First-Order Modal Systems of Partial Ambiguous Predicates

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Abstract

The concepts and methods of mathematical logic find wide application in computer science and programming. Among them, modal logics are of particular importance as they are used to model dynamic systems, in artificial intelligence, expert systems and for program specification and verification. This paper explores new classes of program-oriented logical formalisms of modal type – pure first-order modal logics of partial ambiguous quasiary predicates. Modal logics of quasiary predicates combine the expressive capabilities of traditional modal logics with those of the logics of quasiary predicates. The most important class of these logics is transitional modal logics (TML), which allow us to describe the evolution and dynamics of subject domains. At the core of TML lies the notion of a transitional modal system (TMS). Two variants of TML based on logics of partial ambiguous (multi-valued) quasiary predicates of relational type, or *R*-predicates, are proposed: pure first-order general TMS (GMS^Q) and pure first-order multiple-expert modal systems with dominance. This work describes the specific semantic features of GMS^Q of *R*-predicates and demonstrates the connection between TML of *R*-predicates and four-valued modal logics based on Belnap's logic. The second type of the proposed TMS is pure first-order multiple-expert modal systems of *R*-predicates ($MEMS^Q$). On the set of experts, a transitive dominance relation is introduced. Differing from known multiple-expert modal systems, which are oriented towards preserving truth under dominance, the proposed $MEMS^Q$ preserve both truth and falsity under dominance. To avoid inconsistencies in dominance, a separate dominance relation is proposed: an expert may be directly dominated by at most one other expert. The languages of pure first-order MEMS, referred to as $MEMS^Q$, are formally defined. On the set of formulas specified by both experts and states, a number of logical consequence relations are introduced. In $MEMS^Q$ of *P*-predicates, we specify consequence relations $|=_{IR}$, $|=_T$, $|=_F$, and $|=_{TF}$, while in в $MEMS^Q$ of *R*-predicates, there remains a single non-degenerate relation ${}^R|=_{TF}$. A further study of the proposed logics is planned for future work.

Keywords

modal logic, transitional modal system, multiple-expert modal system, partial predicate, logical consequence

1. Introduction

The concepts and tools of mathematical logic are widely used to describe and model various subject domains, as well as in information and software systems (see, for example, [1–3]). Among them, modal logics have found broad application in practical areas [4, 5], especially temporal logics [6] and epistemic logics [7]. Temporal logics are used for dynamic systems modelling, program specification and verification; epistemic logics are applied in artificial intelligence systems, knowledge bases and expert systems (see, e.g., [1–3, 5, 7]).

The capabilities of traditional modal logics and logics of partial quasiary predicates [8] are combined in composition nominative modal logics. The most important class of such logics is transitional modal logics (TML) of partial quasiary predicates. These logics allow describing the

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evolution and change of subject domains. TML have been studied in a number of works. In particular, [9, 10] investigate pure first-order TML without monotonicity condition on quasiary predicates; these logics are referred to as TML^Q. The main focus of those works is on TML^Q with predicates of strong equality and weak equality; the corresponding classes of logics are denoted as $TML^{Q\text{-}}$ and $TML^{Q=}$.

The concept of TML is based [9, 10] on the notion of a transitional modal system (TMS). Pure first-order TMS are called TMS^Q. Common varieties of TMS include general TMS (GMS), temporal TMS (TmMS), and multimodal TMS (MMS). Works [9, 10] focus on GMS, though similar techniques can be applied to study other TMS variants. Pure first-order GMS are referred to as GMS^Q.

The modal systems explored in [9, 10] were based on logics of *single-valued* quasiary predicates, or *P*-predicates, which were described, for example, in [8].

The aim of this work is to investigate new classes of program-oriented logical formalisms of modal type, which are not restricted by conditions of predicate single-valuedness and monotonicity. In this paper, we begin studying modal logics of *ambiguous* (multi-valued, non-deterministic) quasiary predicates of relational type, or *R*-predicates. Such predicates are described, for example, in [8]. We consider two variants of pure first-order TMS: the GMS^Q of *R*-predicates, and special multiple-expert modal systems with dominance ($MEMS^Q$), which may be viewed as a specific type of epistemic logic.

In the second part of the work, we focus on MEMS. Let us examine this in more detail.

Epistemic logics (logics of knowledge and belief) are characterized by the presence of knowledge experts, or intelligent agents. Each of these experts may be in a certain situation (that is, a state of the world). For each expert, a binary relation is defined on the set of such possible situations (states), called the transition relation (or accessibility relation). The accessibility relation $\triangleright_a$ for an expert (intelligent agent) a is interpreted as follows: $\alpha \triangleright_a \beta$ means that in state (situation) α, expert a considers state (situation) β to be possible. Each expert may have their own opinion about a given situation. If an expert's opinion is independent of others, conflicting situations may arise in multiple-expert systems. Hence, the problem of making a final decision emerges. To address this, a dominance relation is introduced on the set of experts. The notion of truth then depends on the dominance relation between experts: in a given possible situation, it may differ for different experts. Accessibility relations between possible situations may also vary across experts. Multiple-expert modal models of the propositional level with a dominance relation are described in [11]. If expert a dominates expert b and believes a statement S is true, then b must also accept S as true. Similarly, if a considers situation β to be possible from α (i.e., $\alpha \triangleright_a \beta$), then b must also consider β possible from α (i.e., $\alpha \triangleright_b \beta$). This means that the multiple-expert modal models in [11] are designed to preserve truth under dominance, and are thus studied basing on relational models of propositional intuitionistic logic. For such relational multiple-expert models, equivalent many-valued modal models are constructed based on Heyting algebra.

The interaction of knowledge experts (intelligent agents) is studied in [12–14]. The problem of reaching the consensus by a group of communicating intelligent agents is examined in [14]. Notably, the ordering of agents by sharpness of perception in [14] is opposite to the ordering considered in [11].

In this paper, we consider multiple-expert modal systems based on logics of pure first-order quasiary predicates. The truth values of such logics are *true* and *false*, denoted T and F. We require that truth values be preserved under dominance, that is, both T and F must be preserved. This means: if expert a dominates b and considers statement S to be true, then b must also consider S to be true; if a considers statement S to be false, then b must also consider S to be false. The dominance relation is assumed to be transitive. If an expert is directly dominated by more than one expert, a conflict can arise: for instance, dominant expert a may require S to be true, while another dominant expert b may require S to be false. Note that such a conflict cannot arise in the models from [11], since dominance there only requires preservation of truth, not falsity. In a multiple-expert system based on the logic of quasiary predicates, such inconsistency can be avoided by restricting the dominance relation, requiring that an expert may be directly dominated by only one expert.

The modal logical formalisms proposed in this work can be applied to the adequate modeling of complex information and software systems that require accounting for partiality, ambiguity and incompleteness of information. In particular, multiple-expert modal systems based on quasiary predicates may be used in artificial intelligence systems and expert systems to model the interaction between experts (intelligent agents).

We will follow the notation used in works [8–10]. Concepts that are not defined here are interpreted in the sense of [8–10].

2. Transitional Modal Systems of R-predicates

We specify a *V*-*A*-quasiary predicate as a partial ambiguous (non-deterministic) function $Q: {}^{V}A \to \{T, F\}$, where ${}^{V}A$ is the set of *V*-*A*-nominative sets (*V*-*A*-NS), *V* is the set of subject names (variables), *A* is the set of subject values, $\{T, F\}$ is the set of truth values. *V*-*A*-NS is formally defined [8] as a single-valued function of the form $d : V \to A$.

In this paper we interpret ambiguous (multi-valued) *V*-*A*-quasiary predicates as correspondences (relations) between ${}^{V}A$ and $\{T, F\}$. Therefore, we consider *R*-predicates [8].

Each *R*-predicate *Q* is determined by two sets:

– the *truth* domain $T(Q) = \{d\in{}^{V}A \mid T\in Q(d)\}$;

– the *falsity* domain $F(Q) = \{d\in{}^{V}A \mid F\in Q(d)\}$.

The single-valuedness condition for an *R*-predicate *Q* is $T(Q)\cap F(Q) = \varnothing$. Single-valued quasiary predicates are referred to as *P*-predicates [see 8]. Thus, we obtain a *P*-predicate, provided that condition $T(Q)\cap F(Q) = \varnothing$ holds for an *R*-predicate *Q*.

We will denote by PrR^{V-A} and PrP^{V-A} the classes of *V*-*A*-quasiary *R*-predicates and *V*-*A*-quasiary *P*-predicates, respectively.

For an *R*-predicate *Q*, the set $Q(d)$ of values that *Q* can take when applied to $d\in{}^{V}A$, can be one of the following sets: $\varnothing$, $\{T\}$, $\{F\}$, or $\{T, F\}$. These values will be denoted by $\uparrow$, *T*, *F*, and *TF*, respectively. For a *P*-predicate *Q*, the set $Q(d)$ can be one of the following: $\varnothing$, $\{T\}$, or $\{F\}$; this will be denoted by $Q(d)\uparrow$, $Q(d) = T$, and $Q(d) = F$, respectively.

Pure first-order TMS, or TMS^{Q}, is defined [9, 10] as the object $\boldsymbol{M} = ((St, \boldsymbol{R}, Pr, C), Fm, Im)$, where

– *St* is the set of states of the world; we specify states as algebraic systems (structures) of the form $\alpha = (A_\alpha, Pr_\alpha)$, where A_α is the set of basic data of state α, Pr_α is the set of quasiary predicates ${}^{V}A_\alpha \to \{T, F\}$, called predicates of state α;

– $\boldsymbol{R}$ is a set of relations of the form $R \subseteq St \times St$, interpreted as transition relations on states;

– *Pr* is a set of predicates of the system $\boldsymbol{M}$;

– *C* is a set of compositions on *Pr*;

– *Fm* is a set of formulas of the TML^{Q} language;

– *Im* is an interpretation mapping for formulas of the language on states of the world.

The set *Pr* consists of state predicates and *global* predicates; global predicates have the form ${}^{V}A \to \{T, F\}$, where $A = \bigcup_{\alpha\in S} A_\alpha$.

The set *C* is defined by basic logical compositions $\neg$, $\vee$, $R_{\bar{x},\perp}^{\bar{v},\bar{u}}$, $\exists x$, *Ez*, and basic modal compositions.

As noted above, one can define [9, 10] the following classes of TMS: General TMS (GMS), Temporal TMS (TmMS), and Multimodal TMS (MMS).

GMS^{Q} is a TMS^{Q} with $\boldsymbol{R} = \{\rhd\}$ and a basic modal composition $\Box$.

$TmMS^{Q}$ is a TMS^{Q} with $\boldsymbol{R} = \{\rhd\}$ and basic modal compositions $\Box_{\uparrow}$ and $\Box_{\downarrow}$.

MMS^{Q} is a TMS^{Q} with $\boldsymbol{R} = \{\rhd_i \mid i\in I\}$ and corresponding basic modal compositions M_i, $i\in I$.

Epistemic MMS^Q ($EpMS^Q$) is a MMS^Q with finite sets of same-type relations $\rhd_i$.

In the next section of the paper, we propose multiple-expert modal systems with dominance, which may be seen as a special type of epistemic logic systems. In this section, we restrict our attention to GMS^Q.

Let us describe a language of a GMS^Q. The alphabet: a set V of variables (subject names), a set Ps of predicate symbols, a set of basic logical compositions' symbols $\{\neg, \vee, R_{\bar{x},\perp}^{\bar{v},\bar{u}}, \exists x; Ez\}$, a set $Ms = \{\Box\}$ of basic modal compositions' symbols.

The set Fm of language formulas is given as follows:

Fa) $Ps \subseteq Fm$; formulas of the form $p \in Ps$ will be called *atomic*;

$F_\neg$) $\Phi \in Fm \Rightarrow \neg\Phi \in Fm$;

$F_\vee$) $\Phi, \Psi \in Fm \Rightarrow \vee\Phi\Psi \in Fm$;

F_R) $\Phi \in Fm \Rightarrow R_{\bar{x},\perp}^{\bar{v},\bar{u}}\Phi \in Fm$;

$F_\exists$) $\Phi \in Fm \Rightarrow \exists x\Phi \in Fm$;

$F_\Box$) $\Phi \in Fm \Rightarrow \Box\Phi \in Fm$.

Formulas that contain symbols of modal compositions (in our case, the symbol $\Box$), are called modalized.

Formulas that do not contain modal composition symbols are called non-modalized.

The symbols Ez of predicates-indicators (see [8, 10]) form the set Frs of *singular formulas* of the language: $Frs = \{Ez \mid z \in V\}$. Such symbols are not components of complex formulas and may only appear as elements of sets of formulas of the language. The predicates-indicators are special 0-ary compositions; they are used for quantifier elimination (see [8, 10]).

The set $Fr = Fm \cup Frs$ will be called the *extended* set of formulas.

So far we have defined formulas in prefix notation. Going forward, the traditional infix notation and the symbols of derived compositions $\rightarrow$, &, $\forall x$, and $\Diamond$ will be used (see [8, 10]). In particular, the formulas $\neg\exists x\neg\Phi$ and $\neg\Box\neg\Phi$ will be abbreviated as $\forall x\Phi$ and $\Diamond\Phi$, respectively.

Note that from a syntactic point of view, the languages GMS^Q of *R*-predicates and GMS^Q of *P*-predicates are identical. The difference lies in the different classes of semantic models and different interpretation mappings.

The distinction of certain classes of quasiary predicates induces the distinction of the corresponding interpretation classes, or semantics. We consider the general class of *R*-predicates, within which a subclass of single-valued *R*-predicates, or *P*-predicates, is identified. Therefore, in the context of GMS^Q, we may further refer to *R*-semantics and *P*-semantics.

Let us specify an interpretation mapping for formulas on states of the world of GMS^Q.

We start by defining $Im: Ps \times St \rightarrow Pr$, where it must hold that $Im(p, \alpha) \in Pr_\alpha$. Therefore, basic predicates are states predicates. Composition symbols are interpreted as the corresponding logical or modal compositions. The mapping $Im: Ps \times St \rightarrow Pr$ is extended to a full interpretation mapping $Im: Fm \times St \rightarrow Pr$ as follows:

I¬) $Im(\neg\Phi, \alpha) = \neg(Im(\Phi, \alpha))$.

I∨) $Im(\vee\Phi\Psi, \alpha) = \vee(Im(\Phi, \alpha), Im(\Psi, \alpha))$.

IR) $Im(R_{\bar{x},\perp}^{\bar{v},\bar{u}}\Phi, \alpha) = \mathrm{R}_{\bar{x},\perp}^{\bar{v},\bar{u}}(Im(\Phi, \alpha))$.

I∃) $Im(\exists x\Phi, \alpha) = \exists x(Im(\Phi, \alpha))$.

I□) $d \in T(Im(\Box\Phi, \alpha)) \Leftrightarrow$ there exists $\beta \in St: \alpha \rhd \beta$; and for any $\delta \in St$ we have:

$$\alpha \rhd \delta \Rightarrow d \in T(Im(\Phi, \delta));$$

$d \in F(Im(\Box\Phi, \alpha)) \Leftrightarrow$ there exists $\delta \in St: \alpha \rhd \delta$ and $d \in F(Im(\Phi, \delta))$.

We will abbreviate the predicate $Im(\Phi, \alpha)$ as Φ_α.

Statement 1. The clauses I¬, I∨, IR, and I∃ can be reformulated in terms of the truth and falsity domains of the respective predicates as follows:

I¬d) $T((\neg\Phi)_\alpha) = F(\Phi_\alpha)$;

$F((\neg\Phi)_\alpha) = T(\Phi_\alpha)$.

I∨d) $T((\vee\Phi\Psi)_\alpha) = T(\Phi_\alpha)\cup F(\Phi_\alpha)$;

$F((\vee\Phi\Psi)_\alpha) = F(\Phi_\alpha)\cap F(\Phi_\alpha)$.

IRd) $T((R^{\bar{v},\bar{u}}_{\bar{x},\perp}\Phi)_\alpha) = \{d\in {}^V A \mid r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)\in T(\Phi_\alpha)\}$;

$F((R^{\bar{v},\bar{u}}_{\bar{x},\perp}\Phi)_\alpha) = \{d\in {}^V A \mid r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)\in F(\Phi_\alpha)\}$.

I∃d) $T((\exists x\Phi)_\alpha) = \bigcup_{a\in A_\alpha} \{d \mid d\|_{-x} \cup x\mapsto a\in T(\Phi_\alpha)\}$;

$F((\exists x\Phi)_\alpha) = \bigcap_{a\in A_\alpha} \{d \mid d\|_{-x} \cup x\mapsto a\in F(\Phi_\alpha)\}$.

Statement 2. For the negations of membership in the truth and falsity domains of the predicate $(\Box\Phi)_\alpha$, we have:

I¬□) $d\notin T((\Box\Phi)_\alpha) \Leftrightarrow$ there doesn't exist $\beta\in St$: $\alpha\rhd\beta$,

or there exists $\delta\in St$: $\alpha\rhd\delta$ and $d\notin T(\Phi_\delta)$;

$d\notin F((\Box\Phi)_\alpha) \Leftrightarrow$ for any $\delta\in St$ we have: $\alpha\rhd\delta \Rightarrow d\notin F(\Phi_\delta)$.

The algebra of partial ambiguous quasiary predicates of relational type (*R*-predicates) is isomorphic [15] to the algebra of total single-valued predicates of Belnap's 4-valued logic [16]. Therefore, *R*-predicates can be modelled as predicates of Belnap's 4-valued logic with the set of truth values {↑, *T*, *F*, *TF*}.

Thus, for *R*-predicates we have:

$Q(d) = T$, if $T\in Q(d)$ and $F\notin Q(d)$;

$Q(d) = F$, if $T\notin Q(d)$ and $F\in Q(d)$;

$Q(d) = TF$, if $T\in Q(d)$ and $F\in Q(d)$;

$Q(d) = \uparrow$, or $Q(d)\uparrow$, if $T\notin Q(d)$ and $F\notin Q(d)$.

Note that the algebra of total single-valued predicates of Kleene's strong 3-valued logic and the algebra of partial single-valued quasiary predicates (*P*-predicates) are also isomorphic [15], therefore, *P*-predicates can be modelled as predicates of Kleene's strong 3-valued logic with the set of truth values {↑, *T*, *F*}.

Hence, modal logics of quasiary *P*-predicates can be modelled as 3-valued modal logics based on Kleene's strong 3-valued logic, and modal logics of quasiary *R*-predicates can be modelled as 4-valued modal logics based on Belnap's 4-valued logic.

Statement 3. The clauses I¬, I∨, and IR can be presented as follows:

$$\text{I}\neg_{\text{B}})\ (\neg\Phi)_\alpha(d) = \begin{cases} T, & \text{if } \Phi_\alpha(d) = F; \\ F, & \text{if } \Phi_\alpha(d) = T; \\ TF, & \text{if } \Phi_\alpha(d) = TF; \\ \uparrow, & \text{if } \Phi_\alpha(d) = \uparrow. \end{cases}$$

$$\text{I}\vee_{\text{B}})\ (\vee\Phi\Psi)_\alpha(d) = \begin{cases} T, & \text{if } \Phi_\alpha(d) = T \text{ or } \Psi_\alpha(d) = T \text{ or} \\ & (\Phi_\alpha(d) = TF \text{ and } \Psi_\alpha(d) = \uparrow) \text{ or } (\Phi_\alpha(d) = \uparrow \text{ and } \Psi_\alpha(d) = TF); \\ F, & \text{if } \Phi_\alpha(d) = F \text{ and } \Psi_\alpha(d) = F; \\ TF, & \text{if } (\Phi_\alpha(d) = TF \text{ and } \Psi_\alpha(d) = TF) \text{ or} \\ & (\Phi_\alpha(d) = TF \text{ and } \Psi_\alpha(d) = F) \text{ or } (\Phi_\alpha(d) = F \text{ and } \Psi_\alpha(d) = TF); \\ \uparrow, & \text{if } (\Phi_\alpha(d) = \uparrow \text{ and } \Psi_\alpha(d) = \uparrow) \text{ or} \\ & (\Phi_\alpha(d) = \uparrow \text{ and } \Psi_\alpha(d) = F) \text{ or } (\Phi_\alpha(d) = F \text{ and } \Psi_\alpha(d) = \uparrow). \end{cases}$$

$$\text{IR}_{\text{B}})\ (R^{\bar{v},\bar{u}}_{\bar{x},\perp}\Phi)_\alpha(d) = \begin{cases} T, & \text{if } \Phi_\alpha(r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)) = T; \\ F, & \text{if } \Phi_\alpha(r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)) = F; \\ TF, & \text{if } \Phi_\alpha(r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)) = TF; \\ \uparrow, & \text{if } \Phi_\alpha(r^{\bar{v},\bar{u}}_{\bar{x},\perp}(d)) = \uparrow. \end{cases}$$

Similar concretizations of the interpretation mapping in the style of Belnap's 4-valued logic can be made for the clauses I∃ and I□, basing on the following theorems.

Theorem 1. For any $\Phi\in Fm$, $\alpha\in St$, and $a, b, c\in A$, and $d\in{}^{V}\!A$, we have:

1) $(\exists x\Phi)_\alpha(d) = T \Leftrightarrow d\in T((\exists x\Phi)_\alpha)$ and $d\notin F((\exists x\Phi)_\alpha) \Leftrightarrow \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = T$ for some $a\in A_\alpha$ or ($\Phi_\alpha(d\,\|_{-x} \cup x \mapsto c) = TF$ for some $c\in A_\alpha$ and $\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow$ for some $b\in A_\alpha$);

2) $(\exists x\Phi)_\alpha(d) = F \Leftrightarrow d\notin T((\exists x\Phi)_\alpha)$ and $d\in F((\exists x\Phi)_\alpha) \Leftrightarrow \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = F$ for any $a\in A_\alpha$;

3) $(\exists x\Phi)_\alpha(d) = TF \Leftrightarrow d\in T((\exists x\Phi)_\alpha)$ and $d\in F((\exists x\Phi)_\alpha) \Leftrightarrow \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = TF$ for some $a\in A_\alpha$ and for any $b\in A_\alpha$ we have ($\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = TF$ or $\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow$);

4) $(\exists x\Phi)_\alpha(d) = \uparrow \Leftrightarrow d\notin T((\exists x\Phi)_\alpha)$ and $d\notin F((\exists x\Phi)_\alpha) \Leftrightarrow$ for any $b\in A_\alpha$ we have ($\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = F$ or $\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow$) and $\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow$ for some $a\in A_\alpha$.

Theorem 2. For any $\Phi\in Fm$, and $\alpha, \beta, \delta\in St$, and $d\in{}^{V}\!A$, we have:

1) $\Box\Phi_\alpha(d) = T \Leftrightarrow d\in T(\Box\Phi_\alpha)$ and $d\notin F(\Box\Phi_\alpha) \Leftrightarrow$
$\Leftrightarrow$ there exists β: $\alpha \rhd \beta$, and for any δ ($\alpha \rhd \delta \Rightarrow \Phi_\delta(d) = T$);

2) $\Box\Phi_\alpha(d) = F \Leftrightarrow d\notin T(\Box\Phi_\alpha)$ and $d\in F(\Box\Phi_\alpha) \Leftrightarrow$
$\Leftrightarrow$ there exists δ ($\alpha \rhd \delta$ and $\Phi_\delta(d) = F$) or (there exists δ ($\alpha \rhd \delta$ and $\Phi_\delta(d) = \uparrow$) and β ($\alpha \rhd \beta$ and $\Phi_\beta(d) = TF$));

3) $\Box\Phi_\alpha(d) = TF \Leftrightarrow d\in T(\Box\Phi_\alpha)$ and $d\in F(\Box\Phi_\alpha) \Leftrightarrow$
$\Leftrightarrow$ there exists δ ($\alpha \rhd \delta$ and $\Phi_\delta(d) = TF$), and for any δ ($\alpha \rhd \delta \Rightarrow \Phi_\delta(d) = T$ or $\Phi_\delta(d) = TF$);

4) $\Box\Phi_\alpha(d) = \uparrow \Leftrightarrow d\notin T(\Box\Phi_\alpha)$ and $d\notin F(\Box\Phi_\alpha) \Leftrightarrow$
$\Leftrightarrow$ there doesn't exist $\beta: \alpha \rhd \beta$, or (there exists δ ($\alpha \rhd \delta$ and $\Phi_\delta(d) = \uparrow$) and for any δ ($\alpha \rhd \delta \Rightarrow$
$\Rightarrow \Phi_\delta(d) = T$ or $\Phi_\delta(d) = \uparrow$)).

Corollary. The clauses I∃ and I□ can be presented in the following manner:

$\mathrm{I}\exists_{\mathrm{B}}$) $(\exists x\Phi)_\alpha(d) =$

$$= \begin{cases} T, \text{ if } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = T \text{ for some } a\in A_\alpha, \text{ or} \\ \quad \Phi_\alpha(d\,\|_{-x} \cup x \mapsto c) = TF \text{ for some } c\in A_\alpha \text{ and } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow \text{ for some } b\in A_\alpha; \\ F, \text{ if } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = F \text{ for any } a\in A_\alpha; \\ TF, \text{ if } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = TF \text{ for some } a\in A_\alpha, \text{ and} \\ \quad (\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = TF \text{ or } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = F) \text{ for any } b\in A_\alpha; \\ \uparrow, \text{ if } (\Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = F \text{ or } \Phi_\alpha(d\,\|_{-x} \cup x \mapsto b) = \uparrow) \text{ for any } b\in A_\alpha, \text{ and} \\ \quad \Phi_\alpha(d\,\|_{-x} \cup x \mapsto a) = \uparrow \text{ for some } a\in A_\alpha. \end{cases}$$

$\mathrm{I}\Box_{\mathrm{B}}$) $\Box\Phi_\alpha(d) =$

$$= \begin{cases} T, \text{ if there exists } \beta: \alpha \rhd \beta, \text{ and for any } \delta\ (\alpha \rhd \delta \Rightarrow \Phi_\delta(d) = T); \\ F, \text{ if there exists } \delta\ (\alpha \rhd \delta \text{ and } \Phi_\delta(d) = F), \\ \quad \text{or there exists } \delta\ (\alpha \rhd \delta \text{ and } \Phi_\delta(d) = \uparrow) \text{ and } \beta\ (\alpha \rhd \beta \text{ and } \Phi_\delta(d) = TF); \\ TF, \text{ if there exists } \delta\ (\alpha \rhd \delta \text{ and } \Phi_\delta(d) = TF), \text{ and} \\ \quad \text{for any } \delta\ (\alpha \rhd \delta \Rightarrow \Phi_\delta(d) = T \text{ or } \Phi_\delta(d) = TF); \\ \uparrow, \text{ if there doesn't exist } \beta: \alpha \rhd \beta, \text{ or (there exists } \delta\ (\alpha \rhd \delta \text{ and } \Phi_\delta(d) = \uparrow), \text{ and} \\ \quad \text{for any } \delta\ (\alpha \rhd \delta \Rightarrow \Phi_\delta(d) = T \text{ or } \Phi_\delta(d) = \uparrow)). \end{cases}$$

For modal logics of quasiary *P*-predicates, a similar concretization of the interpretation mapping can also be made in the style of Kleene's strong 3-valued logic.

Thus, modal logics of quasiary *R*-predicates can be described in terms of 4-valued modal logics based on Belnap's logic, and vice versa; modal logics of quasiary *P*-predicates can be described in terms of 3-valued modal logics based on Kleene's strong 3-valued logic.

Depending on the properties of the accessibility relation $\rhd$, one can define different classes of $\mathrm{GMS}^{\mathrm{Q}}$. This can be done in the same way as for GMS of *P*-predicates (see [9, 10]).

Depending on how the value of $\Phi_\delta(d)$ is assigned under the condition $d\notin{}^{V}\!A_\delta$, one can distinguish (see [9]) TMS with a strong condition of definedness on states and TMS with a general condition of definedness on states. For GMS^Q of R-predicates, the general condition of definedness on states is preferable, since the strong condition of definedness imposes unnecessary constraints on semantic models. The general condition of definedness means that state predicates δ respond only to components with basic data $a{\in}A_\delta$.

Therefore, for non-modalized formulas, under the condition $d{\notin}{}^VA_\delta$, we assume that $\Phi_\delta(d) = \Phi_\delta(d_\delta)$. Here d_δ denotes the nominative set $[v{\mapsto}a{\in}d \mid a{\in}A_\delta]$.

The semantic properties of GMS^Q that are not related to modalities are, in general, analogous to the corresponding properties of the logic of quasiary R-predicates; properties related to the interaction of modal compositions with renominations and quantifiers also hold in GMS^Q (see [9, 10]).

On the set of formulas of the GMS^Q language specified with states, one can define a number of logical consequence relations. In the case of GMS^Q of R-predicates, we do this in the same way as for GMS^Q of P-predicates (see [9, 10]). First, we define the relations of IR-consequence, T-consequence, F-consequence, and TF-consequence within a specific GMS; then, based on this, we specify the corresponding logical consequence relations for a given class M of such GMS. For GMS^Q of P-predicates, we obtain four non-degenerate logical consequence relations: ${}^M|{=}_{IR}$, ${}^M|{=}_T$, ${}^M|{=}_F$, and ${}^M|{=}_{TF}$; they correspond to the analogous relations ${}^P|{=}_{IR}$, ${}^P|{=}_T$, ${}^P|{=}_F$, and ${}^P|{=}_{TF}$ in the traditional logics of quasiary predicates (see [8]). However, in the case of GMS^Q of R-predicates, there is a single non-degenerate relation ${}^{M^R}|{=}_{TF}$, which is analogous to the relation ${}^R|{=}_{TF}$ in logics of quasiary predicates.

The non-modal properties of the relation ${}^{M^R}|{=}_{TF}$ repeat the corresponding properties of the relation ${}^R|{=}_{TF}$ for sets of formulas in the traditional logic of quasiary predicates (for more details, see [8 – 10]). The properties related to modal compositions (e.g., carrying modal compositions through renominations and elimination of modalities) are analogous to the corresponding properties for GMS^Q of P-predicates (see [9]).

3. Multiple-Expert Modal Systems of Quasiary Predicates

We define multiple-expert modal system (MEMS) as an object $\boldsymbol{M} = ((Ex, \succ, St, \rhd, Pr, Cm), Fm, I_M)$, where:

– Ex is a finite non-empty set, which we interpret as a set (group) of experts;

– $\succ \subseteq Ex \times Ex$ is a relation of immediate dominance on the set of experts;

– St is a non-empty set interpreted as the set of possible states of the world, or situations;

– $\rhd \subseteq Ex \times St \times St$ is an accessibility relation $\rhd$ between possible states (situations), which depends on the expert being considered; instead of $(e, \alpha, \beta){\in}\rhd$, we will shortly write $\alpha \rhd_e \beta$;

– Pr is a set of predicates of the system $\boldsymbol{M}$;

– Cm is a set of compositions on Pr;

– Fm is a set of formulas of the MEMS language;

– $I_M : Ps \times Ex \times St \longrightarrow \{T, F\}$ is an interpretation mapping for atomic formulas; I_M depends on the state and expert being considered.

The reflexive-transitive closure of the relation $\succ$ will be denoted by $\gg$.

Instead of $\succ(e, g)$ and $\gg(e, g)$, we will usually write $e \succ g$ and $e \gg g$.

For the first-order MEMS, we specify the set St as a set of algebraic structures $\alpha = (A_\alpha, Pr_\alpha)$, where A_α is a set of basic data of the state α, Pr_α is a set of quasiary predicates ${}^VA_\alpha \longrightarrow \{T, F\}$. Such predicates will be called *predicates of the state* α. The set $A = \bigcup_{\alpha \in S} A_\alpha$ is called the *set of basic data of the system* $\boldsymbol{M}$. The predicates ${}^VA \longrightarrow \{T, F\}$ will be called *global*.

Pure first-order MEMS $\boldsymbol{M} = ((Ex, \succ, St, \rhd, Pr, Cm), Fm, I_M)$ will be shorter denoted by $\boldsymbol{M} = (Ex, \succ, St, \rhd, A, I_M)$.

The previously described GMS can be interpreted as 1-expert MEMS with a trivial dominance relation.

The accessibility relation $\rhd \subseteq Ex \times St \times St$ is connected to the domination relation $\gg\ \subseteq Ex \times Ex$ in the following way: $\alpha \rhd_e \beta$ and $e \gg h \Rightarrow \alpha \rhd_h \beta$.

Informally, if expert e considers that situation β is possible relative to the situation α (i.e., state β is accessible from state α), then any expert h over whom e dominates must also acknowledge this.

The dominance relation in MEMS must preserve both T and F: if expert a dominates b and considers statement (predicate) S to be true, then b must also consider S to be true; if a considers statement S to be false, then b must also consider S to be false. Therefore, a conflict can emerge when an expert is directly dominated by two or more experts. For instance, expert e is directly dominated by experts a and b, and a assumes predicate S on a data d to be true (i.e., $S(d) = T$), while b requires S to be false (i.e., $S(d) = F$). Hence the question arises: how expert e should evaluate $S(d)$?

Thus, for MEMS, it is appropriate to consider the following restriction on the dominance relation: only one expert may directly dominate any given expert. This means that the relation $\succ$ must be injective: for each $e \in Ex$, there exists at most one $g \in Ex$ such that $g \succ e$. An injective relation $\succ\ \subseteq Ex \times Ex$ can be represented as a forest of trees with vertices from Ex.

The reflexive-transitive closure of the injective relation of immediate dominance will be called the relation of *separate dominance.*

From now on, we will concentrate on the case of MEMS with a separate dominance relation.

Further study of MEMS, particularly the properties of dominance relations and accessibility relations is planned in future papers.

Let us describe the language of pure first-order MEMS, or $\mathrm{MEMS}^{\mathrm{Q}}$.

From a syntactic point of view, the languages $\mathrm{GMS}^{\mathrm{Q}}$ and $\mathrm{MEMS}^{\mathrm{Q}}$ are identical. The difference lies in their semantic models. Thus, in the $\mathrm{MEMS}^{\mathrm{Q}}$ language the set of formulas Fm is defined according to the clauses Fa, $\mathrm{F}_{\neg}$, $\mathrm{F}_{\vee}$, F_{R}, $\mathrm{F}_{\exists}$, $\mathrm{F}_{\Box}$ (see Section 2).

We distinguish modalized formulas (that include symbols of modal compositions) and non-modalized formulas (that do not include such symbols).

The set of non-modalized formulas in a $\mathrm{MEMS}^{\mathrm{Q}}$ language coincides with the set of formulas of its basic logical language.

Let us define the interpretation mapping I_M of formulas on states of the world and experts.

First, we specify $I_M : Ps \times Ex \times St \longrightarrow Pr$, where $I_M(p, e, \alpha) \in Pr_\alpha$.

The mapping $I_M : Ps \times Ex \times St \longrightarrow Pr$ is extended to the mapping $I_M : Fm \times Ex \times St \longrightarrow Pr$ as follows:

I¬) $I_M(\neg\Phi, e, \alpha) = \neg(I_M(\Phi, e, \alpha)$.

I∨) $I_M(\vee\Phi\Psi, e, \alpha) = \vee(I_M(\Phi, e, \alpha), I_M(\Phi, e, \alpha))$.

IR) $I_M(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi), e, \alpha) = \mathrm{R}^{\bar{v},\bar{u}}_{\bar{x},\perp}(I_M(\Phi, e, \alpha))$.

I∃) $I_M(\exists x\Phi, e, \alpha) = \exists x(I_M(\Phi, e, \alpha))$.

I□) $d \in T(I_M(\Box\Phi, e, \alpha)) \Leftrightarrow$ there exists $\beta \in St : \alpha \rhd_e \beta$; and for any $g \in Ex$ and $\delta \in St$ we have:

$e \gg g$ and $\alpha \rhd_g \delta \Rightarrow d \in T(I_M(\Phi, g, \delta))$;

$d \in F(I_M(\Box\Phi, e, \alpha)) \Leftrightarrow$ there exists $\delta \in St : \alpha \rhd_e \delta$ and $d \in F(I_M(\Phi, e, \delta))$.

Then $I_M(\Phi, e, \alpha)$ is the predicate that represents the value of the formula Φ for expert e on state α.

We will also denote the predicate $I_M(\Phi, e, \alpha)$ briefly as ${}_M\Phi^e_\alpha$, and also as Φ^e_α if $\boldsymbol{M}$ is taken by default.

Statement 4. For formula abbreviations of the form $\Diamond\Phi$, we have:

I◇) $d \in T(I_M(\Diamond\Phi, e, \alpha)) \Leftrightarrow$ there exists $\delta \in St : \alpha \rhd_e \delta$ and $d \in T(I_M(\Phi, e, \delta))$;

$d \in F(I_M(\Diamond\Phi, e, \alpha)) \Leftrightarrow$ there exists $\beta \in St : \alpha \rhd_e \beta$,

and for any $g \in Ex$ and $\delta \in St$ we have: $e \gg g$ and $\alpha \rhd_g \delta \Rightarrow d \in F(I_M(\Phi, g, \delta))$.

This follows from the clause I□ and the following:

$d \in T(I_M(\Diamond\Phi, e, \alpha)) \Leftrightarrow d \in T(I_M(\neg\Box\neg\Phi, e, \alpha)) \Leftrightarrow d \in F(I_M(\Box\neg\Phi, e, \alpha))$ and
$d \in F(I_M(\Diamond\Phi, e, \alpha)) \Leftrightarrow d \in F(I_M(\neg\Box\neg\Phi, e, \alpha)) \Leftrightarrow d \in T(I_M(\Box\neg\Phi, e, \alpha))$.

We consider the general class of *R*-predicates and its subclass of single-valued *R*-predicates, or *P*-predicates. The distinction between them induces the corresponding distinction between interpretation classes, or semantics: *R*-semantics and *P*-semantics, respectively. For the MEMS^Q language, this is done in the same way as for the GMS^Q language and the corresponding basic logical language of the logic of quasiary predicates (see [8]).

On the set of formulas of the MEMS^Q language specified with experts and states, we introduce consequence relations and logical consequence relations. These relations are defined analogously to the corresponding relations in the logics of quasiary predicates (see [8]).

A formula specified by expert and state names has the form $\Phi^{e,\alpha}$. Here, Φ is a formula of the MEMS^Q language, $e \in$ E and $\alpha \in$ S, where E is a set of expert names and S is a set of states of the world names.

Let Σ be a set of formulas specified with expert and states, with sets of experts' and states' names E and S, respectively.

We say that the set Σ is *consistent* with MEMS^Q $\boldsymbol{M} = (Ex, \succ, St, \triangleright, A, I_M)$, provided that injections from E into *Ex* and from S into *St* are defined.

By using the notation $\Gamma\ _M|\!\!=_* \Delta$, we assume by default the consistency of sets of specified formulas Γ and Δ with MEMS $\boldsymbol{M}$.

For sets of specified formulas Γ and Δ of the MEMS^Q language let us define the following *consequence relations in a fixed MEMS* $\boldsymbol{M} = (Ex, \succ, St, \triangleright, A, I_M)$: these are the traditional relations of irrefutability (*IR*), truth (*T*), falsity (*F*), and strong (*TF*) consequence.

Let Γ and Δ be sets of specified formulas of the MEMS^Q language.

Δ is an *IR*-consequence of Γ in a consistent with them MEMS^Q $\boldsymbol{M}$ (denoted $\Gamma\ _M|\!\!=_{IR} \Delta$), if for any $d \in {}^V\!A$ we have

$$d \in T({}_M\Phi^e_\alpha) \text{ for any } \Phi^{e,\alpha} \in \Gamma \Rightarrow d \notin F({}_M\Psi^g_\delta) \text{ for some } \Psi^{g,\delta} \in \Delta. \quad \text{(I)}$$

Δ is a *T*-consequence of Γ in a consistent with them MEMS^Q $\boldsymbol{M}$ (denoted $\Gamma\ _M|\!\!=_T \Delta$), if for any $d \in {}^V\!A$ we have

$$d \in T({}_M\Phi^e_\alpha) \text{ for any } \Phi^{e,\alpha} \in \Gamma \Rightarrow d \in T({}_M\Psi^g_\delta) \text{ for some } \Psi^{g,\delta} \in \Delta. \quad \text{(T)}$$

Δ is an *F*-consequence of Γ in a consistent with them MEMS^Q $\boldsymbol{M}$ (denoted $\Gamma\ _M|\!\!=_F \Delta$), if for any $d \in {}^V\!A$ we have

$$d \in F({}_M\Psi^g_\delta) \text{ for any } \Psi^{g,\delta} \in \Delta \Rightarrow d \in F({}_M\Phi^e_\alpha) \text{ for some } \Phi^{e,\alpha} \in \Gamma. \quad \text{(F)}$$

Δ is a *TF*-consequence of Γ in a consistent with them MEMS^Q $\boldsymbol{M}$ (denoted $\Gamma\ _M|\!\!=_{TF} \Delta$), if $\Gamma\ _M|\!\!=_T \Delta$ and $\Gamma\ _M|\!\!=_F \Delta$.

In the case of MEMS of *P*-predicates, the conditions (I), (T), and (F) can be presented as follows:

$${}_M\Phi^e_\alpha(d) = T \text{ for any } \Phi^{e,\alpha} \in \Gamma \Rightarrow {}_M\Psi^g_\delta(d) \neq F \text{ for some } \Psi^{g,\delta} \in \Delta. \quad (\text{I}_\text{P})$$

$${}_M\Phi^e_\alpha(d) = T \text{ for any } \Phi^{e,\alpha} \in \Gamma \Rightarrow {}_M\Psi^g_\delta(d) = T \text{ for some } \Psi^{g,\delta} \in \Delta. \quad (\text{T}_\text{P})$$

$${}_M\Psi^g_\delta(d) = F \text{ for any } \Psi^{g,\delta} \in \Delta \Rightarrow {}_M\Phi^e_\alpha(d) = F \text{ for some } \Phi^{e,\alpha} \in \Gamma. \quad (\text{F}_\text{P})$$

The relations of logical *IR*-, *T*-, *F*-, and *TF*-consequence for sets of specified formulas Γ and Δ with respect to a MEMS^Q of a certain type are specified according to the following scheme (σ denotes either *IR*, *T*, *F*, or *TF*).

Δ is a *logical* σ*-consequence* of Γ with respect to a MEMS^Q of a type $\mathbf{M}$, if $\Gamma\ _M|\!\!=_\sigma \Delta$ for any MEMS $\boldsymbol{M} \in \mathbf{M}$; it will be denoted by $\Gamma\ ^{\mathrm{M}}|\!\!=_\sigma \Delta$, and also by $\Gamma\ |\!\!=_\sigma \Delta$, provided that $\mathbf{M}$ is taken by default.

According to the definitions, we have: $\Gamma\ ^{\mathrm{M}}|\!\!=_{TF} \Delta \Leftrightarrow \Gamma\ ^{\mathrm{M}}|\!\!=_T \Delta$ and $\Gamma\ ^{\mathrm{M}}|\!\!=_F \Delta$.

In the case of MEMS of *P*-predicates, we obtain non-degenerate consequence relations $^{\mathrm{P}}|\!\!=_{IR}$, $^{\mathrm{P}}|\!\!=_T$, $^{\mathrm{P}}|\!\!=_F$, and $^{\mathrm{P}}|\!\!=_{TF}$, which correspond to the analogous relations $^{p}|\!\!=_{IR}$, $^{p}|\!\!=_T$, $^{p}|\!\!=_F$, and $^{p}|\!\!=_{TF}$ in the traditional logics of quasiary predicates (see [8]). In the case of MEMS of *R*-predicates, a single non-degenerate consequence relation ${}^{\mathbf{R}}|\!=_{TF}$ remains, and it is analogous to the relation ${}^{R}|\!=_{TF}$ in logics of quasiary *R*-predicates.

As in the case of the traditional logics of quasiary predicates and $\mathrm{GMS}^{\mathrm{Q}}$, in $\mathrm{MEMS}^{\mathrm{Q}}$ we have the following relationships between the introduced logical consequence relations:

Theorem 3. ${}^{\mathbf{R}}|\!=_{TF} \subset {}^{\mathrm{P}}|\!=_{TF}$, ${}^{\mathrm{P}}|\!=_{TF} \subset {}^{\mathrm{P}}|\!=_{T} \subset {}^{\mathrm{P}}|\!=_{IR}$, ${}^{\mathrm{P}}|\!=_{TF} \subset {}^{\mathrm{P}}|\!=_{F} \subset {}^{\mathrm{P}}|\!=_{IR}$; ${}^{\mathrm{P}}|\!=_{T} \neq {}^{\mathrm{P}}|\!=_{F}$.

The non-modal properties of the relations ${}^{\mathrm{P}}|\!=_{IR}$, ${}^{\mathrm{P}}|\!=_{T}$, ${}^{\mathrm{P}}|\!=_{F}$, ${}^{\mathrm{P}}|\!=_{TF}$, and ${}^{\mathbf{R}}|\!=_{TF}$ in $\mathrm{MEMS}^{\mathrm{Q}}$ repeat the corresponding properties of the relations ${}^{P}|\!=_{IR}$, ${}^{P}|\!=_{T}$, ${}^{P}|\!=_{F}$, ${}^{P}|\!=_{TF}$, and ${}^{R}|\!=_{TF}$ for sets of formulas of the traditional logic of quasiary predicates (see [8]). The general properties related to modal compositions are analogous to the corresponding properties for $\mathrm{GMS}^{\mathrm{Q}}$ of *P*-predicates (see [9, 10]).

4. Conclusion

The paper investigates new classes of program-oriented logical formalisms – pure first-order transitional modal systems (TMS) of partial ambiguous (non-deterministic) quasiary predicates. We propose two varieties of these TMS: pure first-order General Transitional Modal Systems ($\mathrm{GMS}^{\mathrm{Q}}$), and pure first-order Multiple-Expert Modal Systems with dominance ($\mathrm{MEMS}^{\mathrm{Q}}$). These systems are based on logics of R-predicates – partial ambiguous (multi-valued) quasiary predicates of relational type. The work describes the semantic features of $\mathrm{GMS}^{\mathrm{Q}}$ and shows the connection between modal logics of quasiary *R*-predicates and four-valued modal logics based on Belnap's logic. The proposed MEMS preserve both true and falsity under dominance as we introduce a transitive relation of separate dominance on the set of experts which guarantees that no expert can be directly dominated by more than one other expert. The languages of $\mathrm{MEMS}^{\mathrm{Q}}$ are described. On the set of formulas specified with both experts and states, a number of consequence relations in MEMS and logical consequence relations are defined.

In the future works, we will focus on studying MEMS with different properties of the dominance and accessibility relations, investigating logical consequence relations in MEMS, and constructing sequent-type calculi for MEMS.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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