A method for determining the weighting coefficients of alternative characteristics based on lexicographic ordering analysis

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Abstract

This paper addresses the problem of ordering multi-attribute alternatives using lexicographic criteria. It proposes a method for determining the numerical values of criteria weights when applying a lexicographic approach to ranking such alternatives. The method includes a rationale for deriving weighting coefficients based on a posteriori analysis of criterion values. The concept of exact approximation of weights calculated in cardinal scales is introduced. The paper explores the potential applications and directions for using these relative importance coefficients. A formalized approach to computing the rankings of alternatives—based on solving a multicriteria problem with lexicographic criteria—is presented. Depending on the chosen heuristic for criteria aggregation, metrized values of criteria are derived. In this context, methods for the automated recovery of numerical criteria weights are proposed. The paper also provides a justification for the proposed approach and outlines prospects for future research in this area.

Keywords

multi-attribute ordering, lexicographic criterion, weighting coefficients, rating, exact approximation

1. Introduction

Lexicographic ordering problems are a well-established area of scientific research [1, 2]. They are widely and successfully applied in various practical contexts, attracting attention from both domestic and international researchers across different classes of such problems [3, 4]. Applications include the optimization of complex systems composed of interdependent subsystems, marketing research, the determination of ratings or rankings in competitive sports, admissions to higher education institutions, and many other areas of human activity.

The method presented in this paper can be interpreted as a solution to a recovery problem. It may also be viewed as a technique for the a posteriori determination of attribute weights for alternatives, or the relative importance of decision-making criteria in multicriteria optimization problems [5, 6].

Developing tools to address common applied problems is both a pressing and often underappreciated task [7]. There will always be critics who question the necessity or usefulness of introducing yet another method, especially when dozens of established approaches already exist and appear to perform adequately. As a result, it becomes essential to justify the new method's effectiveness, highlight its advantages, and demonstrate its superiority over existing approaches [8].

However, for so-called instrumental methods—that is, auxiliary techniques intended to support, rather than directly solve, primary problems—traditional evaluation criteria such as "better," "more efficient," or "more accurate" may not always apply. Nonetheless, it is clear that in certain unique situations, the development or application of equally unique methods is both reasonable and necessary.

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An important stage in expert evaluation involves determining the balance between experts' subjective opinions and "objective" data [9, 10]. Numerous studies have demonstrated that seemingly obvious subjective judgments can differ significantly from objective or a posteriori assessments. In such cases, indirect methods have shown particular promise [11].

In many real-world problems across various applied fields, the subjective perceptions of experts and decision-makers often differ significantly from the corresponding "objective" assessments. In such cases, identifying the heuristics an expert used to organize alternatives becomes a problem of inference. The resulting mathematical model is considered more objective when the outcomes of applying a particular aggregation of criteria closely align with the expert's ranking of alternatives.

2. Formulation of the Lexicographic Ordering Problem and Its Mathematical Model

The lexicographic method is based on the principle that partial optimization criteria are first ranked according to their relative importance. Then, using expert-defined preference ratios among the criteria, a sequence of single-criterion optimization problems is solved iteratively, starting with the most important criterion. It is worth noting that the initial auxiliary task of ranking the criteria on an ordinal scale is generally nontrivial [12, 13], although numerous methods have been developed to address this challenge [14, 15].

Lexicographic ordering of alternatives follows an approach in which criteria are considered in descending order of importance, with the most important criterion taking precedence over equal values of all others. Such strict prioritization often emerges when additional criteria are introduced sequentially into conventional scalar optimization problems that may lack a unique solution. An optimization problem with strictly ordered criteria is referred to as a lexicographic optimization problem.

A preliminary step in solving the multicriteria optimization problem underlying lexicographic ordering is to rank the partial criteria in decreasing order of importance. By assigning numerical labels to the criteria, we can—without loss of generality—assume that the first criterion is the most important. In many practical cases, the lexicographic method produces a unique solution after optimizing with respect to the first criterion, allowing the process to terminate at that point [16, 17]. However, such trivial cases are not considered in this paper.

Problems in which the criteria are ranked by importance and renumbered such that each preceding criterion is incomparably more important than all subsequent ones are referred to as lexicographic optimization problems. In such problems, the resulting non-strict preference relation adheres to a lexicographic order [12, 18].

Suppose we have a multi-attribute selection problem

$$x_i^j \in X, i \in I = \{1,...,k\}, j \in J = \{1,...,n\}, , X \in R^k \tag{1}$$

$$F = (f_1,...,f_k), \ f_i = f_i(x) = x, i = 1,...,k \ . \tag{2}$$

It is necessary to rank the alternatives of type (1) using the criteria of type (2). Moreover, the number of attributes for each alternative is assumed to be equal to the number of criteria in the problem. Suppose the decision-maker chooses to apply the lexicographic criterion to solve this problem. Without loss of generality, we assume that the criteria are ordered by importance as follows:

$$f_1 \succ f_2 \succ ... \succ f_{k-1} \succ f_k \ . \tag{3}$$

Accordingly, the weighting factors that represent the relative importance of the criteria in ranking (3) must satisfy the following inequalities:

$$\rho_1 > \rho_2 > ... > \rho_{k-1} > \rho_k \ . \tag{4}$$

The vectors of criteria values for different alternatives will be denoted by

$$x^j = \left(x_1^j, \ldots, x_k^j\right), j \in J = \{1, \ldots, n\},\ x_i^j \in \{0,1,\ldots\} \text{ for } \forall i,j : i \in I, j \in J \tag{5}$$

Based on the application of the lexicographic criterion, the alternatives in set (1) are ranked by importance. In other words, the ordering of the alternatives in set (1) is known. However, no further information is available about the vectors of the form (5). Nevertheless, in this decision-making context, some additional information about the structure of set (1) is available.

Without diminishing the generality, we will assume that as a result of applying the lexicographic criterion (2), the structure of which is reflected in formula (3), the following order takes place:

$$x^1 \succ x^2 \succ \ldots \succ x^{n-1} \succ x^n$$

The task is to determine the quantitative values of the weighting coefficients (4). To clarify the problem, we will introduce several heuristics.

Heuristic H1. The aggregating criterion for ordering the alternatives is the additive convolution of the weighted values of the attributes of the alternatives. That is, to determine the generalized indicators of each alternative of the form (4), experts use a linear convolution:

$$F^j = F\left(x^j\right) = \sum_{i=1}^{k} \rho_i x_i^j.$$

Heuristic H2. When applying the quantitative weighting coefficients of the criteria [19, 20], the order established by applying the lexicographic criterion should be preserved. That is, the system of inequalities must be fulfilled:

$$F^1 \geq F^2 \geq \ldots \geq F^{n-1} \geq F^n$$

or

$$\sum_{i=1}^{k} \rho_i x_i^1 \geq \sum_{i=1}^{k} \rho_i x_i^2 \geq \ldots \geq \sum_{i=1}^{k} \rho_i x_i^{n-1} \geq \sum_{i=1}^{k} \rho_i x_i^n. \tag{6}$$

When applying lexicographic criteria, it is believed that the difference between the ordered criteria is so great that the next criterion in the series is considered only if it is not possible to find an answer according to the older criteria and there is no question of concessions at all [17]. In this case, the optimization problem or the problem of ordering a set of alternatives with strictly ordered criteria is called lexicographic.

Heuristic H3. When applying quantitative weighting coefficients of the criteria, the system of inequalities can be fulfilled:

$$\sum_{i=1}^{k} \rho_i \sqrt{x_i^1} \geq \sum_{i=1}^{k} \rho_i \sqrt{x_i^2} \geq \ldots \geq \sum_{i=1}^{k} \rho_i \sqrt{x_i^{n-1}} \geq \sum_{i=1}^{k} \rho_i \sqrt{x_i^n}. \tag{7}$$

We will analyze all the ratios specified in heuristics H2-H3 in order to determine the system (6)-(8) that is closest to the subjectively ordered alternatives.

From a mathematical point of view, there is no ideal method or way to solve such problems. Each of them has its own specific advantages and disadvantages and scope. According to the set of goals $\left(C = \{C_1, \ldots, C_k\}\right)$, facing the researcher, the following class of expert evaluation tasks is distinguished [12] - ranking, i.e., ordering the entire set of alternatives [21, 22]. The lexicographic method is based on the fact that the criteria are taken into account in descending order of importance, with the advantage of the extremum of the highest criteria over the same value of all others [16, 23]. It is clear that this method has disadvantages that limit the application of this method, such as the complexity and subjectivity of ranking criteria, inapplicability in case of their equivalence, etc.

3. Examples of application of the lexicographic criterion

Lexicographic multi-criteria ordering is widely used in various practical applications [22, 23]. In certain fields, it represents the only natural method for structuring a set of alternatives. Some examples of its application are presented in Table 1. Clearly, this list is not exhaustive and could be significantly extended. Providing a comprehensive overview of all possible applications of the lexicographic criterion was not the author's objective.

Table 1
Areas of application of lexicographic ordering of alternatives

№	Field of application	Criterion 1	Criterion 2	Criterion 3	Criterion 4
1	Optimizing a complex system consisting of interdependent subsystems	Optimizing the top-level subsystem of the hierarchy	Optimization of mid-level hierarchy subsystems	Optimize the subsystem of the lower level of the hierarchy	Optimization of lower-level subsystems if the hierarchical system has more than three hierarchical levels
2	Sort files in file systems	File type - lexicographic organization,: folders, then files	File extension - lexicographically	File name - lexicographically	
3	Purchase of computers to create a computer classroom	High performance - maximized	Price - minimized	Reliability is maximized	
4	Drawing up a standings in some game sports	Points scored - minimized	The difference between goals scored and conceded is minimized	Team name - lexicographically	
5	Ranking of players in tennis	Rating points - minimized	The number of tournaments won is minimized	Player's name - lexicographically	
6	Sorting movies in a streaming service	Film genre - organized lexicographically	Movie rating - minimized	Movie title - organized lexicographically	
7	Sorting cases in court	Case status: priority or regular	The date of filing the case is minimized	Case number - organized according to codification	
8	Sorting medical records of patients in a hospital	Treatment priority: from critical to planned	The patient's age is minimized	Patient's last name - organized lexicographically	Patient's name - organized lexicographically
9	Admission to higher education institutions	Quotas are the highest priority	Exam grades	Average school certificate score, additional points	Motivation letter

№	Field of application	Criterion 1	Criterion 2	Criterion 3	Criterion 4
10	Student rating by learning outcomes	GPA - minimized	The number of excellent grades is minimized	Surname, name of the student - lexicographically	Determination by the scholarship committee of the possibilities to take into account additional points
11	Accommodation in a student dormitory with a limited number of beds	Benefit categories - lexicographical organization, depending on the category of benefits	Remoteness of the place of residence is minimized	Student success is minimized	
12	Marketing: Analyzing advertising campaigns	ROI (Return on Investment) is minimized	The campaign budget is minimized	Campaign name - organized lexicographically	
13	Ranking of companies by revenue and name	The company's income is minimized	Company name - lexicographically in ascending order		
14	Sorting bank customers by account balance and last name	The account balance is minimized	Client's surname - lexicographically in ascending order		
15	Sort products by category, brand, and price	Category - lexicographical	Brand - lexicographically	Price - maximized	
16	Creating a list of vacancies	Wages are minimized	Company name - lexicographically	Position - lexicographically	
17	Sorting orders by priority	Order status: from "urgent" to "regular"	Date of order - primarily early orders are preferred	Client's name - lexicographically	
18	Sorting logistics delivery routes	Delivery priority: from highest to lowest	The length of the route is maximized	Name of the destination - lexicographically	

Remark 1. Lexicographic ordering is often used to establish rules of precedence, priority, etc.

Remark 2: In lexicographic organization, ordinal scales with several gradations are often used.

The method is used in problems in which individual goals have different weights and can be arranged in a certain hierarchical order. In such problems, the first stage of optimization determines the set of solutions that optimizes the highest-ranked objective. The resulting set D of solutions is narrowed at the second stage by optimizing the second most important objective. This process continues until there is only one single solution. If a single solution cannot be found when optimizing the lowest ranked objective, a subjective choice is made from the set of remaining solutions [24], or an additional criterion is introduced. This method is widely used, but it assumes a hierarchy of goals [25, 26].

It should be noted that partial criteria can be qualitative, quantitative, and lexicographic [27, 28].

Remark 3: For situations of ordering by qualitative criteria, there may be variations with a different number of gradations or clusters, depending on the specifics of the application area, approaches developed by the authors of the task, and other factors.

Remark 4. Of course, there may be various variations of multicriteria lexicographic organization along the lines shown in the table. But the table demonstrates the breadth and diversity of the application of multicriteria lexicographic organization.

Remark 5. In order to achieve fairness, it is sometimes correct to move from quantitative criteria to interval criteria, to the construction of membership functions, to the creation of clusters, and to solve auxiliary problems beforehand. It is clear that such procedures should be supported by a strong justification. In addition, for sound work in these classes of problems, appropriate mathematical support must be developed.

4. Accurate approximation of the preference structure

Definition 1. The special points for the exact approximation of the preference structure on the set of criteria (2) are the points generated by the following situations:

$$\sum_{i=1}^{k} \rho_i x_i^{j-1} \geq \sum_{i=1}^{k} \rho_i x_i^{j}, j = 2,...,n$$

but the corresponding alternatives remain ordered according to the lexicographic criterion.

Definition 2. An exact approximation is a situation where inequalities are generated at special points, and to transform them into an order ratio with respect to the corresponding weighting coefficients, some sufficiently small number must be subtracted.

Heuristic H4. In order to apply an accurate approximation and comply with the requirements of strict ordering, a sufficiently small fixed deviation from the found values of the weighting coefficients can be expertly set, if necessary: . $\rho_i = \rho_i - \varepsilon, i \in \{1,...,k\}, \varepsilon > 0$

Heuristic H5. The value of the correction to the respective weighting factors can be calculated, for example, as the inverse of the highest ranking of alternatives determined by the results of solving the lexicographic ordering task.

Remark 6. With an inaccurate approximation, the ordering of alternatives on the set (1) is preserved, but the constructed ratings of alternatives differ significantly from those determined with an accurate approximation. This creates additional opportunities for manipulating the ratings.

It is worthwhile to examine the correlation between subjective perceptions and the "objective" data derived from accurately approximating the structure of preferences induced by the ordering of alternatives in set (1) [29, 30]. As we will demonstrate, these two perspectives differ significantly—at least when evaluated using the following two heuristics [31, 32].

Heuristic H6. We will use the lexicographic criterion to descend the ordering of some structure.

Heuristic H7. The aggregating criterion for ordering alternatives is the additive convolution of the weighted values of the attributes of alternatives.

Heuristic H8. Equality in the weighted convolution values of alternative attributes is possible only when the corresponding attribute vectors are completely equivalent. In all other cases, once the weights of the partial criteria have been determined, they should be incremented by a specified small value. The resulting adjusted weights are then considered an accurate approximation of the structure of the set of alternatives.

5. Algorithm of the method for determining the weighting coefficients of the characteristics of alternatives

Let's assume that, according to formula (4), the first partial criterion is the most important and the last one is the least important. We will calculate the weights of the partial criteria in the calibration form. That is, we will assume that the weighting coefficient of the least important criterion is equal to one: $\rho_k = 1$. All other criteria depend on the structure of the lexicographic ordering of the set of alternatives (1).

The algorithm is based on analyzing the structure of preferences, beginning with the comparison of attribute values for the least important criteria. Below, we present the algorithm for determining the weighting coefficients of criteria in a lexicographic ordering of alternatives as a sequence of steps.

Step 1. Set the year to 1.

Step 2. Cycle through $i = k - 1$ to 2.

Step 2.1. Set the counter of special situations for the index i: $s = 0$.

Step 2.2. Loop for $j = 1$ to $n - 1$.

Step 2.2.1. If the values of the attributes of the alternatives of set (1) correspond to the inequality $x_j^i \geq x_{j+1}^i$, then go to step 2.2.3. - to the end of the cycle for j.

Step 2.2.2. If there is an inequality $x_j^i < x_{j+1}^i$, then we determine the value of the ratio: $\rho_i^s = \left(x_{j+1}^i - x_j^i\right) / \left(x_j^{i-1} - x_{j+1}^{i-1}\right)$

Step 2.2.3. Increase the counter of special situations $s = s + 1$.

Step 2.3. If the counter of special situations is $s = 0$, then the presence of a quasi-series is stated - a loose ordering of partial criteria. In this case, equivalent criteria are combined and $k = k - 1$ is assumed.

Step 2.4. Determine the $\rho_i^M = \max\limits_{l=1,\dots,s} \rho_i$.

Step 2.5. Set the counter of special situations for the index $(i-1)$: $s = 0$.

Step 2.6. Cycle through $j = 1$ to $n - 1$.

Step 2.6.1. If the values of the attributes of the alternatives of set (1) correspond to the inequality $x_j^{i+1} \geq x_{j+1}^{i+1}$, then go to step 2.6.3. - to the end of the cycle for j.

Step 2.6.2. If there is an inequality $x_j^{i+1} < x_{j+1}^{i+1}$, then we determine the value of the ratio: $\rho_{i+1}^s = \left(\rho_i^M \cdot \left(x_{j+1}^{i-1} - x_j^{i-1}\right) + x_{j+1}^i - x_j^i\right) / \left(x_j^{i+1} - x_{j+1}^{i+1}\right)$.

Step 2.6.3. Increase the counter of special situations $s = s + 1$.

Step 2.7. Determination of the $\rho_{i+1}^M = \max\limits_{l=1,\dots,s} \rho_{i+1}$.

Step 3. End of the cycle for i.

Step 4. Increase the values of partial weighting coefficients by some small predefined real number: $\rho_i = \rho_i + \varepsilon$, for $i = 1,\dots,k-1$.

Step 5. Displaying the calculated values of the partial criteria: ρ_i, $i = 1,\dots,k$..

Step 6. End of the algorithm.

6. A computational experiment

Let us consider the team medal standings from one edition of the Olympic Games—the 2024 Paris Olympics. Using publicly available data [32], we analyzed the medal standings of both the Olympic and Paralympic Games from 2008 to 2024.

Table 2
Coding of Olympiad participating countries

Country number	Country name	Country number	Country name	Country number	Country name
N1	Albania	N32	Germany	N63	Norway
N2	Algeria	N33	Great Britain	N64	Pakistan
N3	Argentina	N34	Greece	N65	Panama
N4	Armenia	N35	Grenada	N66	Peru
N5	Australia	N36	Guatemala	N67	Philippines
N6	Austria	N37	Hong Kong	N68	Poland
N7	Azerbaijan	N38	Hungary	N69	Portugal
N8	Bahrain	N39	India	N70	Puerto Rico
N9	Belgium	N40	Individual Neutral Athletes	N71	Qatar
N10	Botswana	N41	Indonesia	N72	Refugee Olympic Team
N11	Brazil	N42	Iran	N73	Romania
N12	Bulgaria	N43	Ireland	N74	Saint Lucia
N13	Canada	N44	Israel	N75	Serbia
N14	Cape Verde	N45	Italy	N76	Singapore
N15	Chile	N46	Ivory Coast	N77	Slovakia
N16	China	N47	Jamaica	N78	Slovenia
N17	Chinese Taipei	N48	Japan	N79	South Africa
N18	Colombia	N49	Jordan	N80	South Korea
N19	Croatia	N50	Kazakhstan	N81	Spain
N20	Cuba	N51	Kenya	N82	Sweden
N21	Cyprus	N52	Kosovo	N83	Switzerland
N22	Czech Republic	N53	Kyrgyzstan	N84	Tajikistan
N23	Denmark	N54	Lithuania	N85	Thailand
N24	Dominica	N55	Malaysia	N86	Tunisia
N25	Dominican Republic	N56	Mexico	N87	Turkey
N26	Ecuador	N57	Moldova	N88	Uganda
N27	Egypt	N58	Mongolia	N89	Ukraine
N28	Ethiopia	N59	Morocco	N90	United States
N29	Fiji	N60	Netherlands	N91	Uzbekistan
N30	France	N61	New Zealand	N92	Zambia
N31	Georgia	N62	North Korea		

Table 3
Results of the medal standings of the 2024 Olympic Games in Paris

Rank	Number	Country number	Gold	Silver	Bronze	Total
1	1	N90	40	44	42	126
2	2	N16	40	27	24	91
3	3	N48	20	12	13	45
4	4	N5	18	19	16	53
5	5	N30	16	26	22	64
6	6	N60	15	7	12	34
7	7	N33	14	22	29	65
8	8	N80	13	9	10	32
9	9	N45	12	13	15	40

Rank	Number	Country number	Gold	Silver	Bronze	Total
10	10	N32	12	13	8	33
11	11	N61	10	7	3	20
12	12	N13	9	7	11	27
13	13	N91	8	2	3	13
14	14	N38	6	7	6	19
15	15	N81	5	4	9	18
16	16	N82	4	4	3	11
17	17	N51	4	2	5	11
18	18	N63	4	1	3	8
19	19	N43	4	0	3	7
20	20	N11	3	7	10	20
21	21	N42	3	6	3	12
22	22	N89	3	5	4	12
23	23	N73	3	4	2	9
24	24	N31	3	3	1	7
25	25	N9	3	1	6	10
26	26	N12	3	1	3	7
27	27	N75	3	1	1	5
28	28	N22	3	0	2	5
29	29	N23	2	2	5	9
30	30	N7	2	2	3	7
30	31	N19	2	2	3	7
32	32	N20	2	1	6	9
33	33	N8	2	1	1	4
34	34	N78	2	1	0	3
35	35	N17	2	0	5	7
36	36	N6	2	0	3	5
37	37	N37	2	0	2	4
37	38	N67	2	0	2	4
39	39	N2	2	0	1	3
39	40	N41	2	0	1	3
41	41	N44	1	5	1	7
42	42	N68	1	4	5	10
43	43	N50	1	3	3	7
44	44	N47	1	3	2	6
44	45	N49	1	3	2	6
44	46	N85	1	3	2	6
-	-	N40	1	3	1	5
47	47	N28	1	3	0	4
48	48	N83	1	2	5	8
49	49	N26	1	2	2	5
50	50	N68	1	2	1	4
51	51	N34	1	1	6	8
52	52	N3	1	1	1	3
52	53	N27	1	1	1	3
52	54	N86	1	1	1	3
55	55	N10	1	1	0	2
55	56	N15	1	1	0	2
55	57	N74	1	1	0	2
55	58	N88	1	1	0	2

Rank	Number	Country number	Gold	Silver	Bronze	Total
59	59	N25	1	0	2	3
60	60	N36	1	0	1	2
60	61	N59	1	0	1	2
62	62	N24	1	0	0	1
62	63	N64	1	0	0	1
64	64	N87	0	3	5	8
65	65	N56	0	3	2	5
66	66	N4	0	3	1	4
66	67	N18	0	3	1	4
68	68	N53	0	2	4	6
68	69	N62	0	2	4	6
70	70	N54	0	2	2	4
71	71	N39	0	1	5	6
72	72	N57	0	1	3	4
73	73	N52	0	1	1	2
74	74	N21	0	1	0	1
74	75	N29	0	1	0	1
74	76	N49	0	1	0	1
74	77	N58	0	1	0	1
74	78	N65	0	1	0	1
79	79	N84	0	0	3	3
80	80	N1	0	0	2	2
80	81	N35	0	0	2	2
80	82	N55	0	0	2	2
80	83	N70	0	0	2	2
84	84	N14	0	0	1	1
84	85	N46	0	0	1	1
84	86	N66	0	0	1	1
84	87	N71	0	0	1	1
84	88	N72	0	0	1	1
84	89	N76	0	0	1	1
84	90	N77	0	0	1	1
84	91	N92	0	0	1	1

In Table 3, the situations that give rise to special points of lexicographic ordering of alternatives are marked on the yellow background, and the analysis of these points allows for an accurate approximation of the structure of preferences created by applying the lexicogaphy ordering of the set of multi-attribute alternatives.

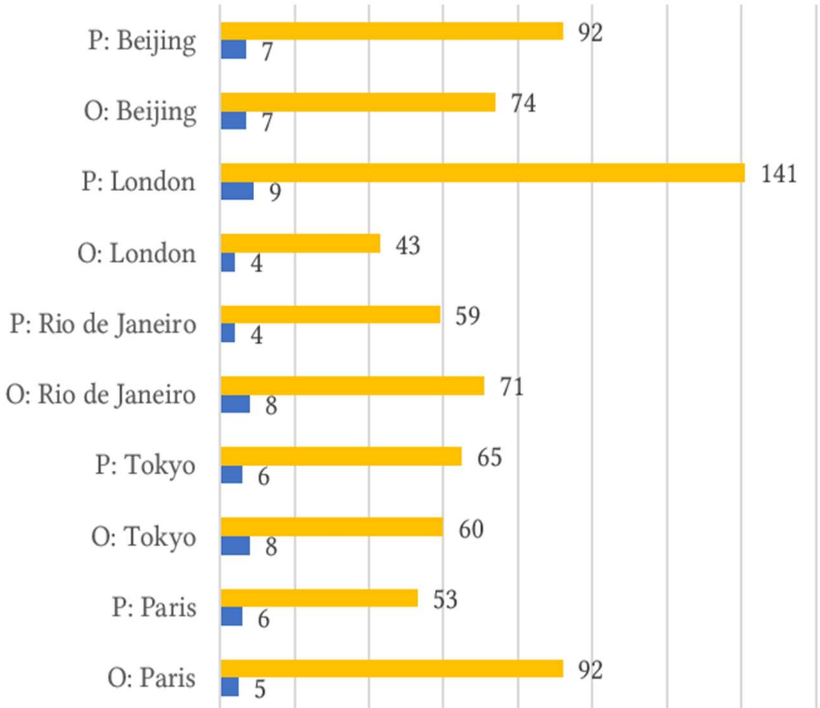

Figure1: Weighting coefficients of gold and silver medals when applying the H2 heuristic and formula (6)

Figure 2 shows an illustration of the ratio of the weighting coefficients of gold and silver medals in relation to bronze medals at different Olympics when applying the H3 heuristic according to formula (7).

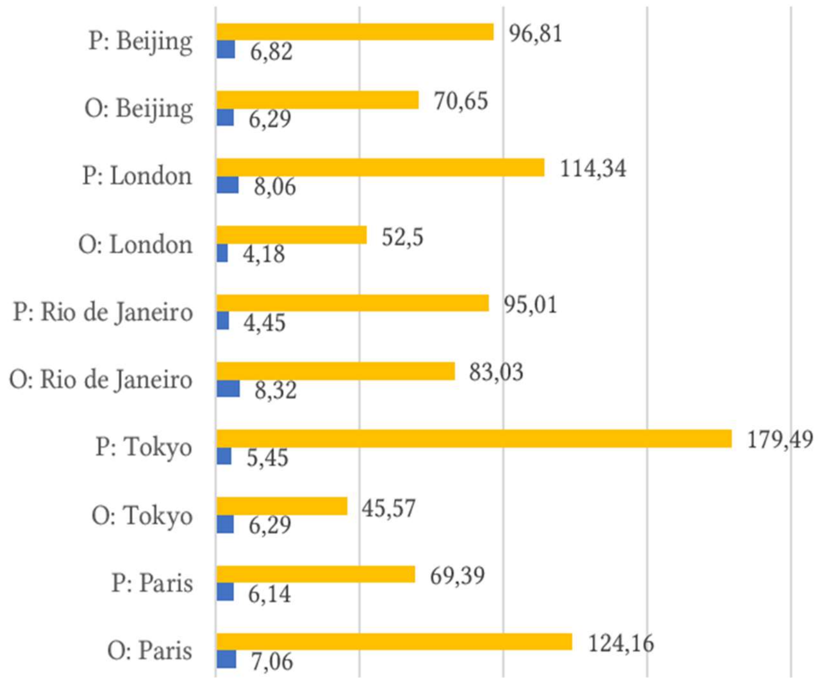

Figure 2: Weighting coefficients of gold and silver medals when applying heuristic H3 and formula (7)

To calculate the metrized values of the criteria weights when analyzing the structure of the set of Olympic medalists by their affiliation with different countries at the 2024 Olympics, it is advisable to use an exact approximation. We will use the algorithm described above. Based on open data [33], we analyzed the results of team medal competitions at the Olympic and Paralympic Games from 2008 to 2024 and summarized them in Table 4.

When applying the H2 heuristic to fulfill the system of inequalities of the form (6), the weighting factor of silver medals should be 5 times greater than the weighting factor of bronze medals, and the "weight of gold" should be 92 times greater than the "weight of bronze". Further research has shown that such large values of the "older" weight coefficients are not limiting. The ratio of the number of medals won by Olympic teams in other years sometimes gives even larger values to some weight coefficients.

Similarly, we analyzed the tables of the number and nationality of medalists from other Olympics: Beijing 2008, London 2012, Rio 2016, and Tokyo 2020. An analysis of the results of the relevant Paralympic Games from 2008 to 2024 was also carried out. The results of all the calculations are summarized in Table 4.

Table 4
The ratio of medal weights at different Olympics

Name of the Olympics	Heuristic H2			Heuristic H		
	Gold/ Bronze	Silver/B ronze	Gold/ Silver	Gold/ Bronze	Silver/B ronze	Gold/Sil ver
Olympics 2024 (Paris)	92	5	18,4	124,16	7,06	17,59
Paralympic Games 2024 (Paris)	53	6	8,8	69,39	6,14	11,30
Olympics 2020 (Tokyo)	60	8	7,5	45,57	6,29	7,24
Paralympic Games 2020 (Tokyo)	160	5	32	179,49	5,45	32,93
Olympics 2016 (Rio de Janeiro)	71	8	8,9	83,03	8,32	9,98
Paralympic Games 2016 (Rio de Janeiro)	59	4	14,8	95,01	4,45	21,35
Olympics 2012 (London)	43	3	14,3	52,5	4,18	12,56
Paralympic Games 2012 (London)	141	9	15,7	114,34	8,06	14,19
Olympics 2008 (Beijing)	74	6	12,3	70,65	6,29	11,23
Paralympic Games 2008 (Beijing)	92	7	13,1	96,81	6,82	14,20

In Table 4, when applying the H3 heuristic, taking into account the system of inequalities of the form (7), the "weight of silver" is 2.2 times higher than the "weight of bronze", and the "weight of gold" exceeds the "weight of bronze" by 9.6 times.

The ratio of the weighting coefficients of gold and silver medals in relation to bronze medals at different Olympics when applying the H2 heuristic according to formula (5) is shown in Figure 1.

Based on the data presented in Table 4, using the H2 heuristic and formula (6), we calculated the values of the corresponding normalized weights and summarized them in Table 5.

Table 5
Normalized values of weighting coefficients when applying the H2 heuristic

Year of the event	City of the event	Olympics			Paralympics		
		Gold	Silver	Bronze	Gold	Silver	Bronze
2024	Paris	0,939	0,051	0,010	0,883	0,100	0,017
2020	Tokyo	0,870	0,116	0,014	0,903	0,083	0,014
2016	Rio de Janeiro	0,888	0,100	0,013	0,922	0,063	0,016
2012	London	0,896	0,083	0,021	0,934	0,060	0,007
2008	Beijing	0,902	0,085	0,012	0,920	0,070	0,010

If we apply the heuristic H3 and formula (7) to the data from Table 5, we obtain different values of the normalized weights, which are presented in Table 6.

Table 6
Normalized values of the weighting coefficients when applying the H3 heuristic

Year of the event	City of the event	Olympics			Paralympics		
		Gold	Silver	Gold	Silver	Gold	Silver
2024	Paris	0,939	0,053	0,008	0,907	0,080	0,013
2020	Tokyo	0,862	0,119	0,019	0,965	0,029	0,005
2016	Rio de Janeiro	0,899	0,090	0,011	0,946	0,044	0,010
2012	London	0,910	0,072	0,017	0,927	0,065	0,008
2008	Beijing	0,906	0,081	0,013	0,925	0,065	0,010

When applying the linear convolution of the form (6), the ratings of the teams participating in the Olympic and Paralympic Games were determined on the basis of the calculated weighting coefficients from Table 5.

Table 7
The ratio of medal weights at different Olympics

Name of the Olympics	Paris		Tokyo		Rio de Janeiro		London		Beijing	
	Olimp	ParaO	Olimp	ParaO	Olimp	ParaO	Olimp	ParaO	Olimp	ParaO
Maximum rating	3942	5488	2701	15711	3600	6688	2094	14099	3928	8730
Rating of Ukraine	305	1366	120	4102	186	2606	282	4756	563	2366
Number of medalist countries	91	85	93	86	86	83	85	75	87	76
The place of Ukraine	22	7	44	6	31	3	14	4	11	4

If we build a linear convolution of the form (7) using the normalized weighting coefficients from Table 6, we will get the ratings of the Olympic teams presented in Table 8.

Table 8
The ratio of medal weights at different Olympics

Name of the Olympics	Paris		Tokyo		Rio de Janeiro		London		Rio de Janeiro	
	Olimp	ParaO	Olimp	ParaO	Olimp	Olimp	Olimp	ParaO	Olimp	ParaO
Maximum rating	838,42	733,35	930,65	1749,2	619,96	1036,2	384,0	1190,4	538,65	977,6
Rating of Ukraine	232,79	363,61	64,45	892,45	138,04	645,53	140,96	691,59	204,86	508,57
Number of medalist countries	91	85	93	86	86	83	85	75	87	76
The place of Ukraine	22	7	44	6	31	3	14	4	11	4

7. Conclusions

This paper addresses the problem of lexicographic ordering of multi-attribute alternatives. A method for precisely approximating the structure of the ordered set of alternatives is proposed. Based on the analysis of this structure, a method for determining the quantitative values of the weighting coefficients is presented, ensuring an accurate approximation of the preference relations on the set of criteria in an ordinal scale.

A calibration method for representing weighting coefficients is proposed, offering a convenient visualization for certain classes of expert evaluation tasks. The calibration of the presented weighting coefficients is achieved by dividing the normalized coefficients by the smallest value among all the coefficients.

Sorting by multiple lexicographical criteria enables the organization of data across several parameters simultaneously, a method commonly used in real-world systems such as databases, user interfaces, logistics, accounting, banking, and many other industries. This approach facilitates easy navigation and quick access to relevant information. The paper illustrates this by presenting the unofficial team standings of national Olympic teams, ranked by the number of medals of various types they won during the competition. The attribute weights calculated by the author help explain why these standings are considered unofficial.

The author presents a well-reasoned illustration of the inefficiency of using a linear convolution of criteria. In fact, the "infinity" of the advantages attributed to more important criteria in lexicographic ordering, as often claimed in many textbooks, is critically examined. The analysis demonstrates that logical and reasonable heuristics can lead to results that sharply contrast with subjective human perceptions.

The author demonstrates that when the ranking of alternatives is determined using weighting coefficients set a priori by experts, the ordering established by the lexicographic criterion may be violated. This underscores the potentially misleading nature of commonly held notions such as fairness and validity. The findings suggest that, in many practical situations, widely accepted standards are neither always logical nor universally acceptable. Ultimately, subjective perceptions often differ significantly from "objective" evaluations—those defined by two heuristics: the lexicographic ordering of partial criteria and the conventional additive convolution of weighted attributes of alternatives.

Declaration on Generative AI

The author has not employed any Generative AI tools.

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