A mathematical model of phosphorus cycle in rhizosphere*

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Abstract
Phosphorus (P) plays a vital role in global crop production and food security. The aim of this paper is to describe mathematically the rhizosphere phosphorus cycle in order to predict and to estimate the concentrations of different phosphorus forms. Our model is a system of first order differential equations for which we show the existence, uniqueness, positivity and boundedness of solution. We also derive one equilibrium point of the model and we show its asymptotic stability. Furthermore numerical simulations are done to show the behavior of soil phosphorus content in time and to confirm the stability of the equilibrium point. Finally, we investigate which plant uptake rate can maintain the amount of available phosphorus in rhizosphere.

Keywords
dynamic system, phosphorus cycle, rhizosphere

1. Introduction

The size of world population is expected to reach 10 billion by 2050, hence it is necessary to double global crop yields to ensure food security. Therefore, all resources that improve agricultural production must be used efficiently. As a crucial macronutrient, phosphorus (P) contributes significantly to plant growth, it is taking part in essential metabolic processes such as photosynthesis, energy transfer, signal transduction, macromolecular biosynthesis, respiration and nitrogen fixation in legumes [1, 2, 3]. Soil phosphorus (P) exists in large quantities in both organic and inorganic forms, but the phosphorus supplies from soils remains a major constraint for plants because its assimilable form $(\mathrm{HPO_4}^{2-}$ or $\mathrm{H_2PO_4}^{-})$ that can be absorbed by the plant roots, appear to be very low in soil [4, 5]. These ions react highly with numerous soil mineral constituents through fixation phenomena and become unavailable for plant uptake [4, 1, 6, 7]. Assimilable phosphorus is released to plants through various physico-chemical and biological phenomena such as desorption, solubilization and mineralization. Many soil microorganisms and microfauna have been investigated and reported for their phosphorus solubilization and mineralization [5, 7, 4, 8].

In this work, we present the different forms of phosphorus, its cycle and different phenomena that influence phosphorus availability in rhizosphere as well as build a mathematical model of the phosphorus cycle. We analyze qualitatively our model and simulate it to justify our theoretical results. These simulations are interpreted for efficient phosphorus management in the rhizosphere.

2. Phosphorus cycle in rhizosphere

2.1. Phosphorus cycle

The phosphorus cycle shows the forms of phosphorus in rhizosphere and the pathway by which phosphorus may taken up by plants. It can be summarised as follows dead plants and animals constitute soil organic matter, which can be degraded and transformed into organic phosphorus by soil microorganisms. Organic phosphorus is mineralized by soil microorganisms to release assimilable phosphorus

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(orthophosphate ions). Assimilable phosphorus with immobilization returns to organic form, and is converted into mineral phosphorus by adsorption or precipitation. By desorption or solubilization, mineral phosphorus become assimilable phosphorus. Plants explore soil through their roots and absorb assimilable phosphorus. Animals eat phosphorus in plant leaves and fruits. After death, these become sources of organic phosphorus in the rhizosphere.

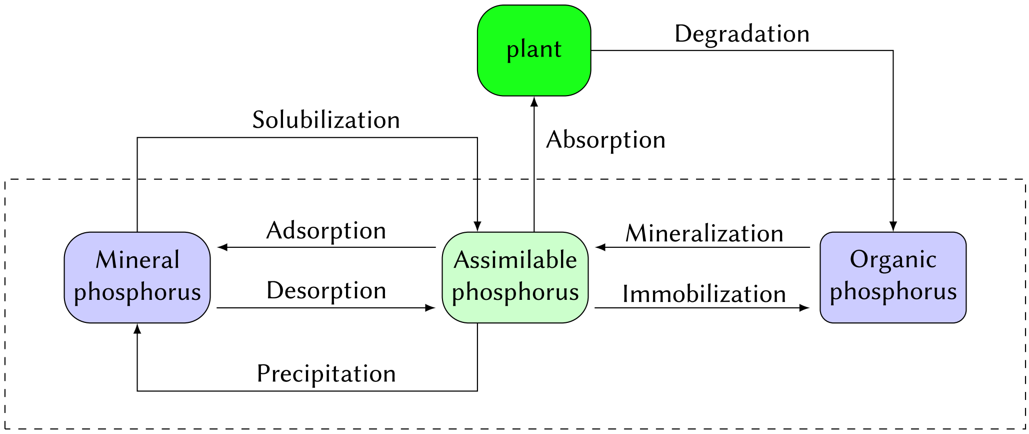

Figure 1: Phosphorus cycle in rhizosphere (Plassard et al., 2016)

2.2. Phosphorus forms

Phosphorus occurs in soil in both organic and inorganic forms [1, 6, 4, 5]. Organic phosphorus is a form of phosphorus present as a constituent of organic compounds. It can represent 30 to 90% of the total soil phosphorus and it is grouped into phosphate esters, phosphonates and phosphoric acid anhydrides [9, 10, 11]. Inorganic phosphorus includes soluble inorganic phosphorus (assimilable phosphorus) and mineral phosphorus. Assimilable phosphorus is known as orthophosphate ions ${H_2PO_4}^-$ or ${HPO_4}^{2-}$, it is in small amounts in soil although that total phosphorus amount is high in soil, this amount is controlled by soil pH and soil organic matter. Assimilable phosphorus is the only form of phosphorus that is available for plant uptake. Mineral phosphorus includes primary phosphate compounds (apatite, strengite, variscite) and secondary phosphorus compounds (calcium, iron, aluminum) phosphate.

2.3. Mechanisms that control phosphorus concentration

The availability of phosphorus to plants depends on the mechanisms that control its concentration in solution. These mechanisms are physico-chemical, biochemical and biological in nature, they are degradation of organic matter, mineralization of organic phosphorus and absorption, adsorption, immobilization, precipitation of assimilable with desorption and solubilization of mineral phosphorus.

- Degradation of organic matter is a process by which microorganisms fungi and bacteria break down dead plants and animals to release organic phosphorus.
- Mineralization of organic phosphorus is a process by which microorganisms release enzymes like phosphatase or phosphohydrolase, phytases, phosphonatase to convert organic phosphorus into assimilable phosphorus [9, 7, 5, 12].
- Absorption of available P is a process by which the plants explore soil through its roots and uptake an available phosphorus for their growth, health and development.
- Adsorption of available P is a chemical fixation of available phosphorus by soil components such as iron (Fe) and aluminum (Al), which makes phosphorus unavailable to plants. Throughout adsorption available phosphorus can be converted to mineral phosphorus.

- Immobilization of available P is a process by which available phosphorus is converted into organic phosphorus by certain soil microorganisms. It occurs when microorganisms consume available phosphorus , these microorganisms die later and produce organic phosphorus which is unavailable for plant uptake [13, 7].
- Precipitation of assimilable phosphorus is a process by which metal ions such as Al^{3+} and Fe^{3+} (acidic soils) and Ca^{2+} (neutral to alkaline soils) react with phosphate ions in the soil solution to form phosphate minerals such as Ca phosphate dicalcium or octacalcium phosphate, hydroxyapatite, Fe and Al phosphate such as strengite, vivianite, variscite and plumbogummite group minerals [14].
- Desorption of mineral phosphorus is a process by which mineral phosphorus is converted to assimilable phosphorus for plants uptake, it is a reverse process of adsorption.
- Solubilization of mineral phosphorus is a process by which mineral phosphorus is converted to assimilable phosphorus by microorganisms actions. It results from mechanisms such as the production of mineral-dissolving compounds and inorganic acids as well as the release of enzymes or enzymolyses by microorganisms [9, 7, 5].

3. Mathematical modeling

Different pools of phosphorus are generally distinguished in soil. This study focuses on the phosphorus cycle that was developed in [7] which consists of three pools organic, assimilable, and mineral phosphorus. We denote the variables as concentrations of organic, mineral and assimilable phosphorus respectively by $[P_{org}](t)$, $[P_{min}](t)$, $[P_i](t)$, and assimilable phosphorus content of the field as $[CP_i](t)$, all concentrations are time dependent.

The parameters include mineralization of organic phosphorus rate k_{min}, immobilization of assimilable phosphorus rate k_{imm}, degradation of organic matter rate k_{deg}, desorption of mineral phosphorus rate k_{des}, solubilization of mineral phosphorus k_{sol}, adsorption of assimilable k_{ads}, precipitation of assimilable k_{pre} and uptake of assimilable phosphorus rate k_{abs}, all parameters range between 0 and 1. To estimate the concentrations of different pools of phosphorus, we build four differential equations described by the following compartmental model:

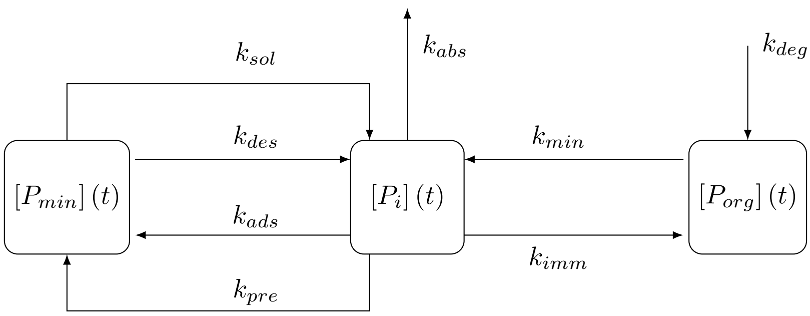

Figure 2: Soil phosphorus compartmental model

$$\frac{d[P_{org}]}{dt} = -k_{min}[P_{org}](t) + k_{imm}[P_i](t) + k_{deg}C \tag{1}$$

$$\frac{d[P_{min}]}{dt} = (k_{ads} + k_{pre})[P_i](t) - (k_{sol} + k_{des})[P_{min}](t) \tag{2}$$

$$\frac{d[P_i]}{dt} = k_{min}[P_{org}](t) + (k_{sol} + k_{des})[P_{min}](t) - (k_{imm} + k_{ads} + k_{abs} + k_{pre})[P_i](t) \tag{3}$$

$$\frac{d[CP_i]}{dt} = [P_i](t) - k_{abs}[CP_i](t). \tag{4}$$

By combining the four differential equations, we obtain the following compartmental system :

$$\begin{cases} \frac{d\left[P_{org}\right]}{dt} = -k_{min}\left[P_{org}\right](t) + k_{imm}\left[Pi\right](t) + k_{deg}C \\ \frac{d\left[P_{min}\right]}{dt} = -(k_{sol}+k_{des})\left[P_{min}\right](t) + (k_{ads}+k_{pre})\left[P_i\right](t) \\ \frac{d\left[P_i\right]}{dt} = k_{min}\left[P_{org}\right](t) + (k_{sol}+k_{des})\left[P_{min}\right](t) - (k_{imm}+k_{ads}+k_{abs}+k_{pre})\left[P_i\right](t) \\ \frac{d\left[CP_i\right]}{dt} = \left[P_i\right](t) - k_{abs}\left[CP_i\right](t) \end{cases} \tag{5}$$

We set $X(t) = \left([P_{org}](t), [P_{min}](t), [P_i](t), [CP_i](t)\right)$ so that the system (5) can be written as the following linear homogeneous differential equation of the first order :

$$\frac{dX(t)}{dt} = F(X(t)),$$

with

$$F(X(t)) = AX(t) + B \tag{6}$$

$$A = \begin{pmatrix} -k_{min} & 0 & k_{imm} & 0 \\ 0 & -(k_{des}+k_{sol}) & (k_{ads}+k_{pre}) & 0 \\ k_{min} & (k_{des}+k_{sol}) & -(k_{imm}+k_{ads}+k_{abs}+k_{pre}) & 0 \\ 0 & 0 & 1 & -k_{abs} \end{pmatrix} \tag{7}$$

$$B = \begin{pmatrix} k_{deg}C \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

4. Mathematical analysis

Let Ω be an open subset of $\mathbb{R}^4$. We consider the Cauchy problem defined by:

$$\begin{cases} \frac{dX(t)}{dt} & = & F(X(t)) \\ X(t_0) & = & X_0 \end{cases} \tag{9}$$

where:

- $X_0 \in \Omega$ and $t_0 \in \mathbb{R}^+$
- $F : \Omega \longrightarrow \mathbb{R}^4$ is defined by (6)
- $X : \mathbb{R}^+ \longrightarrow \mathbb{R}^4$

4.1. Existence and uniqueness of the solution

Since the function F is C^1 on $\mathbb{R}^4$. Thereby, for any non-negative initial condition the Cauchy problem (9) associated to the differential equation (5) admits a unique solution [15].

Proposition 4.1. *[16].*
The solution $X : \mathbb{R}^+ \longrightarrow \mathbb{R}^4$ of the differential equation (9) with initial condition $X(t_0) = X_0$ is unique and is generaly defined by :

$$X(t) = e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-s)}Bds, \quad \forall t \in [t_0; +\infty[\tag{10}$$

4.2. Positivity of the solution

Since system (5) represents an amount system, it is important that the solution remains non-negative values. Thus, we must prove the positivity of the solution.

Definition. .
A square real matrix $M = (m_{ij})_{1\leq i,j\leq n}$ is called a Metzler-matrix if all its off-diagonal entries are nonnegative, $m_{ij} \geq 0$ for $i \neq j\, i, j = 1, \cdots, n$.

Lemma 1 (Metzler [17]). .
Let $M \in \mathbb{R}^{n\times n}$, then $e^{Mt} \geq 0$ for $t \geq 0$ if and only if M is a Metzler-matrix.

Theorem 1. *Let $X(t_0)$ be a positive constant vector and A be a Metzler matrix. The solution $X(t)$ of system (9), defined by (10) remains positive forall $t \geq t_0$.*

Proof. of Theorem(1)
According to Lemma (1), the matrix A is a Metzler-matrix, then for $X_0 \geq 0$ we have $e^{A(t-t_0)}X_0 \geq 0, \quad \forall t \geq t_0$. Since $B \geq 0$ and $e^{A(t-s)} \geq 0, \quad \forall t \geq s$ then $\int_{t_0}^{t} e^{A(t-s)}Bds \geq 0$. we deduce that

$$e^{A(t-t_0)}X_0 + \int_{t_0}^{t} e^{A(t-s)}Bds = X(t) \geq 0.$$

□

4.3. Equilibrium point

We find equilibrium points of the system (5) by making its right-hand side equal zero.

$$\frac{dX(t)}{dt} = 0 \Longleftrightarrow \begin{cases} -k_{min}[P_{org}](t) + k_{imm}[P_i](t) = -k_{deg}C \\ -(k_{des} + k_{sol})[P_{min}](t) + (k_{ads} + k_{pre})[P_i](t) = 0 \\ k_{min}[P_{org}](t) + (k_{des} + k_{sol})[P_{min}](t) - (k_{imm} + k_{ads} + k_{abs} + k_{pre})[P_i](t) = 0 \\ [P_i](t) - k_{abs}[CP_i](t) = 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -k_{min}[P_{org}](t) + k_{imm}[P_i](t) + k_{deg}C = 0 \\ -(k_{des} + k_{sol})[P_{min}](t) + (k_{ads} + k_{pre})[P_i](t) = 0 \\ k_{abs}[P_i](t) = k_{deg}C \\ k_{abs}^2[CP_i](t) = k_{deg}C \end{cases}$$

We obtain the equilibrium point of our model (5), denoted by

$$X^* = \big([P_{org}]^*, [P_{min}]^*, [P_i]^*, [CP_i]^*\big)$$

where

$$\begin{cases} [CP_i]^* = \frac{k_{deg}C}{k_{abs}^2} \\ \\ [P_i]^* = \frac{k_{deg}C}{k_{abs}} \\ \\ [P_{min}]^* = \frac{(k_{ads}+k_{pre})k_{deg}C}{(k_{des}+k_{sol})k_{abs}} \\ \\ [P_{org}]^* = \frac{k_{deg}C}{k_{min}}\big(\frac{k_{imm}}{k_{abs}} + 1\big). \end{cases} \tag{11}$$

4.4. Stability

The stability of $X^* = 0$ of $n \times n$ linear homogeneous system $\dot{X}(t) = AX(t)$ depends on the sign of the eigenvalues of matrix A.

Lemma 2. *[15].*
The origin $X^* = 0$ is asymptotically stable equilibrium point if the real parts of all eigenvalues of the matrix A are less than zero.

We use the Routh-Hurwitz criterion to find out the sign of the real parts of matrix eigenvalues, given the following characteristic polynomial

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n \quad \text{where } a_i \in \mathbb{R}, \quad i = 1, \cdots, n.$$

The coefficients are arranged in descending order of degrees. Thus, the Routh-Hurwitz criterion stated as follows:

Lemma 3. *.[18]*
All solutions of equation

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

have negative real parts if and only if the following inequalities are satisfied

$$a_1 > 0, \quad \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ 1 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & . & . & \cdots & a_n \end{vmatrix} > 0.$$

Theorem 2. *The equilibrium point $X^* = \left([P_{org}]^*, [P_{min}]^*, [P_i]^*, [CP_i]^*\right)$ is asymptotically stable.*

Proof. From (6), we write the system as :

$$\dot{X}(t) = AX(t) + B$$

. with

$$A = \begin{pmatrix} -k_{min} & 0 & k_{imm} & 0 \\ 0 & -(k_{des} + k_{sol}) & (k_{ads} + k_{pre}) & 0 \\ k_{min} & (k_{des} + k_{sol}) & -(k_{imm} + k_{ads} + k_{abs} + k_{pre}) & 0 \\ 0 & 0 & 1 & -k_{abs} \end{pmatrix}$$

$\det(A) = k_{abs}^2 k_{min}(k_{sol} + k_{abs})$, the determinant of matrix A does not equal zero, then this matrix is invertible. We use the results from [15, 16] and apply inverse matrix of A to get the following change variable :

$$Y(t) = X(t) + A^{-1}B,$$

with $X(t) = \left([P_{org}](t), [P_{min}](t), [P_i](t), [CP_i](t)\right)$, we make a deduction that

$$\dot{Y}(t) = \dot{X}(t),$$

then we obtain the following (12) that is equivalent system of (6):

$$\dot{Y}(t) = AY(t) \tag{12}$$

In order to use Routh-Hurwitz criterion, we define the characteristic polynome of matrix A as :

$$P(\lambda) = \det(\lambda I - A)$$

$$P(\lambda) = \begin{vmatrix} k_{min}+\lambda & 0 & -k_{imm} & 0 \\ 0 & \lambda+(k_{des}+k_{sol}) & -(k_{ads}+k_{pre}) & 0 \\ -k_{min} & -(k_{des}+k_{sol}) & \lambda+(k_{imm}+k_{ads}+k_{abs}+k_{pre}) & 0 \\ 0 & 0 & -1 & \lambda+k_{abs} \end{vmatrix}$$

thus, we obtain

$$P(\lambda) = (\lambda + k_{abs})(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3)$$

where

$a_1 = k_{min} + k_{des} + k_{sol} + k_{imm} + k_{ads} + k_{abs} + k_{pre},$

$a_2 = (k_{des} + k_{sol})(k_{imm} + k_{abs}) + k_{min}\big(k_{des} + k_{sol} + k_{ads} + k_{abs} + k_{pre}\big)$

$a_3 = k_{imm}k_{abs}\big(k_{des} + k_{sol}\big)$

$$\begin{cases} a_1 > 0 \\ a_2a_1 - a_3 > 0 \\ a_3 > 0 \end{cases}$$

According to Routh Hurwitz's criterion, the polynomial $P(\lambda)$ has all its roots with negative real parts. Then the origin $Y^* = 0$ of the system (12) is asymptotically stable, we deduce that the equilibrium point $X^* = ([P_{org}]^*, [P_{min}]^*, [P_i]^*, [CP_i]^*)$ is asymptotically stable. Thus, the theorem has been proved.

□

4.5. Boundedness of the solution

We consider the system (5) and we set : $U(t) = X(t) - X^*$, where X^* is the equilibrium point (11), we obtain the equivalent system

$$\begin{cases} \frac{dU(t)}{dt} &= AU(t) \\ U(t_0) &= U_0. \end{cases} \tag{13}$$

From the results in [15, 16], we draw the conclusion that the system has a unique solution generaly defined as follow :

$$U(t) = U_0 e^{(t-t_0)A} \qquad \forall t \in [t_0, +\infty[\,. \tag{14}$$

Thus, it resuts that :

$$X(t) - X^* = U_0 e^{(t-t_0)A} \qquad \forall t \in [t_0, +\infty[\tag{15}$$

The boundedness of the solution $X(t)$, is given in the next theorem:

Theorem 1. *If every eigenvalue of A has negative real part, then $U(t)$ is uniformly bounded. Therefore the solution $X(t)$ of model (5) is bounded.*

Lemma 4. *([15]).*
If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of a $n \times n$ square matrix M, the set of associated eigenvectors $\{v_1, \cdots, v_n\}$ form a basis and the matrix

$$P = [v_1, \cdots, v_n]$$

is invertible. We write

$$P^{-1}MP = diag[\lambda_1, \cdots, \lambda_n].$$

Proof. of Theorem 1.
In this work, the matrix A defined by (7) is a square matrix of order 4, with real coefficients. So, A is diagonalizable because it has 4 distinct eigenvalues. Then, there is an invertible matrix P such that

$A = PDP^{-1}$ where D is the diagonal matrix. According to results from [15, 16], which allow us to write :

$$e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!} = P \left(\sum_{k=0}^{+\infty} \frac{D^k}{k!} \right) P^{-1},$$

it follows then

$$e^A = P\, e^D\, P^{-1}.$$

We deduce that , the solution of the system (13) verify

$$U(t) = P\, e^{(t-t_0)D}\, P^{-1}\, U_0 \qquad \forall t \in [t_0, +\infty[\,.$$

Given triangle inquality in [15, 16] it appears that :

$$\begin{array}{rcl} \|U(t)\| & = & \|P\, e^{(t-t_0)D}\, P^{-1}\, U_0\| \\ \|U(t)\| & \leq & \|U_0\|\, \|P\|\, \|P^{-1}\|\, e^{\alpha(t-t_0)} \\ \|U(t)\| & \leq & \|U_0\|\, K e^{\alpha(t-t_0)} \qquad\qquad \forall t \in [t_0, +\infty[\,. \end{array}$$

where $K \in \mathbb{R}_+$, such that $\|P\|\, \|P^{-1}\| \leq K$ and α is the largest real part of the eigenvalues of the matrix A, here α is a negative real number.
Since $\alpha < 0$ then $\|U(t)\| \leq K\|U_0\|$. Which implies

$$\|X(t) - X^*\| \leq K\|U_0\|$$

According to the consequence of the triangle inequality in [16, 15], it results that :

$$|\ \|X(t)\| - \|X^*\|\ | \leq K\, \|U_0\|$$

we draw the conclusion that :

$$\forall t \in [t_0, +\infty[\,, \quad \|X(t)\| \leq \|X^*\| + K\, \|U_0\|$$

Consequently the solution $X(t)$ of the system (9) is bounded.
Note that for a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and a square real matrix $P = (P_{ij})_{1 \leq i,j \leq n}$:

$$\|x\| = \sqrt{\sum_{k=1}^{n} x_k^2} \text{ and } \|P\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |P_{ij}|.$$

□

5. Results and discussions

In this section, we justify our theoretical results by considering the concentrations of different phosphorus forms in Imeko soil, a region in Nigeria. This region is located less than 30 kilometers from Ketou, one of a town in Benin where agriculture is significant. According to the study in [19], the concentration of assimilable phosphorus is $[P_i]_0 = 0.145 g/kg$, concentration of organic phosphorus is $[P_{org}]_0 = 98.64 g/kg$, concentration of mineral phosphorus is determined by summation of Saloid P, Occuled P, reductant P, residual P, we have $[P_{min}]_0 = 288.36 g/kg$ and the total phosphorus content in soil is TP $= 387.0 g/kg$. Soluble inorganic phosphorus represents 0.1% of total soil content phosphorus [5, 20], then P_i content of field is $[CP_i]_0 = 0.387 g/kg$. Carbon makes up 58% of concentration of organic matter, so the organic matter content is $[MO]_0 = 48.28 g/kg$.

We use transformation rates between stable and labile phosphorus pools presented by [21], which considers two pools of phosphorus, labile pool which represents an assimilable phosphorus and the

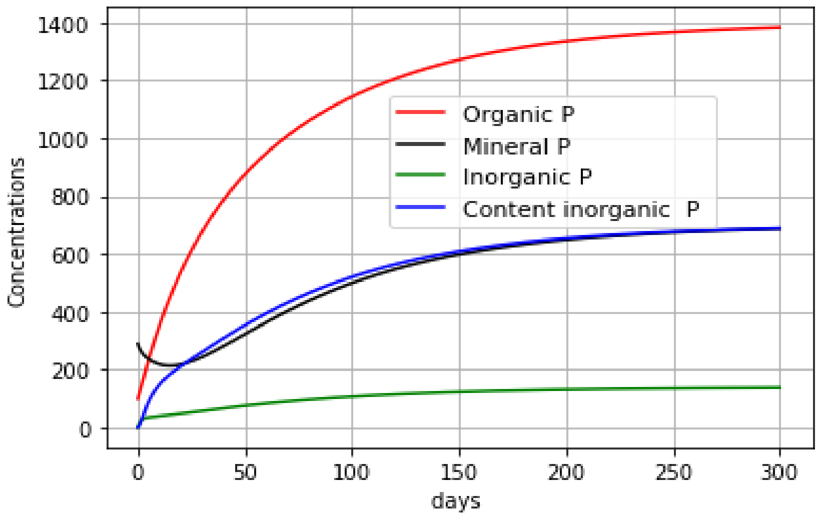

Figure 3: plots of the concentrations of organic P, inorganic P , mineral P and Pi content CPi with $k_{sol} = k_{des} = k_{min} = 0.04,\ k_{ads} = k_{pre} = k_{imm} = k_{abs} = 0.2.$

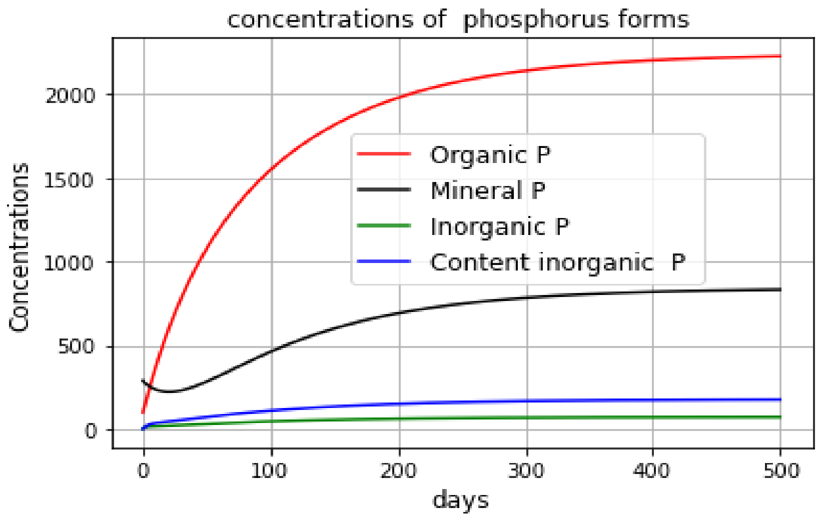

Figure 4: plots of concentrations of organic P, inorganic P , mineral P and Pi content CPi with $k_{sol} = k_{des} = k_{min} = 0.025,\ k_{ads} = k_{pre} = k_{imm} = 0.4,\ k_{abs} = 0.2$.

stable pool which include both mineral and organic phosphorus. We set the rates of all phenomena that reduce assimilable phosphorus as transformation rates of labil to the stable pool and the rates of the phenomena that mobilize assimilable phosphorus as transformation rate of the stable pool to the labile pool. We set $k_{deg} = 0.58$ as degradation rate of organic matter and we varied the uptake rate k_{abs} of assimilable phosphorus between 0 to 1.

To identify which uptake rate allows us to maintain an available phosphorus concentration in rhizosphere, we evaluated the percentage of available phosphorus as a function of the uptake rate.

Depending on the different types of soil considered, the soil achieves some sustainability in phosphorus (P), as follows :

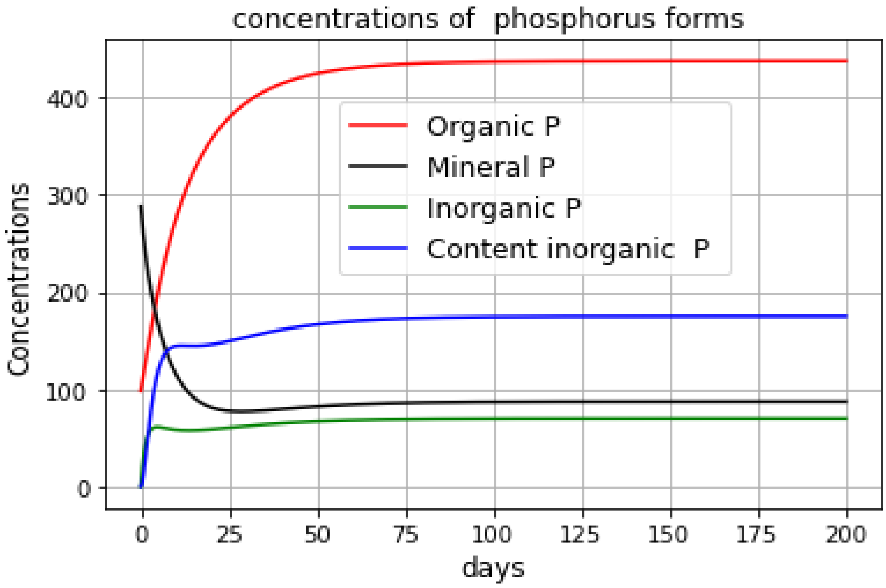

Figure 5: plots of the concentrations of organic P, inorganic P , mineral P and Pi content CPi with $k_{sol} = k_{des} = k_{min} = 0.08,\ k_{ads} = k_{pre} = k_{imm} = 0.1,\ k_{abs} = 0.4$.

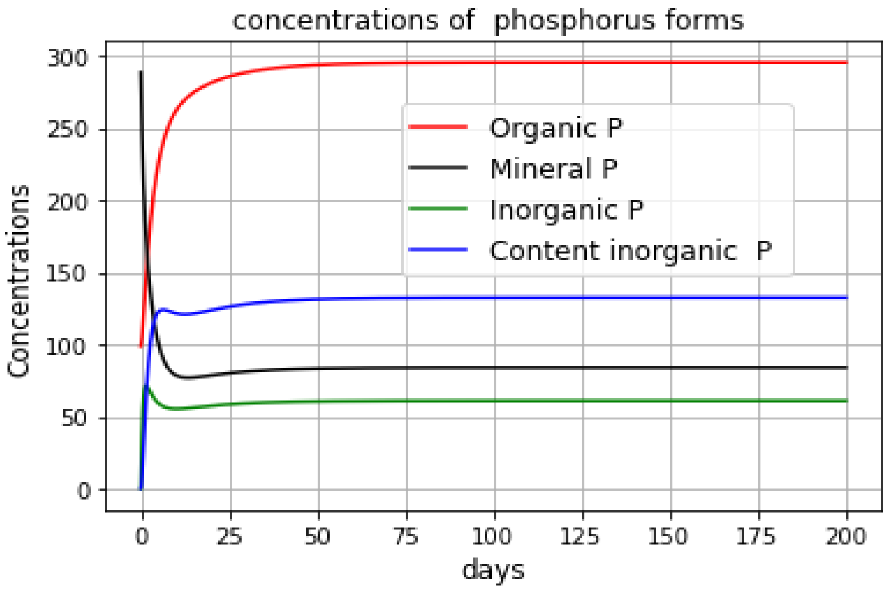

Figure 6: plots of concentration of organic P, inorganic P , mineral P and Pi content CPi with $k_{min} = 0.2$, $k_{sol} = 0.23,\ \ k_{des} = 0.3,\ \ k_{deg} = 0.58,\ \ k_{imm} = 0.51,\ \ k_{ads} = 0.4,\ \ k_{pre} = 0.33,\ \ k_{abs} = 0.46.$

- Figure 4: $X^* = (2240.19,\ 840.07,\ 70.00,\ 175.01)$
- Figure 3: $X^* = (1400.12,\ 700.06,\ 140.01,\ 700.05)$
- Figure 5: $X^* = (437.53,\ 87.50,\ 70.00,\ 175.01)$
- Figure 6: $X^* = (295.24,\ 83.84,\ 60.87,\ 132.33)$

It can be seen that to achieve significant sustainability in phosphorus (P), a significant number of days are also required, specifically :

- Figure 4: $>$ 500 days.
- Figure 3: $>$ 200 days.
- Figure 5: $\approx$ 70 days.
- Figure 6: $\approx$ 40 days.

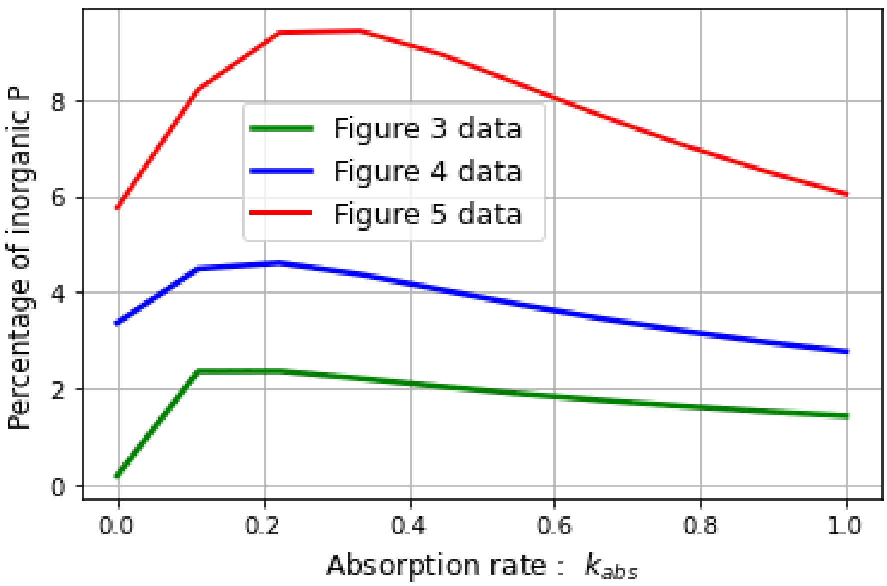

Figure 7: Fraction $[P_i]$ of total phosphorus concentration vs uptake rate k_{abs}

During the time, we observe that the concentrations of organic phosphorus $[P_{org}]$ and inorganic phosphorus content $[CP_i]$ follow a strong increase, and concentrations of different phosphorus forms converge to the equilibrium point $X^* = ([P_{org}]^*, [P_{min}]^*, [P_i]^*, [CP_i]^*)$ of model (5) which is asymptotically stable.

As the rate of mineralization k_{min} increases, the soil becomes less fertile in organic phosphorus $[P_{org}]$. Moreover, the rates of immobilization k_{imm}, precipitation k_{pre} and adsorption k_{ads} do not directly influence the sustainability of the mineral phosphorus $[P_{min}]$ since the origin which is $[P_i]$ has a small fraction of the total phosphorus in the soil.

Figure 7 shows the effect of the uptake rate k_{abs} on the availability of $[P_i]$. Note that if the uptake rate increases,then the availability of phosphorus $[P_i]$ decreases. To maintain available phosphorus in the soil, it suggest that the uptake rate must be between 0.2 and 0.3.

6. Conclusion

. In this work, we described a phosphorus dynamics in the rhizosphere using a compartmental mathematical model of the phosphorus cycle with constant transformation rates of phosphorus cycle phenomena. This model is a system of first order differential equations, a qualitative study was conducted to establish an uniqueness, positivity and boundedness of the solution, and we show that the equilibrium point of the model is asymptotically stable. Computer simulations were performed to justify theoretical results. Finally, we suggest suitable uptake rate values which can not remove assimilable phosphorus in the rhizosphere.

Future research could consider using Artificial Intelligence in particulary Machine Learning to predict the concentration of assimilable phosphorus in soil.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

References

[1] A. Kafle, K. R. Cope, R. Raths, J. Krishna Yakha, S. Subramanian, H. Bücking, K. Garcia, Harnessing soil microbes to improve plant phosphate efficiency in cropping systems, Agronomy 9 (2019) 127.

[2] F. Khan, A. B. Siddique, S. Shabala, M. Zhou, C. Zhao, Phosphorus plays key roles in regulating plants' physiological responses to abiotic stresses, Plants 12 (2023) 2861.

[3] N. A. Agbodjato, T. Mikpon, O. O. Babalola, D. Dah-Nouvlessounon, O. Amogou, H. Lehmane, M. Y. Adoko, A. Adjanohoun, L. Baba-Moussa, Use of plant growth promoting rhizobacteria in combination with chitosan on maize crop: Promising prospects for sustainable, environmentally friendly agriculture and against abiotic stress, Agronomy 11 (2021) 2205.

[4] G. Kalayu, Phosphate solubilizing microorganisms: promising approach as biofertilizers, International Journal of Agronomy 2019 (2019) 1–7.

[5] S. B. Sharma, R. Z. Sayyed, M. H. Trivedi, T. A. Gobi, Phosphate solubilizing microbes: sustainable approach for managing phosphorus deficiency in agricultural soils, SpringerPlus 2 (2013) 1–14.

[6] R. Prasad, D. Chakraborty, Phosphorus basics: Understanding phosphorus forms and their cycling in the soil, Ala. Coop. Ext. Syst (2019).

[7] C. Plassard, A. Robin, E. Le Cadre, C. Marsden, J. Trap, L. Herrmann, K. Waithaisong, D. Lesueur, E. Blanchart, L. Lardy, et al., Améliorer la biodisponibilité du phosphore: comment valoriser les compétences des plantes et les mécanismes biologiques du sol, Innovations Agronomiques 43 (2015) 115–138.

[8] N. Prabhu, S. Borkar, S. Garg, Phosphate solubilization by microorganisms: Overview, mechanisms, applications and advances (chapter 11) (2019).

[9] E. T. Alori, B. R. Glick, O. O. Babalola, Microbial phosphorus solubilization and its potential for use in sustainable agriculture, Frontiers in microbiology 8 (2017) 971.

[10] B. L. Turner, B. J. Cade-Menun, L. M. Condron, S. Newman, Extraction of soil organic phosphorus, Talanta 66 (2005) 294–306.

[11] H. Rodriguez, R. Fraga, Phosphate solubilizing bacteria and their role in plant growth promotion, Biotechnology advances 17 (1999) 319–339.

[12] A. E. Richardson, R. J. Simpson, Soil microorganisms mediating phosphorus availability update on microbial phosphorus, Plant physiology 156 (2011) 989–996.

[13] H. Charles, K. Quirine, D. Dale, S. Kristen, C. Karl, A. Greg, Phosphorus basics–the phosphorus cycle, Agronomy Fact Sheet Series. Fact Sheet 12 (2005).

[14] P. Hinsinger, Bioavailability of soil inorganic p in the rhizosphere as affected by root-induced chemical changes: a review, Plant and soil 237 (2001) 173–195.

[15] L. Perko, Differential equations and dynamical systems, volume 7, Springer Science & Business Media, 2013.

[16] R. Thompson, W. Walter, Ordinary Differential Equations, Graduate Texts in Mathematics, Springer New York, 1998. URL: https://books.google.bj/books?id=111gqKlSnEIC.

[17] W. Mitkowski, Metzler cyclic electric systems, Maritime Technical Journal 209 (2017) 109–118.

[18] M. Bodson, Explaining the routh-hurwitz criterion, A tutorial presentation (Inglés)[Explicación del criterio de Routh-Hurwitz. Un tutorial de presentación] 15 (2019).

[19] A. Soremi, M. Adetunji, J. Azeez, C. Adejuyigbe, J. Bodunde, Speciation and dynamics of phosphorus in some organically amended soils of southwestern nigeria, Chemical Speciation & Bioavailability 29 (2017) 42–53.

[20] M. S. Khan, A. Zaidi, M. Ahemad, M. Oves, P. A. Wani, Plant growth promotion by phosphate solubilizing fungi–current perspective, Archives of Agronomy and Soil Science 56 (2010) 73–98.

[21] S. Z. Sattari, A. F. Bouwman, K. E. Giller, M. K. van Ittersum, Residual soil phosphorus as the missing piece in the global phosphorus crisis puzzle, Proceedings of the National Academy of Sciences 109 (2012) 6348–6353.