Modeling changes of opinions as transition probabilities within one- and two-level model "State-Probability of Action"

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Abstract
Approaches to modeling and simulating processes related to elections and changing voters' opinions in bipartisan democracies on the basis of Markov chains are discussed. The basic approach suggested by the "state-probability of action" model (SPA model) is combined with ideas featured by pairwise comparisons and the Analytic Hierarchy Process. The one-level SPA model, focusing on election results, and the two-level model regarding criteria which affect decisions are considered.
Some modifications of traditional homogenous Markov chains, such as switching roles or random transition probabilities, are explored. Some approaches to using non-homogenous Markov chains are outlined.

Keywords
Markov chains, switching roles, "state-probability of action" model, pairwise comparisons, Analytic Hierarchy Process, making decisions, elections

1. Introduction

In [1, 2] an approach to modeling elections in an established bipartisan democracy combining uncertain decisions with the Analytic Hierarchy Process (AHP) [3-7] has been introduced. The main idea is to apply what is referred to as the "state-probability of action" model, or SPA-model, which has been suggested in [8]. In [8] a parametrized version of this basic model, which acquired ideas of pairwise comparisons and of the AHP, and of taking into account degrees of decisiveness typical for reinforcement learning techniques [9, 10] as well, has been considered.

Decisions are typically made on the ground of some criteria, which leads us to using a very common and ubiquitous two-level scheme of AHP, in which the top level corresponds to eventual decisions, and the bottom level corresponds to the specific criteria. Some ways to implement similar ideas for the two-level SPA-model of decision making on the base on the approach suggested in [11] have been discussed in [1] as well.

The question of modeling changes in voters' opinions resulting in different outcomes of elections and therefore in changing parties which are in power, is a question of great interest. The basic and the simplest point is finding and exploring situations of so-called dynamic equilibrium, when no alternative has advantages over the others. Some sufficient conditions for an equilibrium between alternatives within the SPA-model for the case when there are two alternatives only were reported in [8]; in [2] some ways to how agents of influence might move the situation away from the equilibrium were described. But the approach considering only equilibrium situations is very basic

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and not sufficient one, the question is much more profound. So, in this paper we are going to point out some other approaches to modeling changes in voters' opinions.

2. Methodology

The basic one-level SPA-model, as it was described in [8], is as follows. Let there be n alternatives forming a set $A = \{a_1, \dots, a_n\}$, one of which is to be chosen. Let ξ be a random variable corresponding to a certain choice. Then we can consider a vector $p = \{p_1, \dots, p_n\}$, where p_i is a probability that the alternative a_i is chosen, that is $p_i = P(\xi = a_i)$.

Then we regard some states, each of them corresponds to a certain distribution of probabilities of making certain choices. More formally, let $S = \{s_1, \dots, s_m\}$ be a given set of states, and η be a random variable denoting a state in which the agent is at the specific moment. We are considering conditional probabilities $h_{ij} = P(\xi = a_i | \eta = s_j)$. These probabilities form a matrix denoted as *H*, which is called the SPA-matrix. Eventually, we have to specify the vector of input probabilities $\bar{p} = (\overline{p_1}, \dots, \overline{p_m})$, where $\overline{p_J} = P(\eta = s_j)$.

Then, as it was shown in [8],

$$p = \bar{p}H$$

In the paper we are regarding the most important case *n=2*, when there are two competing alternatives only. This particular case is typical for bipartisan democracies, where there are two competing political parties, which are gaining power in turn according to results of elections. For this case, equilibrium means that $p = (0.5, 0.5)$.

If *n=2*, equilibrium between alternatives holds if the $\bar{p}$ vector is symmetric, and the *H* matrix is centrosymmetric [8], this follows directly from the properties of centrosymmetric matrices [12, 13].

The *H* matrix might be specified in very different ways. The $M(q, \tau, \beta)$ model suggested in [1] appears to be quite suitable for our purposes. Within this approach, each state of the SPA-model corresponds to a certain level of preference of an alternative over the other in terms of pairwise comparisons. The q parameter reflects the granularity of preference levels, we pose there are *2q+1* preference levels been numbered from *-q* to *q*. For quantifying, that is for ascribing certain values to those levels, we are using so-called transitive scales of comparisons [14]. Then the value ascribed to the *k*-th grade of preference shall equal τ^k.

The β parameter reflects the agent's level of confidence. Given the preference values $(v_{k1}, \dots, v_{kn})$ for the *k*-th state, the corresponding probabilities should be obtained as follows [2]:

$$p_{kj} = \frac{e^{\beta v_{kj}}}{\sum_{j=1}^{n} e^{\beta v_{kj}}}$$

The bigger is β, the more decisive the agent is. This approach is typical for reinforcement learning techniques [9, 10].

For example, the $M(q, \tau, \beta)$ model with the parameters $q = 3, \tau = 1.4, \beta = 4.5$ yields the following SPA-matrix (approximately):

$$H = \begin{pmatrix} 1.0 & 0.0 \\ 0.9985 & 0.0015 \\ 0.9563 & 0.0437 \\ 0.5 & 0.5 \\ 0.0437 & 0.9563 \\ 0.0015 & 0.9985 \\ 0.0 & 1.0 \end{pmatrix}$$

In the paper we are going to use this matrix as a starting point.

Given the input probabilities $\bar{p} = (0.1,\ 0.1, 0.15, 0.3, 0.15,\ 0.1, 0.1)$, we obtain the resulting probabilities $p = (0.5, 0.5)$. Equilibrium holds; no alternative has advantages.

The two-level multicriteria decision making model, based on combining two-level SPA-model [11] with classical two-level AHP, has been suggested in [1]. Shortly, we consider a two-level state system: the bottom level corresponds to separate criteria, and the top one corresponds to the eventual choice. For connecting these levels, the following logical rules are suggested: *if the alternative A has a preference over the other alternative B by the separate k-th criterion, then A has the overall preference over B.* For reflecting these rules, we are introducing transitional matrices for each criterion. For the reason of simplicity, we may consider the same transitional matrix R for all criteria as long as we believe that the actual measures of influence for different criteria may be adequately reflected by weighting coefficients for these criteria. These weighting coefficients form the vector $\lambda = (\lambda_1, \dots, \lambda_K)$, where K is the total number of criteria, λ_k is the importance of the k-th criterion, and

$$\sum_{k=1}^{K} \lambda_k = 1$$

Like for the top-level SPA-matrix, the states for the bottom-level matrices related to specific criteria correspond to grades of preference between alternatives. For each k-th criteria we have to specify probabilities for being in certain states. Let's introduce the matrix $D = (d_{ki}, k = \overline{1, K}, i = \overline{1, m})$, where $d_{ki}$is the probability of being in the i-th state with respect to the k-th criteria.

Then [1]

$$p = \lambda \cdot D \cdot R \cdot H$$

Equilibrium holds if λ is a symmetric vector, and D, R, H are centrosymmetric matrices [1]. It is important to mention that, like the similar condition for the one-level model, this is the sufficient condition for equilibrium only, but not the necessary one.

There might be many approaches to specifying the matrix R [1]. However, now for our purposes it appears enough to take it as the unit matrix: R=I. Then

$$p = \lambda \cdot D \cdot H$$

It is necessary to mention that if the number of voters is large enough, then a losing alternative has almost no chances to be chosen. So, if equilibrium holds permanently, both alternatives have equal chances, and they are chosen by turn. Indeed, political parties in bipartisan democracies are chosen by turn and changing each other in power. But the assumption of permanently held equilibrium seems to be unrealistic. To the moment of elections either one or other party should get an advantage, but afterwards this shall change.

In our previous papers we mainly considered input probabilities as something given. But for exploring and modeling processes of changing voters' opinions more rigorously we should look at a Markov chain which these probabilities result from.

3. Regarding homogenous Markov chain

Let's consider a one-level SPA model with m states. Let's introduce a Markov chain having the transition matrix $\Pi = (\pi_{ij}(t), i, j = \overline{1, m})$, where $\pi_{ij}(t)$ is the probability that an agent who is in the i-th state at the moment t jumps to the j-th one. Note that meaningfully transitional probabilities $\pi_{ij}(t)$ describe changes in voters' opinions, since each state reflects a certain opinion about which alternative is better and to which extent. A Markov chain is said to be homogenous if transitional probabilities π_{ij} don't depend on t.

Provided a Markov chain is homogenous, the stationary probability distribution exists under certain conditions; it can be found from the following known equation:

$$p\Pi = p$$

But can we really describe changing power in bipartisan democracies on the basis of homogenous Markov process within the SPA model? Seemingly not, since if we consider a homogenous process in the traditional sense, the only situation when both parties win elections by turn is a situation of the permanent equilibrium of alternatives, which is unrealistic.

4. Homogenous chain with switching roles

However, we can try to retain the assumption about homogenous features if we consider a process which is homogenous in some generalized sense. Namely, we can consider a Markov chain, in which the states are bound not with the certain party but with the role of being in power.

Let's take the matrix H from Section 2, but the alternatives now will be of another sort. The first alternative is to vote for the party which is currently in power, and the other is to vote for their opponents. We can consider two important cases.

Case 1. Let's take the transition matrix as follows:

$$\Pi = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

This case means that an agent has more chances to improve their opinion about the party which is in power than to worsen it.

This yields the stationary distribution as follows:

$$\bar{p} = (0.1834 \quad 0.3057 \quad 0.2038 \quad 0.1359 \quad 0.0906 \quad 0.0604 \quad 0.0201)$$

The resulting probabilities of voting results are as follows:

$$p = \bar{p}H = (0.7556 \quad 0.2444)$$

This means that the ruling party will win again and will win permanently.

Case 2. This is the situation opposite to the previous one.

Let the transition matrix be as follows:

$$\Pi = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}$$

Such a matrix means that the rating of the ruling party tends to decrease. This yields the following stationary distribution

$$\bar{p} = (0.0230 \quad 0.0575 \quad 0.0863 \quad 0.1294 \quad 0.1941 \quad 0.2912 \quad 0.2184)$$

and the resulting probabilities

$$p = \bar{p}H = (0.2366 \quad 0.7634)$$

Now the ruling party loses, and the power moves on to their opponents.

Summarizing the section and combining it with the considerations about the equilibrium of alternatives, we can formulate the following statement.

Let $p^{(R)}$ be the probability that an agent will vote for the ruling party. If within the SPA model changes of voters' opinions are described by a homogenous Markov chain with switching roles and the chain always reaches its stationary mode before the moment of election occurs, then repeated changing of power is possible if and only if $p^{(R)} \leq 0.5$.

5. Why the winners are to lose: some explaining models

It appears interesting to explore how the fact that the party which recently won the election promptly gets into the losing situation can be explained within the SPA model. In this section we are going to provide some considerations about this issue.

Firstly, modern bipartisan democracies are typically considered to be mature and established democracies. On the other hand, they probably have already harnessed most available resources to reach an overall situation which is near to being as good as possible. No party can improve the social situation drastically. Some betterments surely happen. But good things are usually taken for granted whereas faults in governance, which are absolutely inevitable, are rapidly getting into the focus of public discussion.

So, if the ruling party does things similarly to what their opponents did while being in power, and their mistakes are similar as well, the overall situation likely can be described in terms of the homogenous Markov chain with changing roles. In addition to this, in established democracies most people, generally satisfied with their life, are usually conservative and are not going to approve of drastic changes.

People usually tend to estimate ongoing betterment or worsening not just as they are, but rather in comparison with what they think other politicians could secure.

And last, but not least, the situation is near to being Pareto-optimal with respect to criteria indicating the quality of social well-being. This means that it is hard or even impossible to improve any criterion without worsening some others.

We are going to illustrate the latter with the following example.

Let's regard the two-level model. We are considering two criteria: average income and environmental issues. Of course, there actually are many other criteria, but this example is very basic and illustrative.

Average income is considered to be more important. So are the weighting coefficients: *0.6* for average income and *0.4* for environmental issues. Then

$$\lambda = (0.6, \qquad 0.4)$$

The average income gradually rises permanently. Voters take this for granted. Nevertheless, the transitional matrix with respect to this criterion indicates gradual movement in favor of the ruling party. Let it be as follows:

$$\Pi_1 = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.5 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0.5 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

However, environmental issues are deteriorating, and people are really worried about this. So, the transitional matrix with respect to this criterion reflects sharper movement in favor of the opponents. Let it be as follows:

$$\Pi = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0.3 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0.3 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0.3 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

This yields the following matrix D formed by the eigenvectors of reciprocal transitional matrices as of rows:

$$D = \begin{pmatrix} 0.1834 & 0.3057 & 0.2038 & 0.1359 & 0.0906 & 0.0604 & 0.0201 \\ 0.0000 & 0.0002 & 0.0014 & 0.087 & 0.0521 & 0.3125 & 0.6250 \end{pmatrix}$$

Then the resulting vector of probabilities is

$$p = \lambda \cdot D \cdot H = (0.4569 \quad 0.5431)$$

The opponents win.

Considering homogenous Markov chain with switching roles as a basis for modeling changes in voters' opinions appears to be a sound starting point. But there is an issue which hardly can be explained within such a model.

If real-word processes strictly complied with assumptions about homogeneity, changing of power would occur with perfect regularity. The party which won the latest election definitely would lose the next one. However, that doesn't hold in the real world. So, we a going to discuss possible non-homogenous enhancements.

6. Non-homogenous modeling

Generally, non-homogenous Markov chains, featuring varying transitional probabilities, are not studied very well. However, some approaches, which at list could be used for modeling and simulation, might be outlined.

We are considering discrete moments of time $t_1, t_2, \ldots$. For each moment of time the reciprocal transitional matrix $\Pi^{(k)} = (\pi_{ij}(t_k), i, j = \overline{1,m}, k = 1,2, \ldots)$ exists.

For a one-level SPA-model, given SPA-matrix H, current input probabilities $\overline{p^{(k)}}$, and current transitional matrix $\Pi^{(k)}$, we can calculate current probabilities for choosing alternatives as follows:

$$p^{(k)} = \overline{p^{(k)}} H,$$

input probabilities at the next moment of time are as follows:

$$\overline{p^{(k+1)}} = \overline{p^{(k)}} \Pi^{(k)}$$

and probabilities for choosing alternative at the next moment of time are as follows:

$$p^{(k+1)} = p^{(k+1)}H = \overline{p^{(k)}}\Pi^{(k)}H$$

Both homogenous Markov chains with switching roles and non-homogenous Markov chains are suitable for modeling changes in voters' opinions, but their areas of relevance are quite different. Whereas homogenous Markov chains appear to describe rather established situations, non-homogenous ones are to reflect unstable dynamics in situations featuring significant changes. What appears to be a trigger for changing dynamics of opinions is various events occurring in real life and information occasions, which may relate to actual ongoing flow of events, or may not.

A simple view on the matter may be as follows. When an information occasion arises, transitional probabilities start changing. Eventually they are stabilized, and then the Markov chain reaches the stationary distribution of probabilities across the states corresponding to different opinions in our case. There are various approaches to modeling dissemination of information itself [15-22]. The question of how information occasions really affect changes in opinions has not been studied well so far, but some approaches can be outlined.

However, this simple view is not sufficient. The issue is that information occasions may arise too frequently, so there may not be enough time for the Markov chain to be established and to reach the new stationary point. Moreover, agents of influence often are addicted to creating information occasions, sometimes far-fetched, scripted and choreographed or even merely fake. Their opponents are doing the same but aiming to affect the situation in the opposite direction.

Such things can be considered as an informational noise causing occasional fluctuations in voters' opinions which don't affect the situation very much unless something crucial happens. This prompts an idea to consider a homogenous model but that features random transition probabilities.

7. Homogenous modeling with random transition probabilities

For simplicity, let's consider the one-level SPA-model with random transition probabilities having the following view:

$$\Pi^k = \Pi + E$$

where Π is a constant matrix, and E is a random matrix. The important thing to be cared is that Π^k must remain stochastic. That's why components of E must not be distributed normally or so.

Under some conditions, this assumption allows us to estimate upper and lower bounds of resulting probabilities of choice, for instance based on the following evident inequality for the components of the eigenvector v of the given stochastic matrix A:

$$\min_i a_{ij} \le v_j \le \max_i a_{ij}$$

Such a model based on the homogenous Markov chain with switching roles and with random transition probabilities can explain how both sides can win elections by turn even if equilibrium of alternatives does not hold. Now we are going to illustrate this with the following experiment.

For simplicity, we took the one-level model with three states only (*m=3*) and the SPA matrix as follows:

$$H = \begin{pmatrix} 1.0 & 0.0 \\ 0.5 & 0.5 \\ 0.0 & 1.0 \end{pmatrix}$$

We started with the transition matrix

$$\Pi = \begin{pmatrix} 0.05 & 0.8 & 0.15 \\ 0.05 & 0.8 & 0.15 \\ 0.05 & 0.8 & 0.15 \end{pmatrix}$$

having the main left eigenvector

$$\bar{p} = (0.0500 \quad 0.8000 \quad 0.1500)$$

As was explained above, this situation means that the opposition wins the next election. Indeed, the vector of choice probabilities equals

$$p = (0.4500 \quad 0.5500)$$

We are starting from the vector $\bar{p}$. On each step we are forming the randomized transition matrix Π' in the following way:

- either state 1 or 3 is randomly chosen; let's denote the chosen state as L
- $\pi'_{iL} = \pi_{il} + \varepsilon_i, i = \overline{1,m}$, where ε_i are random values from the interval *[0,1]* multiplied by 0.2
- $\pi'_{i2} = \pi_{i2} - \varepsilon_i, i = \overline{1,m}$

After 15 steps we have got the following series of choice probability distributions:

0.3729	0.6271
0.3867	0.6133
0.5285	0.4715
0.4131	0.5869
0.5154	0.4846
0.3805	0.6195
0.5181	0.4819
0.3728	0.6272
0.3985	0.6015
0.3586	0.6414
0.5074	0.4926
0.3988	0.6012
0.4737	0.5263
0.5292	0.4708
0.5333	0.4667

This illustrates that within the described model competing parties really can win elections by turn and change each other in power.

8. Conclusions and discussion

The paper addresses issues related to modeling processes in bipartisan democracies including changes in voter's opinions and elections within the model named "state-probability of action" (SPA model). Changes in opinions are described in terms of transition matrices of Markov chains, where the states represent different levels of preference of one alternative over another, which is typical for pairwise comparisons and the Analytic Hierarchy Process. Both one-level model regarding final choices only and two-level model regarding criteria influencing possible choices are considered.

Possibilities related both to homogenous models considering constant transition matrices and non-homogenous ones considering varying matrices are discussed and illustrated by provided examples. Homogenous models turn out to be very constrained, and under pure homogenous assumptions the only situation when two competing parties have chances to win elections in turn, that is to change each other in power, is the situation of permanent equilibrium between alternatives. This means that the probability for each alternative must permanently equal 0.5, however in practice this seems to be unrealistic.

The model based on homogenous Markov chains with switching roles, which bounds the states of the SPA model not just to specific parties but to their roles, that is to whether they are in power or not, is considered. This model doesn't require an equilibrium of alternatives for modeling changeable results of elections. So, the homogenous model can be used as a basis for modeling, but it has its own constraints. If an equilibrium isn't held permanently, then either one party wins each time, or parties change each other in power with perfect regularity. This means that according to the model a party which won the previous election must lose just the next one. Evidently, any of those options doesn't take place in the real world.

Using Markov chains with switching roles, basically homogenous, but featuring random transitional probabilities, is considered in the paper. It was shown experimentally that such a model performs proper behavior in the sense that it demonstrates that alternatives can be chosen by turn.

Possible approaches to modeling and simulation on the basis of non-homogenous Markov chains are discussed as well.

In the paper only possibilities related to describing changes of opinions on the basis of transition matrices across the states are discussed. However, other parameters affecting opinions, such as measures of confidence or weighting coefficients reflecting importance of different criteria should be considered.

Last but not least, it is very important to consider models of spreading information influence and how information influences and information occasions affect changing opinions. Since agents of influence are counteracting each other, game aspects of the matter should be considered as well.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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