iCK4 enjoys the finite model property (for a more general discussion see e.g. [11]). A *(bi-relational) strong frame* $\mathcal{F}$ is a tuple $\langle W, \leq, R, r \rangle$, where W is a finite non-empty set (worlds), $\leq$ (the intuitionistic relation) is a partial order with minimum element r (the root of $\mathcal{F}$), $R \subseteq W \times W$ is the modal relation satisfying the following properties: (normality) if $w_0 \leq w_1$ and $w_1 R w_2$, then $w_0 R w_2$; (strongness[1]) $R \subseteq \leq$. Note that these two properties imply the transitivity of R.

A *(bi-relational) strong model* $\mathcal{K}$ is a tuple $\langle W, \leq, R, r, V \rangle$ where $\langle W, \leq, R, r \rangle$ is a bi-relational strong frame, and V (the valuation function) is a map associating a subset of $\mathcal{V}$ to every $w \in W$ and satisfying the *persistence property*: $w_0 \leq w_1$ implies $V(w_0) \subseteq V(w_1)$.

Given a strong model $\mathcal{K}$, the *forcing relation* $\Vdash$ between worlds of $\mathcal{K}$ and formulas is defined as follows:

$\mathcal{K}, w \Vdash p$ iff $p \in V(w), \forall p \in \mathcal{V}$ $\qquad$ $\mathcal{K}, w \nVdash \bot$

$\mathcal{K}, w \Vdash \alpha \wedge \beta$ iff $\mathcal{K}, w \Vdash \alpha$ and $\mathcal{K}, w \Vdash \beta$ $\qquad$ $\mathcal{K}, w \Vdash \alpha \vee \beta$ iff $\mathcal{K}, w \Vdash \alpha$ or $\mathcal{K}, w \Vdash \beta$

$\mathcal{K}, w \Vdash \alpha \to \beta$ iff $\forall w' \geq w$, if $\mathcal{K}, w' \Vdash \alpha$ then $\mathcal{K}, w' \Vdash \beta$

$\mathcal{K}, w \Vdash \Box\alpha$ iff $\forall w' \in W$, if wRw' then $\mathcal{K}, w' \Vdash \alpha$.

Hereafter, we write $w \Vdash \varphi$ instead of $\mathcal{K}, w \Vdash \varphi$ when the model $\mathcal{K}$ at hand is clear from the context. It is easy to prove, by induction on the structure of a formula, that the forcing relation is persistent w.r.t. $\leq$ (hence, w.r.t. R, since $R \subseteq \leq$); formally:

Lemma 1 (Strong monotonicity lemma) *Let $\mathcal{K} = \langle W, \leq, R, r, V \rangle$ be a strong model. For every formula α, if $w \Vdash \alpha$ and $w \leq w'$, then $w' \Vdash \alpha$*

A world w *is reflexive* if wRw holds; in a strong model the following holds:

Lemma 2 *Let $\mathcal{K} = \langle W, \leq, R, r, V \rangle$ be a strong model and let $w \in W$.*

(i) If w is a reflexive world then, for every formula α, $w \Vdash \alpha \leftrightarrow \alpha^-$.

(ii) If $w \nVdash \Box\alpha$, then there exists $w^\star \in W$ such that $wRw^\star$, $w^\star \nVdash \alpha$ and either (a) $\forall w' \in W : w^\star R w'$, $w' \Vdash \alpha$ or (b) $w^\star$ is reflexive.

[1] Also called coreflection.

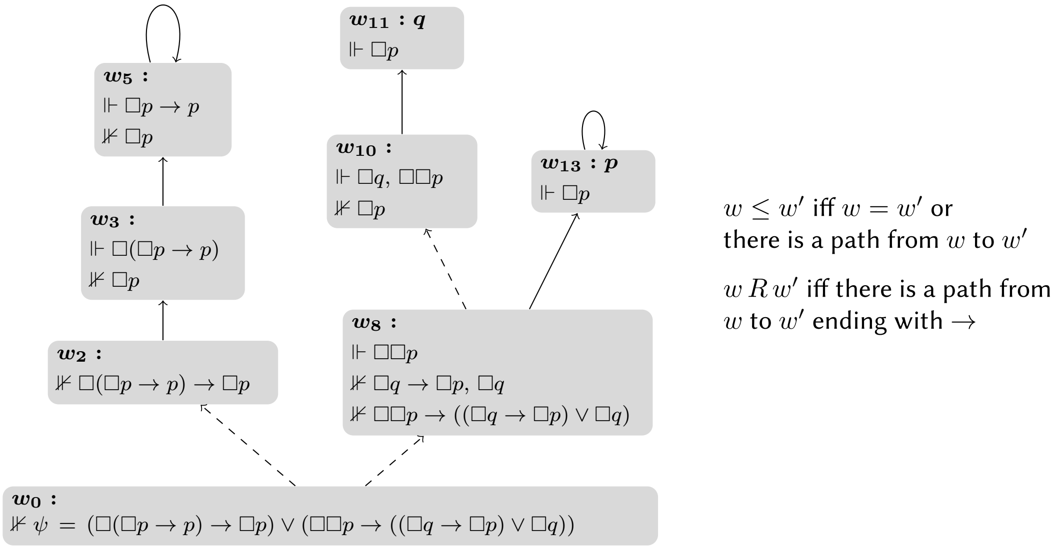

Figure 1: The countermodel for ψ described in Ex. 3, also referenced as $\mathrm{Mod}(\mathcal{D})$ in Ex. 16.

Proof. Point (i). Note that $w \Vdash \varphi \leftrightarrow \Box\varphi$, for every formula φ. Indeed, $w \Vdash \varphi \to \Box\varphi$ follows by strongness of $\mathcal{K}$, $w \Vdash \Box\varphi \to \varphi$ by reflexivity of w. Since α^- is obtained from α by replacing every subformula $\Box\varphi$ with φ, we get , $w \Vdash \alpha \leftrightarrow \alpha^-$.

Point (ii). Since $w \nVdash \Box\alpha$, there exists $w_\alpha \in W$ such that wRw_α and $w_\alpha \nVdash \alpha$. We build a finite sequence $\mathcal{S}$ of pairwise distinct worlds $w_0, \dots, w_n$ of W such that $w_0 R w_1 R \dots R w_n$ and $w_k \nVdash \alpha$ for every $0 \leq k \leq n$. We proceed as follows:

- We set $w_0 = w_\alpha$ (thus, wRw_0).
- Suppose that the last defined world of $\mathcal{S}$ is w_k ($k \geq 0$). If there exists w' such that $w' \notin \{w_0, \dots, w_k\}$ and $w_k R w'$ and $w' \nVdash \alpha$, we set $w_{k+1} = w'$; otherwise, the construction of $\mathcal{S}$ halts and w_k is the last world of $\mathcal{S}$.

Since the worlds in $\mathcal{S}$ are pairwise distinct and W is finite, the construction of $\mathcal{S}$ eventually halts. Let $w^\star$ be the last element of $\mathcal{S}$. We have $w^\star \nVdash \alpha$ and $w_0 R w_1 R \dots R w^\star$ hence, by transitivity of R, $wRw^\star$. If $w^\star$ is reflexive, then $w^\star$ matches (b). Let us assume that $w^\star$ is not reflexive; we show that (a) holds. Let us assume, by contradiction, that there exists w' such that $w^\star R w'$ and $w' \nVdash \alpha$. Note that $w' \notin \mathcal{S}$, otherwise, by transitivity of R, we would get $w^\star R w^\star$, against the hypothesis that $w^\star$ is not reflexive. Since $w' \notin \mathcal{S}$ and $w' \nVdash \alpha$, we can extend $\mathcal{S}$ by adding w', a contradiction ($w^\star$ is the last element of $\mathcal{S}$). This proves that, for every w' such that $w^\star R w'$, $w' \Vdash \alpha$; accordingly, $w^\star$ matches (a). $\Box$

Let Γ be a multiset of formulas. By $w \Vdash \Gamma$ we mean that $w \Vdash \varphi$ for every φ in Γ. The iCK4-consequence relation $\models_{\mathsf{iCK4}}$ is defined as follows:

$$\Gamma \models_{\mathsf{iCK4}} \varphi \quad \text{iff} \quad \forall \mathcal{K}\, \forall w \left(\mathcal{K}, w \Vdash \Gamma \Rightarrow \mathcal{K}, w \Vdash \varphi\right).$$

The logic iCK4 is defined as the set of formulas φ such that $\emptyset \models_{\mathsf{iCK4}} \varphi$. Hence, if $\varphi \notin$ iCK4, there exist a strong model $\mathcal{K}$ such that $r \nVdash \varphi$, with r the root of $\mathcal{K}$; we call $\mathcal{K}$ a *countermodel* for φ.

Example 3 Fig. 6 displays a countermodel $\mathcal{K}$ for the formula

$$\psi = \Box(\Box p \to p) \to \Box p) \vee (\Box\Box p \to ((\Box q \to \Box p) \vee \Box q))$$

ψ is the disjunction between the Gödel-Löb the axiom $\Box(\Box p \to p) \to \Box p$, characterizing logic iSL, and an instance of axiom $\Box r \to ((s \to r) \vee s)$ (with $r = \Box p$ and $s = \Box q$) characterizing the *Modalized*

$$\frac{}{\Gamma \overset{l}{\Rightarrow} \alpha}\ \mathrm{Ax}^{\rhd} \quad \text{if } \Gamma \rhd \alpha \qquad\qquad \frac{}{\bot, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\bot$$

$$\frac{\alpha, \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\wedge \qquad\qquad \frac{\Gamma \overset{l}{\Rightarrow} \alpha \qquad \Gamma \overset{l}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \wedge \beta}\ R\wedge$$

$$\frac{\alpha, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta \qquad \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \vee \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\vee \qquad\qquad \frac{\Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha_k}{\Gamma \overset{l}{\Rightarrow} \alpha_0 \vee \alpha_1}\ R\vee_k$$

$$\frac{\alpha \to \beta, \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha \qquad \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}{\alpha \to \beta, \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L\!\to$$

$$\frac{\Gamma \overset{l}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \to \beta}\ R\overset{\rhd}{\to} \quad \text{if } \Gamma \rhd \alpha \qquad\qquad \frac{\alpha, \Gamma \overset{\mathrm{u}}{\Rightarrow} \beta}{\Gamma \overset{l}{\Rightarrow} \alpha \to \beta}\ R\overset{\not\rhd}{\to} \quad \text{if } \Gamma \not\rhd \alpha$$

$$\frac{\Gamma, \Delta \overset{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Delta \overset{\mathrm{u}}{\Rightarrow} \Box\alpha}\ R^{\Box}_{\mathrm{u}} \qquad \frac{\Box\alpha, \Gamma, \Delta \overset{\mathrm{u}}{\Rightarrow} \alpha \qquad \Gamma^{-}, \Delta^{-} \overset{\mathrm{u}}{\Rightarrow} \alpha^{-}}{\Gamma, \Box\Delta \overset{\mathrm{b}}{\Rightarrow} \Box\alpha}\ R^{\Box}_{\mathrm{b}} \quad \text{if } \Gamma \cup \Box\Delta \not\rhd \Box\alpha$$

Figure 2: The calculus $\mathcal{C} = $ Gbu-iCK4 for iCK4 ($l \in \{\mathrm{b}, \mathrm{u}\}$, $k \in \{0, 1\}$).

Heyting calculus mHC. iSL and mHC are proper extensions of iCK4 and their proper axioms are not valid in iCK4. The worlds of $\mathcal{K}$ are w_0 (the root), w_2, w_3, w_5, w_8, w_{10}, w_{11}, w_{13}. The relations $\leq$ and R can be inferred by the displayed arrows, as accounted for in the figure. E.g., $w_0 \leq w_{10}$, since there is a path from w_0 and w_{10} (actually, a unique path); $w_0 \leq w_{11}$ and $w_0 R w_{11}$, since the path from w_0 and w_{11} ends with the solid arrow $\to$. However, it is not the case that $w_0 R w_{10}$, since the path from w_0 to w_{10} ends with the dashed arrow $\dashrightarrow$. The only reflexive worlds are w_5 and w_{13}. In each world w_k, the first line displays the value of $V(w_k)$; for instance, $V(w_0) = \emptyset$, $V(w_{11}) = \{q\}$. The remaining lines report (separated by commas) some of the formulas forced and not forced in w_k. Since $w_0 \nVdash \psi$, $\mathcal{K}$ is a countermodel for ψ. $\Diamond$

3. The Sequent Calculus Gbu-iCK4

In this section, we introduce the Gbu-iCK4 calculus (Gentzen calculus for iCK4 with b, u-labelled sequents), which we will simply refer to as $\mathcal{C}$ from now on. The calculus $\mathcal{C}$ acts on labelled sequents σ of the form $\Gamma \overset{l}{\Rightarrow} \delta$, with $l \in \{\mathrm{b}, \mathrm{u}\}$ where Γ is a multiset of formulas and δ is a formula; Γ and δ are referred to as the *lhs* ($\mathrm{lhs}(\sigma)$) and the *rhs* ($\mathrm{rhs}(\sigma)$) (left/right hand side) of σ respectively. We call l-sequent a sequent with label l. Let $\sigma = \Gamma \overset{l}{\Rightarrow} \delta$; with σ^{-} we denote the sequent $\Gamma^{-} \overset{l}{\Rightarrow} \delta^{-}$, $\mathrm{Sf}(\sigma) = \mathrm{Sf}(\Gamma \cup \{\delta\})$ (the set of subformulas of σ) and $\mathrm{Sf}^{+}(\sigma) = \mathrm{Sf}(\sigma) \cup \mathrm{Sf}(\sigma^{-})$ (the set of extended subformulas of σ). To define the calculus, we introduce the following evaluation relation:

Definition 4 (Evaluation) *Let Γ be a multiset of formulas and φ a formula.* Γ evaluates φ, *written* $\Gamma \rhd \varphi$, *iff φ matches the following BNF:*

$$\varphi \coloneqq \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \alpha \mid \alpha \vee \varphi \mid \alpha \to \varphi \mid \Box\varphi \qquad \textit{with } \gamma \in \Gamma \textit{ and } \alpha \textit{ any formula.}$$

By $\Gamma \rhd \Delta$ we mean that $\Gamma \rhd \delta$, for every $\delta \in \Delta$. We state some properties of the evaluation relation which are proved in [6].

Lemma 5

(i) If $\Gamma \rhd \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \rhd \varphi$.

(ii) If $\Gamma \cup \Delta \rhd \varphi$ and $\Gamma' \rhd \Delta$, then $\Gamma \cup \Gamma' \rhd \varphi$.

(iii) If $\Gamma \rhd \varphi$, then $\Gamma \cap \mathrm{Sf}(\varphi) \rhd \varphi$.

(iv) If $\Gamma \rhd \varphi$ and $\mathcal{K}, w \Vdash \Gamma$, then $\mathcal{K}, w \Vdash \varphi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p,\, \neg\Box p,\, \alpha \overset{b}{\Rightarrow} \Box p_{(5)}}\,\mathrm{Ax}^{\triangleright} \qquad \dfrac{}{\sigma_6}\,L\bot}{\Box p,\, \neg\Box p,\, \alpha \overset{u}{\Rightarrow} p_{(4)}}\,L\to \qquad \dfrac{\dfrac{}{\neg p,\, \neg\neg p \overset{b}{\Rightarrow} \neg p_{(8)}}\,\mathrm{Ax}^{\triangleright} \qquad \dfrac{}{\sigma_9}\,L\bot}{\neg p,\, \neg\neg p \overset{u}{\Rightarrow} p_{(7)}}\,L\to}{\neg\Box p,\, \alpha \overset{b}{\Rightarrow} \Box p_{(3)}}\,R^{\Box}_{\mathrm{b}} \qquad \dfrac{}{\sigma_{10}}\,L\bot}{\neg\Box p,\, \alpha \overset{u}{\Rightarrow} \bot_{(2)}}\,L\to}{\neg\Box p \wedge \alpha \overset{u}{\Rightarrow} \bot_{(1)}}\,L\wedge}{\overset{u}{\Rightarrow} \neg(\neg\Box p \wedge \alpha)_{(0)}}\,R\overset{\not\triangleright}{\to}$$

$\alpha = \Box\neg\Box\neg p$
$\sigma_6 = \bot, \Box p, \alpha \overset{u}{\Rightarrow} p$
$\sigma_9 = \bot, \neg p \overset{u}{\Rightarrow} p$
$\sigma_{10} = \bot, \alpha \overset{u}{\Rightarrow} \bot$

Figure 3: A $\mathcal{C}$-derivation of $\sigma_0 = \overset{u}{\Rightarrow}\neg(\Box p \wedge \Box\neg\Box\neg p)$.

The rules of the calculus $\mathcal{C}$ are displayed in Fig. 2. They consist of the axiom rules $\mathrm{Ax}^{\triangleright}$ and $L\bot$, together with left/right rules for each logical connective and right rules for $\Box$. For calculi and derivations we use the definitions and notations of [12]. Applications of rules are depicted as trees with sequents as nodes, we call them *$\mathcal{C}$-trees*; a *$\mathcal{C}$-derivation* is a tree where every leaf is an axiom sequent, i.e., a sequent obtained by applying a zero-premise rule of $\mathcal{C}$. A sequent σ is provable in $\mathcal{C}$, and we write $\vdash_{\mathcal{C}} \sigma$, if there exists a $\mathcal{C}$-derivation with root sequent σ.

The calculus is oriented to backward proof search, where rules are applied bottom-up. If the conclusion of a rule has label b, the (bottom-up) application of left rules is blocked. There are two rules for right implication, namely $R\overset{\triangleright}{\to}$ and $R\overset{\not\triangleright}{\to}$; the choice between them is settled by the evaluation relation $\triangleright$. Right $\Box$-formulas are handled by rules $R^{\Box}_{\mathrm{u}}$ and $R^{\Box}_{\mathrm{b}}$; here the choice is determined by the label of the conclusion. We remark that if $\sigma = \Gamma, \Box\Delta \overset{\mathrm{b}}{\Rightarrow} \Box\alpha$ and $\Gamma \cup \Box\Delta \triangleright \Box\alpha$, then σ is an axiom sequent (see rule $\mathrm{Ax}^{\triangleright}$) and an application of rule $R^{\Box}_{\mathrm{b}}$ to σ is prevented by the side condition of $R^{\Box}_{\mathrm{b}}$. In backward proof search, a b-sequent starts the construction of a branch only containing b-sequents, where only right rules are applied. This phase ends either when an axiom sequent is obtained or when no rule can be applied or when one of the rules turning a label b into u is applied (namely, rules $R\overset{\not\triangleright}{\to}$ and $R^{\Box}_{\mathrm{b}}$).

Inspecting the rules of the calculus, one can easily check that every rule, except $R^{\Box}_{\mathrm{b}}$, meets the subformula property, i.e., every formula occurring in the premises is a subformula of a formula occurring in the conclusion. This does not hold for $R^{\Box}_{\mathrm{b}}$ since the formulas in the rightmost premise $\Gamma^-, \Delta^- \overset{\mathrm{u}}{\Rightarrow} \alpha^-$ might not belong to $\mathrm{Sf}(\Gamma, \Box\Delta \overset{\mathrm{b}}{\Rightarrow} \Box\alpha)$. However, every formula in $\Gamma^-, \Delta^- \overset{\mathrm{u}}{\Rightarrow} \alpha^-$ belongs to $\mathrm{Sf}^+(\Gamma, \Box\Delta \overset{\mathrm{b}}{\Rightarrow} \Box\alpha)$. Accordingly, the calculus $\mathcal{C}$ meets a weaker form of the subformula property, we call *extended subformula property*, namely: every formula occurring in a $\mathcal{C}$-tree having σ as root belongs to $\mathrm{Sf}^+(\sigma)$.

Example 6 In Fig.3 we show a $\mathcal{C}$-derivation $\mathcal{D}$ of the u-sequent $\sigma_0 = \overset{\mathrm{u}}{\Rightarrow} \neg(\neg\Box p \wedge \alpha)$, where $\alpha = \Box\neg\Box\neg p$ (we recall that $\neg\beta$ is an abbreviation for $\beta \to \bot$). In $\mathcal{D}$ sequents are marked with an index (n) and, hereafter, are referred to as σ_n. $\mathcal{D}$ highlights some of the peculiarities of $\mathcal{C}$. In backward proof search, σ_3 is obtained by a (backward) application of rule $L\!\to$ to σ_2; the label b in σ_2 is crucial to block the application of rule $L\!\to$, which would generate an infinite branch. In sequent σ_4 (the left premise of rule $R^{\Box}_{\mathrm{b}}$ with conclusion σ_3) the key feature is the presence of the formula $\Box p$ (also called diagonal formula); without it, the sequent σ_4 would be $\neg\Box p, \alpha \overset{\mathrm{u}}{\Rightarrow} p$ and, after the application of $L\!\to$ (the only applicable rule), the left premise would be $\sigma_5 = \neg\Box p, \alpha \overset{\mathrm{b}}{\Rightarrow} \Box p$, which yields a loop ($\sigma_5 = \sigma_3$). We stress that the sequent σ_7, corresponding to the right premise of $R^{\Box}_{\mathrm{b}}$, is a pure intuitionistic sequent, since it is obtained from σ_3 by removing all the boxes. $\Diamond$

The following theorem states the main properties of $\mathcal{C}$:

Theorem 7

(i) $\mathcal{C}$ enjoys the extended subformula property.

(ii) $\mathcal{C}$ is terminating.

(iii) $\vdash_{\mathcal{C}} \Gamma \xRightarrow{l} \delta$ *implies* $\Gamma \models_{\mathsf{iCK4}} \delta$ *(Soundness).*

(iv) $\Gamma \models_{\mathsf{iCK4}} \delta$ *implies* $\vdash_{\mathcal{C}} \Gamma \xRightarrow{\mathrm{u}} \delta$ *(Completeness).*

We remark that in soundness l is any label; instead, in completeness the label is set to u. For instance, since $p \vee q \models_{\mathsf{iCK4}} q \vee p$, completeness guarantees that the u-sequent $\sigma^{\mathrm{u}} = p \vee q \xRightarrow{\mathrm{u}} q \vee p$ is provable in $\mathcal{C}$. A $\mathcal{C}$-derivation of σ^{u} is obtained by first (upwards) applying rule $L\vee$ to σ^{u} and then one of the rules $R\vee_0$ or $R\vee_1$; if we first apply a right rule, we are stuck (e.g., if we apply $R\vee_0$ to σ^{u}, we get the unprovable sequent $p \vee q \xRightarrow{\mathrm{u}} q$). On the contrary, the b-sequent $p \vee q \xRightarrow{\mathrm{b}} q \vee p$ is not provable in $\mathcal{C}$, since the label b inhibits the application of rule $L\vee$ and forces the application of a right rule.

The soundness of the calculus $\mathcal{C}$ can be proved by showing that its rules preserve the iCK4-consequence relation $\models_{\mathsf{iCK4}}$, namely:

Lemma 8 *Let ρ be an application of rule of $\mathcal{C}$ having conclusion $\Gamma \xRightarrow{l} \delta$ and premises $\Gamma_1 \xRightarrow{l_1} \delta_1,\dots, \Gamma_n \xRightarrow{l_n} \delta_n$ ($n \in \{1,2\}$). If $\Gamma_i \models_{\mathsf{iCK4}} \delta_i$ for every $i \in \{1,\dots,n\}$, then $\Gamma \models_{\mathsf{iCK4}} \delta$.*

Proof. We only treat the cases of rule $R^{\Box}_{\mathrm{b}}$, the other cases being straightforward. Let us suppose that $\Gamma, \Box\Delta \not\models_{\mathsf{iCK4}} \Box\alpha$; we show that either $\Box\alpha, \Gamma, \Delta \not\models_{\mathsf{iCK4}} \alpha$ or $\Gamma^-, \Delta^- \not\models_{\mathsf{iCK4}} \alpha^-$. Since $\Gamma, \Box\Delta \not\models_{\mathsf{iCK4}} \Box\alpha$, there exists a model $\mathcal{K}$ and a world w of $\mathcal{K}$ s.t. $w \Vdash \Gamma, \Box\Delta$ and $w \nVdash \Box\alpha$. By Lemma 2(ii) there exists $w^\star$ such that $wRw^\star$, $w^\star \nVdash \alpha$ and one of the conditions (a) or (b) holds. Note that $w^\star \Vdash \Delta$ and, since $w \leq w^\star$, $w^\star \Vdash \Gamma$. Let us assume that (a) holds; then, $w^\star \Vdash \Box\alpha$ and $w^\star \nVdash \alpha$, hence $\Box\alpha, \Gamma, \Delta \not\models_{\mathsf{iCK4}} \alpha$. Let us assume that (b) holds. Then, $w^\star$ is reflexive hence, by Lemma 2(i), we get $w^\star \Vdash \Gamma^-, \Delta^-$ and $w^\star \nVdash \alpha^-$; we conclude $\Gamma^-, \Delta^- \not\models_{\mathsf{iCK4}} \alpha^-$. $\Box$

To prove the termination of $\mathcal{C}$ we need to introduce a proper well-founded relation $\prec_{\mathrm{bu}}$ on labelled sequents. The main problem stems from rule $L\rightarrow$. Let σ and σ' be the conclusion and the left premise of an application of rule $L\rightarrow$; we stipulate that $\sigma' \prec_{\mathrm{bu}} \sigma$ since σ' has label b and σ has label u; thus, we establish that b weighs less than u. Now, we need a way out to accommodate the rules $R\xrightarrow{\not\triangleright}$ and $R^{\Box}_{\mathrm{b}}$ that, read bottom-up, switch b with u. We observe that the lhs of the left premise evaluates a new formula; e.g., in the application of rule $R\xrightarrow{\not\triangleright}$ having premise $\alpha, \Gamma \xRightarrow{\mathrm{u}} \beta$ and conclusion $\Gamma \xRightarrow{l} \alpha \rightarrow \beta$, it holds that $\Gamma \not\triangleright \alpha$ (side condition) and $\Gamma \cup \{\alpha\} \triangleright \alpha$ (definition of $\triangleright$); this suggests that here we can exploit the evaluation relation.

Let Ev be defined as follows:

$$\mathrm{Ev}(\Gamma \xRightarrow{l} \delta) \;=\; \{\, \varphi \mid \varphi \in \mathrm{Sf}(\Gamma \cup \{\delta\}) \text{ and } \Gamma \triangleright \varphi \,\}$$

Note that $\mathrm{Ev}(\sigma) \subseteq \mathrm{Sf}(\sigma)$. We also have to take into account the size of a sequents, where $|\Gamma \xRightarrow{l} \delta| = |\Gamma| + |\delta|$.

Finally, we observe that the conclusion of rule $R^{\Box}_{\mathrm{b}}$ is a modal sequent (it contains at least one $\Box$), while the right premise is intuitionistic (no $\Box$). To convey this property in the definition of $\prec_{\mathrm{bu}}$, we introduce the following function:

$$\mathrm{isModal}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ contains at least one } \Box \\ 0 & \text{otherwise} \end{cases}$$

We write $\sigma' \prec_{\mathrm{bu}} \sigma$ iff one of the following conditions holds:

(a) $\mathrm{isModal}(\sigma') < \mathrm{isModal}(\sigma)$;

(b) $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma)$ and $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$;

(c) $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma)$ and $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$;

(d) $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma)$ and $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ and $\mathrm{label}(\sigma') = \mathrm{b}$ and $\mathrm{label}(\sigma) = \mathrm{u}$;

(e) $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma)$ and $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ and $\mathrm{label}(\sigma') = \mathrm{label}(\sigma)$ and $|\sigma'| < |\sigma|$.

Proposition 9 *The relation $\prec_{\rm bu}$ is well-founded.*

Proof. Assume, by contradiction, that there is an infinite descending chain $\ldots \prec_{\rm bu} \sigma_1 \prec_{\rm bu} \sigma_0$. There is $k \geq 0$ such that $\mathrm{isModal}(\sigma_j)$ stabilizes, namely: $\mathrm{isModal}(\sigma_j) = \mathrm{isModal}(\sigma_k)$ for every $j \geq k$. We have $\mathrm{Sf}(\sigma_k) \supseteq \mathrm{Sf}(\sigma_{k+1}) \supseteq \ldots$; since $\mathrm{Sf}(\sigma_k)$ is finite, the sets $\mathrm{Sf}(\sigma_j)$ eventually stabilize, namely: there is $l \geq k$ such that $\mathrm{Sf}(\sigma_j) = \mathrm{Sf}(\sigma_l)$ for every $j \geq l$. Since $\mathrm{Ev}(\sigma_j) \subseteq \mathrm{Sf}(\sigma_j)$, we get $\mathrm{Ev}(\sigma_l) \subseteq \mathrm{Ev}(\sigma_{l+1}) \subseteq \ \ldots \ \subseteq \mathrm{Sf}(\sigma_l)$. Since $\mathrm{Sf}(\sigma_l)$ is finite, there is $m \geq l$ such that $\mathrm{Ev}(\sigma_j) = \mathrm{Ev}(\sigma_m)$ for every $j \geq m$. This implies that there exists $n \geq m$ such that all the sequents $\sigma_n, \sigma_{n+1}, \ldots$ have the same label; accordingly $|\sigma_n| > |\sigma_{n+1}| > |\sigma_{n+2}| > \ldots \geq 0$, a contradiction. We conclude that $\prec_{\rm bu}$ is well-founded. ☐

To prove that the rules of $\mathcal{C}$ are decreasing w.r.t.$\prec_{\rm bu}$, we need the following:

Lemma 10 *Let ρ be an application of a rule of $\mathcal{C}$, let σ be the conclusion of ρ and σ' any of the premises of ρ. For every formula φ, if $\mathrm{lhs}(\sigma) \triangleright \varphi$ and σ' is not the right premise of $R^{\Box}_{\rm b}$, then $\mathrm{lhs}(\sigma') \triangleright \varphi$.*

Proof. The assertion can be proved by applying Lemma 5. For instance, let $\sigma = \Gamma, \Box\Delta \stackrel{\rm u}{\Rightarrow} \Box\alpha$ and $\sigma' = \Gamma, \Delta \stackrel{\rm u}{\Rightarrow} \alpha$ be the conclusion and the premise of rule $R^{\Box}_{\rm u}$; assume that $\Gamma \cup \Box\Delta \triangleright \varphi$. Since $\Delta \triangleright \Box\Delta$, by Lemma 5(ii) get $\Gamma \cup \Delta \triangleright \varphi$. ☐

Proposition 11 *Every rule of the calculus $\mathcal{C}$ is decreasing w.r.t. $\prec_{\rm bu}$.*

Proof. Let σ and σ' be the conclusion and one of the premises of an application of a rule of $\mathcal{C}$. Note that $\mathrm{isModal}(\sigma') \leq \mathrm{isModal}(\sigma)$. If σ' is not the right premise of $R^{\Box}_{\rm b}$, it holds that $\mathrm{Sf}(\sigma') \subseteq \mathrm{Sf}(\sigma)$; moreover, if $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$, by Lemma 10 we get $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. We can prove $\sigma' \prec_{\rm bu} \sigma$ by a case analysis; we only detail two significant cases.

$$\frac{\sigma' \ = \ \alpha \to \beta, \Gamma \stackrel{\rm b}{\Rightarrow} \alpha \qquad \beta, \Gamma \stackrel{\rm u}{\Rightarrow} \delta}{\sigma \ = \ \alpha \to \beta, \Gamma \stackrel{\rm u}{\Rightarrow} \delta}\ L\!\to$$

If $\mathrm{isModal}(\sigma') < \mathrm{isModal}(\sigma)$, then $\sigma' \prec_{\rm bu} \sigma$ by point (a) of the definition; otherwise we have $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma)$. If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\rm bu} \sigma$ by point (b) of the definition; otherwise, as discussed above, it holds that $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. If $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$, then $\sigma' \prec_{\rm bu} \sigma$ by point (c); otherwise, $\sigma' \prec_{\rm bu} \sigma$ follows by point (d).

$$\frac{\sigma' \ = \ \Box\alpha, \Gamma, \Delta \stackrel{\rm u}{\Rightarrow} \alpha \qquad \sigma'' \ = \ \Gamma^-, \Delta^- \stackrel{\rm u}{\Rightarrow} \alpha^-}{\sigma \ = \ \Gamma, \Box\Delta \stackrel{l}{\Rightarrow} \Box\alpha}\ R^{\Box}_{\rm b} \qquad \Gamma \cup \Box\Delta \not\triangleright \Box\alpha$$

Note that $\mathrm{isModal}(\sigma') = \mathrm{isModal}(\sigma) = 1$. If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\rm bu} \sigma$ by point (b). Otherwise, $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. Note that $\Box\alpha \in \mathrm{Ev}(\sigma')$ and, by the side condition, $\Box\alpha \notin \mathrm{Ev}(\sigma)$. This implies that $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$, hence $\sigma' \prec_{\rm bu} \sigma$ by point (c). As for the right premise, we have $\mathrm{isModal}(\sigma'') = 0$ and $\mathrm{isModal}(\sigma) = 1$, thus $\sigma'' \prec_{\rm bu} \sigma$ by point (a). ☐

By Prop. 9 and 11, we conclude that the calculus $\mathcal{C}$ is terminating.

4. The Refutation Calculus Rbu-iCK4

A common technique to prove the completeness of a sequent calculus consists in showing that, whenever a sequent σ is not provable in the calculus, then a countermodel for σ can be built; we prove the completeness of $\mathcal{C}$ according with this plan. Following the ideas in [13, 7, 8, 14], we formalize the notion of "non-provability in $\mathcal{C}$" by introducing the refutation calculus Rbu-iCK4, a dual calculus to $\mathcal{C}$ = Gbu-iCK4. Hereafter, we will refer to Rbu-iCK4 simply as $\mathcal{R}$. Sequents of $\mathcal{R}$, called *antisequents*, have the form $\Gamma \stackrel{l}{\not\Rightarrow} \delta$. Intuitively, a derivation in $\mathcal{R}$ of $\Gamma \stackrel{l}{\not\Rightarrow} \delta$ witnesses that the sequent $\Gamma \stackrel{l}{\Rightarrow} \delta$ is refutable, that is, not provable, in $\mathcal{C}$. Henceforth, $\Gamma^{\rm at}$ denotes a finite multiset of propositional variables, $\Gamma^{\to}$ denotes a finite multiset of $\to$-formulas (i.e., formulas of the kind $\alpha \to \beta$). The axioms of $\mathcal{R}$ are the *irreducible antisequents*, namely the antisequents $\Gamma \stackrel{l}{\not\Rightarrow} \delta$ such that the corresponding dual sequents

Γ^{at} is a multiset of propositional variables, $\Gamma^{\to}$ is a multiset of $\to$-formulas

$$\frac{}{\sigma}\,\mathrm{Irr}\quad \text{if } \sigma \text{ is irreducible} \qquad \frac{\alpha,\beta,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha\wedge\beta,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L\wedge \qquad \frac{\Gamma \overset{l}{\not\Rightarrow} \alpha_k}{\Gamma \overset{l}{\not\Rightarrow} \alpha_0\wedge\alpha_1}\,R\wedge_k$$

$$\frac{\alpha_k,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha_0\vee\alpha_1,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L\vee_k \qquad \frac{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \qquad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\vee\beta}\,R\vee \qquad \frac{\beta,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha\to\beta,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L{\to}$$

$$\frac{\Gamma \overset{l}{\not\Rightarrow} \beta}{\Gamma \overset{l}{\not\Rightarrow} \alpha\to\beta}\,R\overset{\triangleright}{\to}\ \Gamma\triangleright\alpha \qquad \frac{\alpha,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\Gamma \overset{l}{\not\Rightarrow} \alpha\to\beta}\,R\overset{\not\triangleright}{\to}\ \Gamma\not\triangleright\alpha \qquad \frac{\Gamma^{-} \overset{\mathrm{u}}{\not\Rightarrow} \alpha^{-}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\,\mathrm{Ref}\ \Gamma\not\triangleright\Box\alpha$$

$$\frac{\Box\alpha,\Gamma,\Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\Gamma,\Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\,R^{\Box}_{\mathrm{b}}\ \Gamma\not\triangleright\Box\alpha \qquad \frac{\{\,\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\,\}_{\alpha\to\beta\in\Gamma^{\to}}}{\underbrace{\Gamma^{\mathrm{at}},\Gamma^{\to},\Box\Delta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}\ \begin{array}{l}\Gamma^{\to}\neq\emptyset\\ \delta\in(\mathcal{V}\cup\{\bot\})\setminus\Gamma^{\mathrm{at}}\end{array}$$

$$\frac{\{\,\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\,\}_{\alpha\to\beta\in\Gamma^{\to}} \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0 \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_1}{\underbrace{\Gamma^{\mathrm{at}},\Gamma^{\to},\Box\Delta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \delta_0\vee\delta_1}\,\mathrm{S}^{\vee}_{\mathrm{u}} \qquad \frac{\{\,\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\,\}_{\alpha\to\beta\in\Gamma^{\to}} \quad \Gamma^{\mathrm{at}},\Gamma^{\to},\Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\underbrace{\Gamma^{\mathrm{at}},\Gamma^{\to},\Box\Delta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta}\,\mathrm{S}^{\Box}_{\mathrm{u}}$$

Figure 4: The refutation calculus $\mathcal{R} = $ Rbu-iCK4 ($l \in \{\mathrm{b},\mathrm{u}\}$, $k \in \{0,1\}$).

$\Gamma \overset{l}{\Rightarrow} \delta$ are not the conclusion of any of the rules of $\mathcal{C}$. Irreducible antisequents are characterized as follows:

Definition 12 *An antisequent σ is* irreducible *iff $\sigma = \Gamma^{\mathrm{at}},\Gamma^{\to},\Box\Delta \overset{l}{\not\Rightarrow} \delta$ and both*

(i) $\delta \in (\mathcal{V}\cup\{\bot\})\setminus\Gamma^{\mathrm{at}}$ and

(ii) $l = \mathrm{b}$ or $\Gamma^{\to} = \emptyset$.

The rules of $\mathcal{R}$ are displayed in Fig. 4. In rules $\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$, $\mathrm{S}^{\vee}_{\mathrm{u}}$ and $\mathrm{S}^{\Box}_{\mathrm{u}}$ (we call *Succ rules*) the notation $\{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha\}_{\alpha\to\beta\in\Gamma^{\to}}$ means that, for every $\alpha\to\beta\in\Gamma^{\to}$, the b-antisequent $\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$ is a premise of the rule. Note that all of the Succ rules have at least one premise (in rule $\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$ this is imposed by the condition $\Gamma^{\to}\neq\emptyset$). $\mathcal{R}$-trees and $\mathcal{R}$-derivations are defined analogously to the case of the sequent calculus $\mathcal{C}$. The next theorem, proved below, states the soundness of $\mathcal{R}$:

Theorem 13 (Soundness of $\mathcal{R}$) *If $\vdash_{\mathcal{R}} \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$, then $\Gamma \not\models_{\mathsf{iCK4}} \delta$.*

Example 14 In Fig. 5 we show an $\mathcal{R}$-derivation $\mathcal{D}$ of $\sigma_0 = \ \overset{\mathrm{u}}{\not\Rightarrow} \psi$ where ψ is the same formula considered in Ex. 3. By Th. 13, we get $\not\models_{\mathsf{iCK4}} \psi$, namely $\psi \notin$ iCK4. $\Diamond$

Countermodel extraction. A strong model $\mathcal{K}$ with root r is a *model for* $\sigma = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ (a countermodel for $\overline{\sigma} = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$, respectively) iff $r \Vdash \Gamma$ and $r \nVdash \delta$; thus $\mathcal{K}$ certifies that $\Gamma \not\models_{\mathsf{iCK4}} \delta$. Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of a u-antisequent σ_0^{u}; we show that from $\mathcal{D}$ we can extract a model $\mathrm{Mod}(\mathcal{D})$ for σ_0^{u}. A u-antisequent σ of $\mathcal{D}$ is *prime* iff σ is the conclusion of rule Irr or of a Succ rule. We introduce the relations $\preceq$ and $\prec$ between antisequents occurring in $\mathcal{D}$:

- $\sigma_1 \prec \sigma_2$ iff σ_1 and σ_2 belong to the same branch of $\mathcal{D}$ and σ_1 is below σ_2;
- $\sigma_1 \preceq \sigma_2$ iff either $\sigma_1 = \sigma_2$ or $\sigma_1 \prec \sigma_2$.

Let W be the set of prime antisequents occurring in $\mathcal{D}$. We introduce a map Ψ between the u-antisequents of $\mathcal{D}$ and W.

- $\Psi(\sigma^{\mathrm{u}}) = \sigma_p^{\mathrm{u}}$ iff σ_p^{u} is the $\preceq$-minimum prime antisequent σ such that $\sigma^{\mathrm{u}} \preceq \sigma$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Irr}
\UnaryInfC{$p \to p \overset{b}{\not\Rightarrow} p_{(6)}$}
\RightLabel{$\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$}
\UnaryInfC{$p \to p \overset{u}{\not\Rightarrow} p_{(5)}$}
\RightLabel{Ref}
\UnaryInfC{$\Box p \to p \overset{b}{\not\Rightarrow} \Box p_{(4)}$}
\RightLabel{$\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$}
\UnaryInfC{$\Box p \to p \overset{u}{\not\Rightarrow} p_{(3)}$}
\RightLabel{$\mathrm{S}^{\Box}_{\mathrm{u}}$}
\UnaryInfC{$\Box(\Box p \to p) \overset{u}{\not\Rightarrow} \Box p_{(2)}$}
\RightLabel{$R\overset{\not\triangleright}{\to}$}
\UnaryInfC{$\overset{b}{\not\Rightarrow} \Box(\Box p \to p) \to \Box p_{(1)}$}
\AxiomC{}
\RightLabel{Irr}
\UnaryInfC{$\Box p, q \overset{u}{\not\Rightarrow} p_{(11)}$}
\RightLabel{$\mathrm{S}^{\Box}_{\mathrm{u}}$}
\UnaryInfC{$\Box\Box p, \Box q \overset{u}{\not\Rightarrow} \Box p_{(10)}$}
\RightLabel{$R\overset{\not\triangleright}{\to}$}
\UnaryInfC{$\Box\Box p \overset{b}{\not\Rightarrow} \Box q \to \Box p_{(9)}$}
\AxiomC{}
\RightLabel{Irr}
\UnaryInfC{$p \overset{u}{\not\Rightarrow} q_{(13)}$}
\RightLabel{Ref}
\UnaryInfC{$\Box\Box p \overset{b}{\not\Rightarrow} \Box q_{(12)}$}
\RightLabel{$\mathrm{S}^{\vee}_{\mathrm{u}}$}
\BinaryInfC{$\Box\Box p \overset{u}{\not\Rightarrow} (\Box q \to \Box p) \vee \Box q_{(8)}$}
\RightLabel{$R\overset{\not\triangleright}{\to}$}
\UnaryInfC{$\overset{b}{\not\Rightarrow} \Box\Box p \to ((\Box q \to \Box p) \vee \Box q)_{(7)}$}
\RightLabel{$\mathrm{S}^{\vee}_{\mathrm{u}}$}
\BinaryInfC{$\overset{u}{\not\Rightarrow} (\Box(\Box p \to p) \to \Box p) \vee (\Box\Box p \to ((\Box q \to \Box p) \vee \Box q))_{(0)}$}
\end{prooftree}
$$

Figure 5: The $\mathcal{R}$-derivation $\mathcal{D}$ of $\sigma_0 = \overset{u}{\not\Rightarrow}\Box(\Box p \to p) \to \Box p) \vee (\Box\Box p \to ((\Box q \to \Box p) \vee \Box q))$.

One can easily check that Ψ is well-defined and $\Psi(\sigma_p) = \sigma_p$, for every prime σ_p. Let σ_1 and σ_2 be two elements of W.

- $\sigma_1 R^+ \sigma_2$ iff there is a u-antisequent σ' such that $\sigma_1 \prec \sigma' \preceq \sigma_2$ and σ' is either the premise of rule $R^{\Box}_{\mathrm{b}}$ or the rightmost premise of $\mathrm{S}^{\Box}_{\mathrm{u}}$ or the premise of Ref.
- $\sigma_1 R^* \sigma_2$ iff $\sigma_1 = \sigma_2$ and there exists an u-antisequent σ in $\mathcal{D}$ such that σ is the premise of rule Ref and $\sigma_1 = \Psi(\sigma)$.

We define $\mathrm{Mod}(\mathcal{D})$ as the structure $\langle W, \leq, R, \sigma^{\mathrm{u}}_r, V\rangle$ where:

- W is the set of the prime antisequents of $\mathcal{D}$;
- $\leq$ is the restriction of $\preceq$ to W;
- R is the transitive closure of $R^+ \cup R^*$;
- σ^{u}_r is the $\leq$-minimum prime antisequent of $\mathcal{D}$;
- $V(\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta) = \Gamma \cap \mathcal{V}$.

It is easy to check that $\mathrm{Mod}(\mathcal{D})$ is a strong model; in particular, σ^{u}_r exists since the antisequent at the root of $\mathcal{D}$ has label u. The map Ψ defined above is referred to as the *canonical map* Ψ between the u-antisequents of $\mathcal{D}$ and $\mathrm{Mod}(\mathcal{D})$.

We state the main properties of $\mathrm{Mod}(\mathcal{D})$.

Theorem 15 *Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of a* u*-antisequent σ^{u}_0.*

(i) For every u*-antisequent $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$, $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.*

(ii) $\mathrm{Mod}(\mathcal{D})$ is a model for σ^{u}_0.

Point (ii) follows from (i) and the fact that $\Psi(\sigma^{\mathrm{u}}_0)$ is the root of $\mathrm{Mod}(\mathcal{D})$. The proof of (i) is deferred below. Note that point (ii) of Th. 15 immediately implies the soundness of $\mathcal{R}$ (Th. 13).

Example 16 In Fig. 6 we represent the structure of the $\mathcal{R}$-derivation $\mathcal{D}$ in Fig. 5, displaying the information relevant to the definition of $\mathrm{Mod}(\mathcal{D})$. The model $\mathrm{Mod}(\mathcal{D})$ for σ_0 coincides with the strong model in Fig. 1 (described in Ex. 3), where w_k is an alias for σ_k. Fig. 6 also reports the canonical map Ψ and the relations $\ll$, $\ll_R$ and $\ll^*_R$ defined below. $\Diamond$

Proof search. We investigate more deeply the duality between $\mathcal{C}$ and $\mathcal{R}$. A sequent $\sigma = \Gamma \overset{l}{\Rightarrow} \delta$ is *regular* iff $l = \mathrm{u}$ or $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$; by $\overline{\sigma}$ we denote the antisequent $\Gamma \overset{l}{\not\Rightarrow} \delta$. Let σ be a regular sequent; in the next proposition we show that either σ is provable in $\mathcal{C}$ or $\overline{\sigma}$ is provable in $\mathcal{R}$. The proof conveys a proof search strategy to build the proper derivation, based on backward application of the rules of $\mathcal{C}$. We give priority to the *invertible rules* of $\mathcal{C}$, namely: $L\wedge$, $R\wedge$, $L\vee$, $R\overset{\triangleright}{\to}$, $R\overset{\not\triangleright}{\to}$, $R^{\Box}_{\mathrm{b}}$; as

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\sigma_6\bullet}\,\mathrm{Irr}}{p \to p \stackrel{u}{\not\Rightarrow} p_{(5)}}\,\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}}{\sigma_4\bullet}\,\mathrm{Ref}}{\Box p \to p \stackrel{u}{\not\Rightarrow} p_{(3)}}\,\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}}{\Box(\Box p \to p) \stackrel{u}{\not\Rightarrow} \Box p_{(2)}}\,\mathrm{S}^{\Box}_{\mathrm{u}}}{\sigma_1\bullet}\,R \overset{\not\vdash}{\to}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box p, q \stackrel{u}{\not\Rightarrow} p_{(11)}}\,\mathrm{Irr}}{\Box\Box p, \Box q \stackrel{u}{\not\Rightarrow} \Box p_{(10)}}\,\mathrm{S}^{\Box}_{\mathrm{u}}}{\sigma_9\bullet}\,R \overset{\not\vdash}{\to}
\quad
\dfrac{\dfrac{}{p \stackrel{u}{\not\Rightarrow} q_{(13)}}\,\mathrm{Irr}}{\sigma_{12}\bullet}\,\mathrm{Ref}}{\Box\Box p \stackrel{u}{\not\Rightarrow} (\Box q \to \Box p) \vee \Box q_{(8)}}\,\mathrm{S}^{\vee}_{\mathrm{u}}}{\sigma_7\bullet}\,R \overset{\not\vdash}{\to}
}{\stackrel{u}{\not\Rightarrow} (\Box(\Box p \to p) \to \Box p) \vee (\Box\Box p \to ((\Box q \to \Box p) \vee \Box q))_{(0)}}\,\mathrm{S}^{\vee}_{\mathrm{u}}
$$

$$
\begin{array}{ll}
\Psi(\sigma_0) = \sigma_0,\ \Psi(\sigma_2) = \sigma_2, & \sigma_0 \ll \sigma_2,\ \sigma_0 \ll \sigma_8, \\
\Psi(\sigma_3) = \sigma_3,\ \Psi(\sigma_5) = \sigma_5, & \sigma_8 \ll \sigma_{10},\ \sigma_{10} \ll \sigma_{11}, \\
\Psi(\sigma_{10}) = \sigma_{10},\ \Psi(\sigma_{11}) = \sigma_{11}, & \sigma_2 \ll_R \sigma_3, \\
\Psi(\sigma_{13}) = \sigma_{13} & \sigma_3 \ll_R^* \sigma_5,\ \sigma_8 \ll_R^* \sigma_{13}
\end{array}
$$

Figure 6: Structure of the $\mathcal{R}$-derivation $\mathcal{D}$ in Fig.5 ($\bullet$: label b).

discussed in the proof of Prop. 17, the application of such rules does not require backtracking. If the search for a $\mathcal{C}$-derivation of σ fails, we get an $\mathcal{R}$-derivation of $\overline{\sigma}$.

Proposition 17 *Let σ be a regular sequent. One can build either a $\mathcal{C}$-derivation of σ or an $\mathcal{R}$-derivation of $\overline{\sigma}$.*

Proof. Since $\prec_{\mathrm{bu}}$ is well-founded (Prop. 9), we can inductively assume that the assertion holds for every regular sequent σ' such that $\sigma' \prec_{\mathrm{bu}} \sigma$ (IH). If σ or $\overline{\sigma}$ is an axiom (in the respective calculus), the assertion immediately follows. If an invertible rule ρ of $\mathcal{C}$ is (backward) applicable to σ, we can build the proper derivation by applying ρ or its dual image in $\mathcal{R}$. For instance, let us assume that rule $R^{\Box}_{\mathrm{b}}$ of $\mathcal{C}$ is applicable with conclusion $\sigma = \Gamma, \Box\Delta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha$ and premises $\sigma_0 = \Box\alpha, \Gamma, \Delta \stackrel{\mathrm{u}}{\Rightarrow} \alpha$ and $\sigma_1 = \Gamma^-, \Delta^- \stackrel{\mathrm{u}}{\Rightarrow} \alpha^-$. Let $k \in \{0, 1\}$; since $\sigma_k \prec_{\mathrm{bu}} \sigma$ (see Prop. 11), by (IH) there exists either a $\mathcal{C}$-derivation $\mathcal{D}_k$ of σ_k or an $\mathcal{R}$-derivation $\mathcal{E}_k$ of $\overline{\sigma_k}$. According to the case, we can build one of the following derivations:

$$
\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Box\alpha, \Gamma, \Delta \stackrel{\mathrm{u}}{\Rightarrow} \alpha\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma^-, \Delta^- \stackrel{\mathrm{u}}{\Rightarrow} \alpha^-\end{array}}{\Gamma, \Box\Delta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha}\,R^{\Box}_{\mathrm{b}}
\qquad
\dfrac{\begin{array}{c}\mathcal{E}_0\\ \Box\alpha, \Gamma, \Delta \stackrel{\mathrm{u}}{\not\Rightarrow} \alpha\end{array}}{\Gamma, \Box\Delta \stackrel{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\,R^{\Box}_{\mathrm{b}}
\qquad
\dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma^-, \Delta^- \stackrel{\mathrm{u}}{\not\Rightarrow} \alpha^-\end{array}}{\Gamma, \Box\Delta \stackrel{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\,\mathrm{Ref}
$$

Let us assume that no invertible rule can be applied to σ; then:

- $\sigma = \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ with $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$ and $\delta \in \mathcal{V} \cup \{\, \bot,\ \delta_0 \vee \delta_1,\ \Box\delta_0 \,\}$.

We only discuss the case $\delta = \Box\delta_0$. Let $\sigma_0 = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \stackrel{\mathrm{u}}{\Rightarrow} \delta_0$ be the premise of the application of rule $R^{\Box}_{\mathrm{u}}$ of $\mathcal{C}$ to σ; for every $\alpha \to \beta \in \Gamma^{\to}$, let $\sigma_\alpha = \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha$ and $\sigma_\beta = \Gamma \setminus \{\alpha \to \beta\}, \beta \stackrel{\mathrm{u}}{\Rightarrow} \delta$ be the two premises of an application of rule $L\to$ of $\mathcal{C}$ to σ with main formula $\alpha \to \beta$. By the (IH):

- we can build either a $\mathcal{C}$-derivation $\mathcal{D}_0$ of σ_0 or an $\mathcal{R}$-derivation $\mathcal{E}_0$ of $\overline{\sigma_0}$.
- for every $\alpha \to \beta \in \Gamma^{\to}$ and for every $\omega \in \{\alpha, \beta\}$, we can build either a $\mathcal{C}$-derivation $\mathcal{D}_\omega$ of σ_ω or an $\mathcal{R}$-derivation $\mathcal{E}_\omega$ of $\overline{\sigma_\omega}$.

One of the following four cases holds:

(A) We get $\mathcal{D}_0$.

(B) There is $\alpha \to \beta \in \Gamma^{\to}$ such that we get both $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$.

(C) There is $\alpha \to \beta \in \Gamma^{\to}$ such that we get $\mathcal{E}_\beta$.

(D) We get $\mathcal{E}_0$ and, for every $\alpha \to \beta \in \Gamma^{\to}$, $\mathcal{E}_\alpha$.

According to the case, we can build one of the following derivations:

$$\text{(A)}\ \dfrac{\begin{matrix}\mathcal{D}_0\\ \sigma_0\end{matrix}}{\sigma}\ R_{\mathrm{u}}^{\Box} \qquad \text{(B)}\ \dfrac{\begin{matrix}\mathcal{D}_\alpha & \mathcal{D}_\beta\\ \sigma_\alpha & \sigma_\beta\end{matrix}}{\sigma}\ L\!\to \qquad \text{(C)}\ \dfrac{\begin{matrix}\mathcal{E}_\beta\\ \overline{\sigma_\beta}\end{matrix}}{\overline{\sigma}}\ L\!\to \qquad \text{(D)}\ \dfrac{\begin{matrix} & \mathcal{E}_\alpha & & \mathcal{E}_0\\ \ldots & \overline{\sigma_\alpha} & \ldots & \overline{\sigma_0}\end{matrix}}{\overline{\sigma}}\ \mathrm{S}_{\mathrm{u}}^{\Box}$$

In the proof search strategy, this corresponds to a backtrack point, since we cannot predict which case holds. □

Let us assume $\Gamma \vDash_{\mathsf{iCK4}} \delta$ and let $\sigma = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$. By Soundness of $\mathcal{R}$ (Th. 13) $\overline{\sigma}$ is not provable in $\mathcal{R}$, hence, by Prop. 17, σ is provable in $\mathcal{C}$; this proves the Completeness of $\mathcal{C}$ (Th. 7(iv)).

Properties of $\mathcal{R}$. It remains to prove point (i) of Th. 15. By $\mathrm{Sf}^-(\alpha)$ we denote the set $\mathrm{Sf}(\alpha) \setminus \{\alpha\}$; $w < w'$ means that $w \leq w'$ and $w \neq w'$.

Let $\mathcal{D}$ be an $\mathcal{R}$-derivation having a Succ rule at the root. To display $\mathcal{D}$, we introduce the schema (1) below; at the same time, we define the relations $\ll$, $\ll_R$ and $\ll_R^*$ between u-antisequents in $\mathcal{D}$.

$$\mathcal{D} = \dfrac{\begin{matrix} & \mathcal{D}_\chi & & \vdots\\ \cdots & \sigma_\chi^{\mathrm{b}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \chi & \cdots & \sigma_\psi^{\mathrm{u}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \psi\end{matrix}}{\sigma^{\mathrm{u}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{u}}{\not\Rightarrow} \delta}\ \mathrm{Succ} \tag{1}$$

- σ_χ^{b} is any of the premises of Succ having label b.
- σ_ψ^{u} is only defined if Succ is $\mathrm{S}_{\mathrm{u}}^{\Box}$ (thus $\delta = \Box\psi$); in this case we set $\sigma^{\mathrm{u}} \ll_R \sigma_\psi^{\mathrm{u}}$.
- The $\mathcal{R}$-derivation $\mathcal{D}_\chi$ of σ_χ^{b} has the form

$$\begin{matrix} \dfrac{\begin{matrix}\vdots\\ \sigma_1^{\mathrm{u}}\end{matrix}}{\sigma_1^{\mathrm{b}}}\ \rho_1 & \cdots & \dfrac{\begin{matrix}\vdots\\ \sigma_m^{\mathrm{u}}\end{matrix}}{\sigma_m^{\mathrm{b}}}\ \rho_m & \dfrac{}{\tau_1^{\mathrm{b}}}\ \mathrm{Irr} & \cdots & \dfrac{}{\tau_n^{\mathrm{b}}}\ \mathrm{Irr} \\ & & \mathcal{T}_\chi^{\mathrm{b}} & & & \\ & & \sigma_\chi^{\mathrm{b}} \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi & & & \end{matrix} \qquad \begin{array}{l} m+n \geq 0 \\ \mathcal{T}_\chi^{\mathrm{b}} \text{ only contains} \\ \text{b-antisequents} \\ \Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \end{array}$$

  - The $\mathcal{R}$-tree $\mathcal{T}_\chi^{\mathrm{b}}$ has root σ_χ^{b} and leaves $\sigma_1^{\mathrm{b}}, \ldots, \sigma_m^{\mathrm{b}}, \tau_1^{\mathrm{b}}, \ldots, \tau_n^{\mathrm{b}}$.
  - For every $i \in \{1, \ldots, m\}$, either (A) $\rho_i = R_{\to}^{\not\triangleright}$ or (B) $\rho_i = R_{\mathrm{b}}^{\Box}$ or (C) $\rho_i = \mathrm{Ref}$, namely:

$$\text{(A)}\ \dfrac{\sigma_i^{\mathrm{u}} \;=\; \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\sigma_i^{\mathrm{b}} \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta}\ R_{\to}^{\not\triangleright} \quad \text{or} \quad \text{(B)}\ \dfrac{\sigma_i^{\mathrm{u}} \;=\; \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\sigma_i^{\mathrm{b}} \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\ R_{\mathrm{b}}^{\Box}$$

$$\text{or} \quad \text{(C)}\ \dfrac{\sigma_i^{\mathrm{u}} \;=\; \Gamma^{-} \overset{\mathrm{u}}{\not\Rightarrow} \alpha^{-}}{\sigma_i^{\mathrm{b}} \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha}\ \mathrm{Ref}$$

  We set $\sigma^{\mathrm{u}} \ll \sigma_i^{\mathrm{u}}$ in case (A), $\sigma^{\mathrm{u}} \ll_R \sigma_i^{\mathrm{u}}$ in case (B), $\sigma^{\mathrm{u}} \ll_R^* \sigma_i^{\mathrm{u}}$ in case (C).

Note that, if $\sigma^{\mathrm{u}} \ll_R^* \sigma_i^{\mathrm{u}}$, then the world $\Psi(\sigma_i^{\mathrm{u}})$ of $\mathrm{Mod}(\mathcal{D})$ is reflexive.

Example 18 The relations $\ll$, $\ll_R$ and $\ll_R^*$ induced by the $\mathcal{R}$-derivation $\mathcal{D}$ of Fig. 5 are displayed in Fig. 6. ◊

Now we introduce two technical lemmas which are needed to prove Th. 15.

Lemma 19 *Let $\mathcal{T}^{\mathrm{b}}$ be an $\mathcal{R}$-tree only containing* b*-antisequents having root $\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(I1) $w \nVdash \delta'$, for every leaf $\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta'$ of $\mathcal{T}^{\mathrm{b}}$;

(I2) $w \Vdash (\Gamma^{\to} \cap \mathrm{Sf}^-(\delta)) \cup \Box\Delta$;

(I3) $V(w) = \Gamma^{\mathrm{at}}$.

Then, $w \nVdash \delta$.

Proof. By induction on $\mathrm{depth}(\mathcal{T}^{\mathrm{b}})$. The case $\mathrm{depth}(\mathcal{T}^{\mathrm{b}}) = 0$ is trivial, since the root of $\mathcal{T}^{\mathrm{b}}$ is also a leaf. Let $\mathrm{depth}(\mathcal{T}^{\mathrm{b}}) > 0$; we only discuss the case where

$$\mathcal{T}^{\mathrm{b}} \;=\; \cfrac{\begin{array}{c}\mathcal{T}_0^{\mathrm{b}}\\ \sigma_0^{\mathrm{b}} \;=\; \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta\end{array}}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta}\; R\overset{\triangleright}{\to} \qquad \begin{array}{l}\Gamma \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta\\ \Gamma \triangleright \alpha\end{array}$$

By applying the induction hypothesis to the $\mathcal{R}$-tree $\mathcal{T}_0^{\mathrm{b}}$, having root σ_0^{b} and the same leaves as $\mathcal{T}^{\mathrm{b}}$, we get $w \nVdash \beta$. Let $\Gamma_\alpha = \Gamma \cap \mathrm{Sf}(\alpha)$; by Lemma 5(iii), $\Gamma_\alpha \triangleright \alpha$. Since $\mathrm{Sf}(\alpha) \subseteq \mathrm{Sf}^-(\alpha \to \beta)$, by hypotheses (I2)–(I3) we get $w \Vdash \Gamma_\alpha$, which implies $w \Vdash \alpha$ (Lemma 5(iv)). This proves $w \nVdash \alpha \to \beta$. $\Box$

Lemma 20 *Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ having form (1) where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$; let $\mathcal{K} = \langle W, \leq, R, r, V\rangle$ and $w \in W$ such that:*

(J1) or every $w' \in W$ such that $w < w'$, it holds that $w' \Vdash \Gamma^{\to}$.

(J2) For every $w' \in W$ such that wRw', it holds that $w' \Vdash \Delta$.

(J3) For every $\sigma' = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ such that $\sigma^{\mathrm{u}} \ll \sigma'$, there exists $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$.

(J4) For every $\sigma' = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ such that $\sigma^{\mathrm{u}} \ll_R \sigma'$, there exists $w' \in W$ such that wRw' and $w' \nVdash \alpha$.

(J5) For every $\sigma' = \Gamma^- \overset{\mathrm{u}}{\not\Rightarrow} \alpha^-$ such that $\sigma^{\mathrm{u}} \ll_R^* \sigma'$, there exists $w' \in W$ such that wRw', w' is reflexive and $w' \nVdash \alpha^-$.

(J6) $V(w) = \Gamma^{\mathrm{at}}$.

Then, $w \Vdash \Gamma$ and $w \nVdash \delta$.

Proof. We show that:

(P1) $w \nVdash \chi$, for every premise $\sigma_\chi^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi$ of Succ;

(P2) $w \Vdash \alpha \to \beta$, for every $\alpha \to \beta \in \Gamma^{\to}$.

We introduce the following induction hypothesis:

(IH1) to prove Point (P1) for a formula χ, we inductively assume that Point (P2) holds for every formula $\alpha \to \beta$ such that $|\alpha \to \beta| < |\chi|$;

(IH2) to prove Point (P2) for a formula $\alpha \to \beta$, we inductively assume that Point (P1) holds for every formula χ such that $|\chi| < |\alpha \to \beta|$.

We prove Point (P1). Let σ_χ^{b} be the premise of Succ displayed in schema (1). We show that the $\mathsf{RbuSL}_\Box$-tree $\mathcal{T}_\chi^{\mathrm{b}}$ and w match the hypotheses (I1)–(I3) of Lemma 19, so that we can apply the lemma to infer $w \nVdash \chi$.

We prove (I1). Let $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta$ any leaf of $\mathcal{T}_\chi^{\mathrm{b}}$; we show that $w \nVdash \delta$. By definition of schema (1), one of the following cases holds.

(a) $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \to \beta$ and $\sigma^{\mathrm{u}} = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$ and $\sigma^{\mathrm{u}} \ll \sigma^{\mathrm{u}}$;

(b) $\sigma^{\mathrm{b}} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha$ and $\sigma^{\mathrm{u}} = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Delta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ and $\sigma^{\mathrm{u}} \ll_R \sigma^{\mathrm{u}}$;

(c) $\sigma^{\mathrm{b}} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha$ and $\sigma^{\mathrm{u}} = \Gamma^- \overset{\mathrm{u}}{\not\Rightarrow} \alpha^-$ and $\sigma^{\mathrm{u}} \ll_R^* \sigma_i^{\mathrm{u}}$;

(d) $\sigma^{\mathrm{b}} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta \overset{\mathrm{b}}{\not\Rightarrow} \delta$ is irreducible.

In case (a), by hypothesis (J3) there is $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$, hence $w \nVdash \alpha \to \beta$. In case (b), by hypothesis (J4) there is w' such that wRw' and $w' \nVdash \alpha$, hence $w \nVdash \Box\alpha$. Let us consider case (c). By hypothesis (J5) there exists a reflexive world w' such that wRw' and $w' \nVdash \alpha^-$. By Lemma 2, $w' \Vdash \alpha \leftrightarrow \alpha^-$; it follows that $w' \nVdash \alpha$, hence $w \nVdash \Box\alpha$. In case (d), we have $\delta \in \mathcal{V} \cup \{\bot\}$ and $\delta \notin \Gamma^{\mathrm{at}}$. Since $V(w) = \Gamma^{\mathrm{at}}$ (hypothesis (J6)), we get $w \nVdash \delta$. This proves that hypothesis (I1) holds.

We prove (I2). Let $\gamma \in \Gamma^{\to} \cap \mathrm{Sf}^-(\chi)$; since $|\gamma| < |\chi|$, by (IH1) we get $w \Vdash \gamma$. Moreover, $w \Vdash \Box\Delta$ by (J2), thus (I2) holds. Finally, (I3) coincides with (J6). We can apply Lemma 19 and conclude $w \nVdash \chi$, and this proves Point (P1).

We prove Point (P2). Let $\alpha \to \beta \in \Gamma^{\to}$, let $w' \in W$ be such that $w \leq w'$ and $w' \Vdash \alpha$; we show that $w' \Vdash \beta$. Note that $\sigma^{\mathrm{b}}_{\alpha} = \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha$ is a premise of Succ; since $|\alpha| < |\alpha \to \beta|$, by (IH2) we get $w \nVdash \alpha$. This implies that $w < w'$. By hypothesis (J1), $w' \Vdash \alpha \to \beta$, hence $w' \Vdash \beta$; this proves (P2).

We prove the assertion of the lemma. By (P2) and hypotheses (J2) and (J6), we get $w \Vdash \Gamma$. The proof that $w \nVdash \delta$ depends on the specific rule Succ at hand and follows from Point (P1) and hypothesis (J6). $\Box$

We are now in position to complete the proof of Th. 15.

Proof. [Th. 15(i)] By induction on the depth of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$. Let ρ be the rule of $\mathcal{R}$ having conclusion σ^{u}. We proceed by a case analysis, only detailing some significant cases.

If $\rho = \mathrm{Irr}$, then $\Gamma = \Gamma^{\mathrm{at}}, \Box\Delta$ and $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Note that $V(\sigma^{\mathrm{u}}) = \Gamma^{\mathrm{at}}$, hence $\sigma^{\mathrm{u}} \Vdash \Gamma^{\mathrm{at}}$ and $\sigma^{\mathrm{u}} \nVdash \delta$; it remains to show that $\sigma^{\mathrm{u}} \Vdash \Box\Delta$. If σ^{u} is reflexive, then σ^{u} is the premise of Ref, hence Δ is empty. Otherwise, there is no w in $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} R w$, hence $\sigma^{\mathrm{u}} \Vdash \Box\Delta$.

If $\rho = R\overset{\triangleright}{\to}$, then, $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \alpha \to \beta$, where $\Gamma \triangleright \alpha$, and the premise of ρ is $\sigma^{\mathrm{u}}_1 = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$. By the induction hypothesis, $\Psi(\sigma^{\mathrm{u}}_1) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}_1) \nVdash \beta$. By Lemma 5(iv) we get $\Psi(\sigma^{\mathrm{u}}_1) \Vdash \alpha$, which implies $\Psi(\sigma^{\mathrm{u}}_1) \nVdash \alpha \to \beta$. Since $\Psi(\sigma^{\mathrm{u}}) = \Psi(\sigma^{\mathrm{u}}_1)$, we conclude $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \alpha \to \beta$.

If $\rho = \mathrm{S}^{\Box}_{\mathrm{u}}$, then $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta$, where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Delta$, and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Let $\mathcal{D}^{\mathrm{u}}$ be the subderivation of $\mathcal{D}$ having root σ^{u}; we apply Lemma 20 setting $\mathcal{D} = \mathcal{D}^{\mathrm{u}}$, $\mathcal{K} = \mathrm{Mod}(\mathcal{D})$ and $w = \sigma^{\mathrm{u}}$. We check that hypotheses (J1)–(J6) hold.

We prove hypothesis (J1). Let w' be a world of $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} < w'$; we show that $w' \Vdash \Gamma^{\to}$. There exists an u-antisequent $\sigma' = \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta'$ such that $\sigma^{\mathrm{u}} \prec \sigma' \preceq w'$ and either $\sigma^{\mathrm{u}} \ll \sigma'$ or $\sigma^{\mathrm{u}} \ll_R \sigma'$ or $\sigma^{\mathrm{u}} \ll^*_R \sigma'$ (see the definition of schema 1). Since $\mathrm{depth}(\sigma') < \mathrm{depth}(\sigma^{\mathrm{u}})$, by the induction hypothesis we get $\Psi(\sigma') \Vdash \Gamma'$. If $\sigma^{\mathrm{u}} \ll \sigma'$ or $\sigma^{\mathrm{u}} \ll_R \sigma'$, we get $\Gamma^{\to} \subseteq \Gamma'$, hence $\Psi(\sigma') \Vdash \Gamma^{\to}$. Let $\sigma^{\mathrm{u}} \ll^*_R \sigma'$. Then, $(\Gamma^{\to})^- \subseteq \Gamma'$, hence $\Psi(\sigma') \Vdash (\Gamma^{\to})^-$. Since $\Psi(\sigma')$ is reflexive, by Lemma 2 we get $\Psi(\sigma') \Vdash \Gamma^{\to}$. Having proved $\Psi(\sigma') \Vdash \Gamma^{\to}$, by the fact that $\Psi(\sigma') \leq w'$ we conclude $w' \Vdash \Gamma^{\to}$.

We prove hypothesis (J2). Let w' be a world of $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} R w'$; we show that $w' \Vdash \Delta$. There exists an u-antisequent $\sigma' = \Gamma' \overset{\mathrm{u}}{\not\Rightarrow} \delta'$ such that $\sigma^{\mathrm{u}} \prec \sigma' \preceq w'$ and either $\sigma^{\mathrm{u}} \ll_R \sigma'$ or $\sigma^{\mathrm{u}} \ll^*_R \sigma'$. Reasoning as in the case concerning (J1), we get $\Psi(\sigma') \Vdash \Delta$; since $\Psi(\sigma') \leq w'$, we conclude $w' \Vdash \Delta$.

We prove (J3). Let $\sigma^{\mathrm{u}} \ll \sigma' = \alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$; we show that there exists w' such that $\sigma^{\mathrm{u}} \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$. By the induction hypothesis, $\Psi(\sigma') \Vdash \alpha$ and $\Psi(\sigma') \nVdash \beta$. Since $\sigma^{\mathrm{u}} \leq \Psi(\sigma')$, we can set $w' = \Psi(\sigma')$.

We prove (J4). Let $\sigma^{\mathrm{u}} \ll_R \sigma' = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \alpha$; we show that there exists w' such that $\sigma^{\mathrm{u}} R w'$ and $w' \nVdash \alpha$. By the induction hypothesis, $\Psi(\sigma') \nVdash \alpha$. Since $\sigma^{\mathrm{u}} R\, \Psi(\sigma')$, we can set $w' = \Psi(\sigma')$.

We prove (J5). Let $\sigma^{\mathrm{u}} \ll^*_R \sigma' = \Gamma^- \overset{\mathrm{u}}{\not\Rightarrow} \alpha^-$; we show that there exists w' such that $\sigma^{\mathrm{u}} R w'$, w' is reflexive and $w' \nVdash \alpha^-$. By the induction hypothesis, $\Psi(\sigma') \nVdash \alpha^-$. Since $\sigma^{\mathrm{u}} R\, \Psi(\sigma')$ and $\Psi(\sigma')$ is reflexive, we can set $w' = \Psi(\sigma')$.

Hypothesis (J6), namely $V(\sigma^{\mathrm{u}}) = \Gamma^{\mathrm{at}}$, holds by the definition of V. By applying Lemma 20, we conclude that $\sigma^{\mathrm{u}} \Vdash \Gamma$ and $\sigma^{\mathrm{u}} \nVdash \Box\delta$. $\Box$

Conclusions. In this paper we have presented a terminating sequent calculus Gbu-iCK4 for iCK4 enjoying a weak variant of the subformula property. If a sequent σ is not derivable in Gbu-iCK4, then σ is derivable in the dual calculus Rbu-iCK4, and from the Rbu-iCK4-derivation we can extract a countermodel for σ. We leave as future work the investigation of cut-admissibility for Gbu-iCK4; this is a rather tricky task since labels impose strict constraints on the shape of derivations. We also aim to extend our approach to the other provability logics with the coreflection principle related with iCK4 and iSL, such as iGL, mHC and KM (for an overview, see, e.g., [11]).

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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