Merging Marginalized Total Preorders

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Abstract

Total preorders over possible worlds are often used to represent epistemic states, with the minimal worlds representing an agent's beliefs. Marginalization refers to reducing the signature of an epistemic state, resulting in a smaller total preorder over possible worlds over the chosen subset of the original signature. Together with the syntax splitting principle, which states that only syntactically relevant parts of the epistemic state should be modified during revision, marginalization can be used to make belief revision more efficient: The original preorder is marginalized into smaller preorders, which are revised independently, and then merged back together. This last merging step offers a challenge, however, since some information about the original ordering may be lost during marginalization. In this paper, we explore how this merging step may be performed while preserving as much information as possible, in particular the original syntax splitting.

Keywords

belief revision, epistemic states, total preorders, syntax splitting, marginalization, merging

1. Introduction

Both in belief revision and in non-monotonic reasoning, total preorders over possible worlds play an important role as representations of an agent's epistemic state. By ranking the possible worlds according to their plausibility, with the minimal worlds being most plausible, they encode both propositional and conditional beliefs.

Marginalization refers to reducing the signature of an epistemic state. The concept originates from probability theory, where marginalization means reducing the dimensionality of a probability distribution. Similarly, the marginalization of a total preorder is defined over a chosen subset of the original signature, resulting in an exponentially smaller total preorder over the reduced signature. In conjunction with the so-called *syntax splitting* principle, this technique can be used to make belief revision and non-monotonic reasoning more efficient. The core idea of syntax splitting (which goes back to Parikh [1]) is that only syntactically relevant parts of the epistemic state (resp. background knowledge) should influence a belief revision result (resp. the answer to an inference query).

In this paper, we will focus on the application of syntax splitting to iterated belief revision of total preorders. In [2], a weak and a strong version of Parikh's original syntax splitting postulate were identified, and the strong version was adapted for the iterated revision of epistemic states in [3], resulting in two syntax splitting postulates, (MR) and (P^{it}), the conjunction of which guarantees relevance-sensitive revision of epistemic states. However, while there are various revision operators in the literature, currently there is no known operator that satisfies (MR) and (P^{it}). We fill this gap by proposing a simple variation of lexicographic revision [2] complying with these postulates. Moreover, we will investigate how these two postulates can be exploited in order to perform a global revision, i.e. a revision of the whole total preorder, by first marginalizing it, performing local revisions on the marginalized preorders independently, and then merging the results back together. This final merging step is of particular interest: it is not obvious how the local preorders should be combined, since all

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information about how the local parts were originally connected is lost during the initial marginalization and the subsequent local revisions. Towards a solution of this problem, we propose postulates for a general approach of merging local revision results. In particular, we argue that the original syntax splitting should be preserved by a merging operator.

In summary, the main contributions of this paper are:

- We propose a revision operator that satisfies both (P^{it}) and (MR).
- We propose two postulates for merging epistemic states which are defined over disjunct signatures: (ReMarg) and (MSplit).
- We show that local revision of marginalized total preorders with an additional merging step, with a merging operator satisfying (ReMarg) and (MSplit), amounts to a global revision satisfying the syntax splitting postulates (P^{it}) and (MR).

The rest of this paper is organized as follows. In Section 2, we recall formal basics and notations. Afterwards, related work is discussed in Section 3. In Section 4, we recall syntax splitting for iterated belief revision, and we propose a revision operator satisfying the iterated syntax splitting postulates in Section 5. Section 6 is the core of this paper, where we develop general merging strategies for marginalized total preorders. Finally, Section 7 contains conclusions and pointers to future work.

2. Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over the alphabet $\Sigma = \{a, b, c, \ldots\}$. Formulas $A, B, C, \ldots$ are formed using the standard connectives $\wedge, \vee, \neg$. For conciseness of notation, we will write AB instead of $A \wedge B$ for conjunctions, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. The symbol $\top$ denotes an arbitrary propositional tautology. The set of all *possible worlds* (propositional interpretations) over $\Sigma' \subseteq \Sigma$ is denoted by $\Omega(\Sigma')$. If Σ' is clear from the context we simply write Ω. We denote with $\omega \models A$ that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $\mathrm{Mod}(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $\mathrm{Mod}(A) \subseteq \mathrm{Mod}(B)$, as usual. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise. For a subsignature $\Theta \subseteq \Sigma$ we denote by ω^{Θ} the Θ-part of a world $\omega \in \Omega$, such that $\Omega(\Theta) = \{\omega^{\Theta} \mid \omega \in \Omega(\Sigma)\}$.

As a structure for possible worlds we consider *epistemic states* represented as $\Psi = (\Sigma_\Psi, \Omega_\Psi, \preceq_\Psi)$ with $\Omega_\Psi \subseteq \Omega(\Sigma_\Psi)$, and $\preceq_\Psi \subseteq \Omega_\Psi \times \Omega_\Psi$ being a total preorder (TPO), i.e., a total and transitive relation. We assume that every epistemic state can be uniquely identified by its associated total preorder $\preceq_\Psi$, that is each epistemic state Ψ induces a unique total preorder $\preceq_\Psi$, and each total preorder $\preceq$ induces a unique epistemic state $\Psi_\preceq$. As usual, $\omega_1 \prec \omega_2$ if $\omega_1 \preceq \omega_2$, but not $\omega_2 \preceq \omega_1$, and $\omega_1 \approx \omega_2$ if both $\omega_1 \preceq \omega_2$ and $\omega_2 \preceq \omega_1$. Total preorders represent plausibility orderings, with the most plausible worlds being located in the lowermost layer of $\preceq$ which we denote by $\min(\Omega, \preceq)$. More generally, if $\Omega' \subseteq \Omega$ is a subset of possible worlds, $\min(\Omega', \preceq)$ denotes the set of minimal worlds in Ω' according to $\preceq$. The preorder $\preceq$ is lifted to a relation between propositions in the usual way: $A \preceq B$ if there is $\omega \models A$ such that $\omega \preceq \omega'$ for all $\omega' \models B$. The minimal formulas with respect to a total preorder $\preceq_\Psi$ representing an epistemic state Ψ are the propositional beliefs of Ψ, denoted as $\mathrm{Bel}(\Psi)$. Note that $\mathrm{Mod}(\mathrm{Bel}(\Psi)) = \min(\Omega, \preceq_\Psi)$.

Ordinal Conditional Functions (OCFs, also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$ [4] assign degrees of implausibility, or surprise, to possible worlds. An OCF κ is lifted to formulas by $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. Note that these definitions are in full compliance with corresponding definitions for total preorders.

Every ranking function induces a unique total preorder which can be obtained via the *transformation operator* τ [5] which maps an OCF κ to an epistemic state Ψ_κ, $\tau : \kappa \mapsto \Psi_\kappa$ such that for all $\omega, \omega' \in \Omega$,

$$\omega \preceq_{\Psi_\kappa} \omega' \text{ iff } \kappa(\omega) \leq \kappa(\omega') \tag{1}$$

holds. While each ranking function is associated with a unique total preorder there are potentially infinitely many OCFs complying with a given total preorder. We employ a specific transformation operator ρ [5] that maps an epistemic state Ψ to an OCF κ_Ψ, $\rho : \Psi \mapsto \kappa_\Psi$ such that for all $\omega \in \Omega$

$$\kappa_\Psi(\omega) = \min_{\kappa \in \tau^{-1}(\Psi)} \{\kappa(\omega)\} \tag{2}$$

holds, i.e., ρ maps Ψ to the minimal possible ranking function complying with Ψ. For a more detailed analysis on such transformations we refer to [6].

3. Related Work

There is a fairly large body of work on the general topic of merging information in logic-based frameworks, see e.g. [7] for an overview. There are also approaches dedicated specifically to the merging of epistemic states, e.g. [8, 9, 10]. These works are mainly motivated by the integration of information from different sources. Hence, they consider complicated merging scenarios where conflicting information is expected, and conflict resolution is necessary.

We are concerned with a different scenario. When merging *marginalized* total preorders, the information encoded over pairwise disjoint signatures is interpreted as independent from and irrelevant to each other. Our focus is on constructing a coherent global revision result from the results of local revisions, in order to make the whole process more efficient. To the best of our knowledge, there currently exist no merging strategies for marginalized TPOs. In [3] an operator $\oplus$ for combining OCFs $\kappa_1, \ldots, \kappa_n$ over pairwise disjoint signatures $\Sigma_1, \ldots, \Sigma_n$ was implicitly defined:

$$(\kappa_1 \oplus \cdots \oplus \kappa_n)(\omega_1 \ldots \omega_n) = \kappa_1(\omega_1) + \cdots + \kappa_n(\omega_n)\ . \tag{3}$$

The additivity of OCFs is very similar in spirit to (and in fact our main inspiration for) what we call *level-order merging* of total preorders in this paper.

4. Syntax Splitting for Epistemic States

The notion of syntax splitting we consider in this paper goes back to Parikh [1]. The main idea of syntax splitting is to pay attention to which parts of an agent's knowledge are syntactically relevant for the new information during belief revision. If the agent's knowledge is expressed over two disjoint subsignatures, and the new information is expressed in only one of them, then the agent's knowledge expressed over the other subsignature should not be affected.

Parikh originally formulated this concept as an axiom (P) for propositional revision of belief sets. In [11], two readings of this axiom, namely "weak (P)" and "strong (P)", were identified. The strong version of this axiom was generalized to iterated revision by sets of propositions in [3], giving rise to two syntax splitting postulates for the revision of epistemic states: (MR) and (P^{it}). These postulates rely on two important concepts: *marginalization* and *TPO splittings*.

Definition 1 (marginalization $\Psi_{\downarrow\Theta}$)**.** Let $\Psi = (\Sigma, \Omega, \preceq_\Psi)$ be an epistemic state, and let $\Theta \subseteq \Sigma$. The *marginalization of* Ψ *on* Θ is an epistemic state $\Psi_{\downarrow\Theta} = (\Theta, \Omega^\Theta, \preceq_{\Psi_{\downarrow\Theta}})$ with $\Omega^\Theta = \{\omega^\Theta \mid \omega \in \Omega\}$ and:

$$\omega_1^\Theta \preceq_{\Psi_{\downarrow\Theta}} \omega_2^\Theta \quad \text{iff} \quad \omega_1^\Theta \preceq_\Psi \omega_2^\Theta. \tag{4}$$

When the epistemic state Ψ is clear from context, or when the specific state does not matter and we only focus on the total preorder, we sometimes write $\preceq_{\downarrow\Theta}$ instead of $\preceq_{\Psi_{\downarrow\Theta}}$ in order to ease notation a bit, and talk about the marginalization of the total preorder instead of the marginalization of the corresponding epistemic state.

Definition 2 (TPO splitting [3])**.** Let $\Psi = (\Sigma, \Omega, \preceq_\Psi)$ be an epistemic state, and let $(\Sigma_1, \dots, \Sigma_n)$ be a partition of Σ. We say that Ψ *splits over* $(\Sigma_1, \dots, \Sigma_n)$ if the following condition holds for all $i \in \{1, \dots, n\}$ and for all $\omega_1, \omega_2 \in \Omega$ with $\omega_1^{\Sigma \setminus \Sigma_i} = \omega_2^{\Sigma \setminus \Sigma_i}$:

$$\omega_1 \preceq_\Psi \omega_2 \quad \text{iff} \quad \omega_1^{\Sigma_i} \preceq_{\Psi_{\downarrow \Sigma_i}} \omega_2^{\Sigma_i} \,. \tag{5}$$

We use the symbol $*$ to denote revision operators for epistemic states. We presuppose that $*$ is able to deal with epistemic states and formulas over any signature, i.e., it may be used both for global revision (of epistemic states defined over the whole signature Σ) and local revision (of epistemic states defined over subsignatures $\Sigma' \subseteq \Sigma$). However, $*$ does not have to make a connection between global and local revision scenarios; $(\Psi * A)_{\downarrow \Sigma'}$ and $\Psi_{\downarrow \Sigma'} * A$ may lead to completely different results, even if A only contains information about Σ'. The syntax splitting postulates from [3], which are given below, restrict this arbitrary handling of global and local contexts.

(MR) Let Ψ be an epistemic state defined on Σ that splits over $(\Sigma_1, \dots, \Sigma_n)$. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ with $C_i \in \mathcal{L}(\Sigma_i)$ be the new information. Then

$$(\Psi * \mathcal{C})_{\downarrow \Sigma_i} = (\Psi_{\downarrow \Sigma_i}) * C_i \,. \tag{6}$$

(P^{it}) Let Ψ be an epistemic state defined on Σ that splits over $(\Sigma_1, \dots, \Sigma_n)$. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ with $C_i \in \mathcal{L}(\Sigma_i)$ be the new information. Then $\Psi * \mathcal{C}$ splits over $(\Sigma_1, \dots, \Sigma_n)$.

(MR) makes an explicit connection between global and local revision scenarios, given that the new information $\mathcal{C}$ fits a syntax splitting of Ψ. More precisely, (MR) states that it should not matter whether you first revise the whole epistemic state and then marginalize to Σ_i, or marginalize first and only locally revise with the relevant information. (P^{it}), on the other hand, states that the splitting should be preserved, but not how the individual C_i should (or should not) influence the individual parts of the belief state. This ensures that no unnecessary syntactical dependencies are introduced by the revision. Hence, these two postulates formulate two different requirements for iterated revision operators, and neither of the two postulates implies the other [3]. Therefore, proper syntax splitting for total preorders is captured by the conjunction of (P^{it}) and (MR). We propose to express this by the following postulate.

(P^{MR}) Let Ψ be an epistemic state defined on Σ that splits over $(\Sigma_1, \dots, \Sigma_n)$. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ with $C_i \in \mathcal{L}(\Sigma_i)$ be the new information. Then $\Psi * \mathcal{C}$ splits over $(\Sigma_1, \dots, \Sigma_n)$ such that

$$(\Psi * \mathcal{C})_{\downarrow \Sigma_i} = (\Psi_{\downarrow \Sigma_i}) * C_i. \tag{7}$$

5. Revision Operator for Epistemic Syntax Splitting

In this section we will show that (MR) and (P^{it}) can indeed be brought together by defining a simple iterated revision operator which satisfies both postulates.

First, we define a measure for how much a set of formulas $\mathcal{C}$ and a possible world ω deviate from each other as the number of formulas in $\mathcal{C}$ that are not satisfied by ω:

$$\delta_{\mathcal{C}}(\omega) := |\{C_i \in \mathcal{C} \mid \omega \not\models C_i\}| \,. \tag{8}$$

Next we define a variation of the (simplified) lexicographic revision operator proposed in [2] which revises an epistemic state by a set of formulas (instead of a single proposition). In order to simplify the following definition, we assume that the new information is consistent.

Definition 3 (Multiple SimpLex Revision)**.** Let Ψ be an epistemic state over Σ, and let $\mathcal{C} \subseteq \mathcal{L}$ be a consistent set of formulas. Then the *multiple SimpLex (MSL) revision operator* $*_\ell$ is defined via the following conditions.

(MSL1) If $\delta_{\mathcal{C}}(\omega_1) = \delta_{\mathcal{C}}(\omega_2)$, then: $\omega_1 \preceq_{\Psi *_\ell \mathcal{C}} \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$.

(MSL2) If $\delta_{\mathcal{C}}(\omega_1) < \delta_{\mathcal{C}}(\omega_2)$, then: $\omega_1 \prec_{\Psi *_\ell \mathcal{C}} \omega_2$.

The two conditions given in the definition above uniquely define the posterior TPO $\preceq_{\Psi *_\ell \mathcal{C}}$. Moreover, it is easy to show that this operator fits into the popular AGM framework[1] [12, 13, 14] since the most plausible worlds after revision are the minimal models of the new information prior to revision.

Proposition 1. *Let* Ψ *be an epistemic state and let* $\mathcal{C} = \{C_1, \ldots, C_n\} \subseteq \mathcal{L}$. *Then* $\mathrm{Mod}(\mathrm{Bel}(\Psi *_\ell \mathcal{C})) = \min(\mathrm{Mod}(\mathcal{C}), \preceq_\Psi)$.

Proof. For all $\omega \in \mathrm{Mod}(\mathcal{C})$ it holds that $\delta_{\mathcal{C}}(\omega) = 0$, and for all $\omega' \notin \mathrm{Mod}(\mathcal{C})$ there must be some $C_i \in \mathcal{C}$ such that $\omega' \not\models C_i$, which implies $\delta_{\mathcal{C}}(\omega') > 0$. Hence $\omega \prec_{\Psi *_\ell \mathcal{C}} \omega'$ due do Definition 3. Therefore, if a world is minimal according to $\preceq_{\Psi *_\ell \mathcal{C}}$, then it must be a model of $\mathcal{C}$. Now let $\omega_1, \omega_2 \in \mathrm{Mod}(\mathcal{C})$. Then $\delta_{\mathcal{C}}(\omega_1) = \delta_{\mathcal{C}}(\omega_2)$, which implies $\omega_1 \preceq_{\Psi *_\ell \mathcal{C}} \omega_2$ iff $\omega_1 \preceq_\Psi \omega_2$ according to Definition 3. Therefore, $\min(\mathrm{Mod}(\mathcal{C}), \preceq_{\Psi *_\ell \mathcal{C}}) = \min(\mathrm{Mod}(\mathcal{C}), \preceq_\Psi)$. □

The following example illustrates how the MSL operator works.

Example 1. Let $\Psi = (\Sigma_\Psi, \Omega_\Psi, \preceq_\Psi)$ be an epistemic state over the signature $\Sigma_\Psi = \{a, b, c\}$. The total preorder $\preceq_\Psi$ is given below, with possible worlds represented as conjunctions of literals, and worlds in the same column being considered equally plausible with respect to Ψ.

$$\Psi : \quad abc \quad \prec_\Psi \quad \begin{matrix} ab\overline{c} \\ a\overline{b}c \\ \overline{a}bc \end{matrix} \quad \prec_\Psi \quad \begin{matrix} a\overline{b}\overline{c} \\ \overline{a}b\overline{c} \\ \overline{a}\overline{b}c \end{matrix} \quad \prec_\Psi \quad \overline{a}\overline{b}\overline{c}\,. \tag{9}$$

Now let $\mathcal{C} = \{(\neg a \vee \neg b), \neg c\}$. The revision $\Psi *_\ell \mathcal{C}$ now results in the following TPO:

$$\Psi *_\ell \mathcal{C} : \quad \underbrace{\begin{matrix} a\overline{b}\overline{c} \\ \overline{a}b\overline{c} \end{matrix} \quad \prec_{\Psi *_\ell \mathcal{C}} \quad \overline{a}\overline{b}\overline{c}}_{\text{deviation } 0} \quad \prec_{\Psi *_\ell \mathcal{C}} \quad \underbrace{\begin{matrix} ab\overline{c} \\ a\overline{b}c \\ \overline{a}bc \end{matrix}}_{\text{deviation } 1} \quad \prec_{\Psi *_\ell \mathcal{C}} \quad \underbrace{\overline{a}\overline{b}c \quad \prec_{\Psi *_\ell \mathcal{C}} \quad abc}_{\text{deviation } 2}\,. \tag{10}$$

As shown above, the possible worlds are staggered with respect to their deviation from the new information (given by $\delta_{\mathcal{C}}$), with equally-deviating worlds keeping their original relative ordering.

Proposition 2. *The revision operator* $*_\ell$ *satisfies* (P^{it}).

Proof. Let Ψ be an epistemic state defined over Σ, and let $\{\Sigma_1, \ldots, \Sigma_n\}$ be a TPO-splitting of $\preceq_\Psi$. Furthermore, let $\mathcal{C} = \{C_1, \ldots, C_n\} \subseteq \mathcal{L}$ with $C_i \in \mathcal{L}(\Sigma_i)$. Let $\Phi = \Psi *_\ell \mathcal{C}$ be the revision result.

Now choose $i \in \{1, \ldots, n\}$, and let $\omega_1, \omega_2 \in \Omega$ with $\omega_1^{\Sigma \setminus \Sigma_i} = \omega_2^{\Sigma \setminus \Sigma_i}$. In order to prove satisfaction of (P^{it}), we need to show that the equivalence

$$\omega_1 \preceq_\Phi \omega_2 \quad \text{iff} \quad \omega_1^{\Sigma_i} \preceq_{\Phi_{\downarrow \Sigma_i}} \omega_2^{\Sigma_i} \tag{11}$$

holds. We make a case distinction on whether exactly one of the two worlds ω_1, ω_2 satisfies C_i:

Case 1: Let $\omega_1 \models C_i$ and $\omega_2 \not\models C_i$ (without loss of generality). Then $\delta_{\mathcal{C}}(\omega_1) < \delta_{\mathcal{C}}(\omega_2)$, resulting in $\omega_1 \prec_\Phi \omega_2$. Now let $\omega_2' \in \min(\mathrm{Mod}(\omega_2^{\Sigma_i}), \preceq_\Phi)$. Since $\omega_2 \not\models C_i$, also $\omega_2' \not\models C_i$ holds. Now let ω_1' be defined by

$$\omega_1' := \omega_1^{\Sigma_i} \wedge (\omega_2')^{\Sigma \setminus \Sigma_i}\,. \tag{12}$$

Because of this construction, we have $\delta_{\mathcal{C}}(\omega_1') < \delta_{\mathcal{C}}(\omega_2')$, resulting in $\omega_1' \prec_\Phi \omega_2'$, i.e. there is a model of $\omega_1^{\Sigma_i}$ that is more plausible in $\preceq_\Phi$ than the minimal models of $\omega_2^{\Sigma_i}$. Therefore, $\omega_1 \prec_{\Phi_{\downarrow \Sigma_i}} \omega_2$. Hence, (11) holds for ω_1, ω_2.

[1] Originally, the AGM approach only considered revision of belief sets by single propositions [12]. It has since been extended in various ways, including revision of epistemic states [13] and revision by sets of formulas [14].

Case 2: Let either $\omega_1, \omega_2 \models C_i$, or $\omega_1, \omega_2 \models \neg C_i$. Then $\delta_{\mathcal{C}}(\omega_1) = \delta_{\mathcal{C}}(\omega_2)$ and $\omega_1 \preceq_\Phi \omega_2$ holds iff $\omega_1 \preceq_\Psi \omega_2$ holds. Assume $\omega_1 \preceq_\Psi \omega_2$ (without loss of generality). Using the same construction (12) as in the previous case, it follows that the minimal models of $\omega_1^{\Sigma_i}$ in $\preceq_\Phi$ are at least as plausible as the minimal models of $\omega_2^{\Sigma_i}$. Hence $\omega_1 \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2$, i.e., (11) holds. □

Proposition 3. *The revision operator $*_\ell$ satisfies (MR).*

Proof. Let Ψ be an epistemic state defined over Σ, and let $\{\Sigma_1, \dots, \Sigma_n\}$ be a TPO-splitting of $\preceq_\Psi$. Furthermore, let $\mathcal{C} = \{C_1, \dots, C_n\} \subseteq \mathcal{L}$ with $C_i \in \mathcal{L}(\Sigma_i)$. Let $\Phi = \Psi *_\ell \mathcal{C}$ be the revision result.

Now choose $i \in \{1, \dots, n\}$, and let $\omega_1^i, \omega_2^i \in \Omega(\Sigma_i)$. In order to prove satisfaction of (MR), we need to show that the equivalence

$$\omega_1^i \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2^i \quad \text{iff} \quad \omega_1^i \preceq_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i \tag{13}$$

holds. In this proof, we will make use of Proposition 2, i.e. Ψ having a TPO-splitting implies the same TPO-splitting for Φ.

Direction "⇒": Let $\omega_1^i \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$. We make a case distinction with respect to C_i.

- *Case 1:* $\omega_1^i \models C_i$ and $\omega_2^i \not\models C_i$. Because of $\delta_{\mathcal{C}}(\omega_1^i) < \delta_{\mathcal{C}}(\omega_2^i)$, it follows directly from (MSL2) that $\omega_1^i \prec_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$.
- *Case 2:* $\omega_1^i \not\models C_i$ and $\omega_2^i \models C_i$. This would mean $\delta_{\mathcal{C}}(\omega_1^i) > \delta_{\mathcal{C}}(\omega_2^i)$, which would imply $\omega_1^i \succ_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$ due to (MSL2), contradicting the assumption that $\omega_1^i \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$. Therefore, this case is impossible.
- *Case 3:* Both $\omega_1^i, \omega_2^i \models C_i$, or both $\omega_1^i, \omega_2^i \models \neg C_i$. Because we have a TPO-splitting, $\omega_1^i \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$ implies that $\omega_1 \preceq_\Phi \omega_2$ for all $\omega_1, \omega_2 \in \Omega$ with $\omega_1^{\Sigma_i} = \omega_1^i$, $\omega_2^{\Sigma_i} = \omega_2^i$, and $\omega_1^{\Sigma\setminus\Sigma_i} = \omega_2^{\Sigma\setminus\Sigma_i}$. Now choose $\omega_2 \in \min(\mathrm{Mod}(\omega_2^i), \preceq_\Psi)$, and let ω_1 be defined by

$$\omega_1 := \omega_1^i \wedge \omega_2^{\Sigma\setminus\Sigma_i} \,. \tag{14}$$

Then $\delta_{\mathcal{C}}(\omega_1) = \delta_{\mathcal{C}}(\omega_2)$, and we have $\omega_1 \preceq_\Psi \omega_2$ according to (MSL1). Since ω_2 is a minimal model of ω_2^i in $\preceq_\Psi$, it follows that $\omega_1^i \preceq_{\Psi_{\downarrow\Sigma_i}} \omega_2^i$. Therefore, (MSL1) requires that $\omega_1^i \preceq_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$.

Direction "⇐": Let $\omega_1^i \preceq_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$. We make a case distinction with respect to C_i.

- *Case 1:* $\omega_1^i \models C_i$ and $\omega_2^i \not\models C_i$. Let $\omega_2 \in \min(\mathrm{Mod}(\omega_2^i), \preceq_\Phi)$, and let ω_1 be defined by

$$\omega_1 := \omega_1^i \wedge \omega_2^{\Sigma\setminus\Sigma_i} \,. \tag{15}$$

Then $\delta_{\mathcal{C}}(\omega_1) < \delta_{\mathcal{C}}(\omega_2)$ and, therefore, $\omega_1 \prec_\Phi \omega_2$ due to (MSL2). Since ω_2 is a minimal model of ω_2^i in $\preceq_\Phi$, it follows that $\omega_1^i \prec_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$.
- *Case 2:* $\omega_1^i \not\models C_i$ and $\omega_2^i \models C_i$. This would mean $\delta_{\mathcal{C}}(\omega_1^i) > \delta_{\mathcal{C}}(\omega_2^i)$, which would imply $\omega_1^i \succ_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$ due to (MSL2), contradicting the assumption that $\omega_1^i \preceq_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$. Therefore, this case is impossible.
- *Case 3:* Both $\omega_1^i, \omega_2^i \models C_i$, or both $\omega_1^i, \omega_2^i \models \neg C_i$. Then $\omega_1^i \preceq_{\Psi_{\downarrow\Sigma_i}} \omega_2^i$ follows directly from $\omega_1^i \preceq_{(\Psi_{\downarrow\Sigma_i}) *_\ell C_i} \omega_2^i$ according to (MSL1). Because we have a TPO-splitting, it follows that $\omega_1 \preceq_\Psi \omega_2$ for all $\omega_1, \omega_2 \in \Omega$ with $\omega_1^{\Sigma_i} = \omega_1^i$, $\omega_2^{\Sigma_i} = \omega_2^i$, and $\omega_1^{\Sigma\setminus\Sigma_i} = \omega_2^{\Sigma\setminus\Sigma_i}$. Now choose $\omega_2 \in \min(\mathrm{Mod}(\omega_2^i), \preceq_\Phi)$, and let ω_1 be defined by

$$\omega_1 := \omega_1^i \wedge \omega_2^{\Sigma\setminus\Sigma_i} \,. \tag{16}$$

Then $\delta_{\mathcal{C}}(\omega_1) = \delta_{\mathcal{C}}(\omega_2)$, and $\omega_1 \preceq_\Phi \omega_2$ follows from $\omega_1 \preceq_\Psi \omega_2$ due to (MSL1). Since ω_2 is a minimal model of ω_2^i in $\preceq_\Phi$, it follows that $\omega_1^i \preceq_{\Phi_{\downarrow\Sigma_i}} \omega_2^i$.

□

In order to summarize the main result of this section, we combine Propositions 2 and 3 into the following theorem.

Theorem 4. *The revision operator $*_\ell$ satisfies both (MR) and (P^{it}), and thus (P^{MR}).*

6. Merging of Marginalized Total Preorders

In the previous section, we showed that there is a revision operator which satisfies both (MR) and (P^{it}). In this section, we want to build upon the concept of marginalized revision to obtain more general localized revision operators complying with the syntax splitting postulates. The idea is to first revise the marginalized epistemic states independently from each other, and then merge the results together.

6.1. Postulates for Merging Total Preorders over Disjoint Signatures

In this paper, we use the symbol $\vartriangle$ to denote merging operators for epistemic states that take two epistemic states as input and produce one merged epistemic state as output. In order to simplify notation, we stipulate that $\vartriangle$ is *left-associative*, i.e. $\Psi_1 \vartriangle \Psi_2 \vartriangle \Psi_3 = (\Psi_1 \vartriangle \Psi_2) \vartriangle \Psi_3$. Note that $\vartriangle$ may be commutative, but does not have to be; i.e., we may have $\Psi_1 \vartriangle \Psi_2 \neq \Psi_2 \vartriangle \Psi_1$. For most of the results in this section, this does not make a big difference, since we do not consider any additional preferences over the epistemic states themselves during merging. However, we will briefly talk about non-commutative merging towards the end of this section.

Because we are mainly interested in syntax splitting properties, we want to merge marginalized epistemic states defined over disjoint signatures. This means that the possible worlds which the epistemic states respectively talk about do not share any atoms, whereas the merged epistemic state's possible worlds should be defined over the combined signature. Therefore, we define the set of possible worlds over the combined signature as follows.

Definition 4 (Combined Possible Worlds). Let Σ_1, Σ_2 be signatures with $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let $\Omega_1 \subseteq \Omega(\Sigma_1)$, and $\Omega_2 \subseteq \Omega(\Sigma_2)$. Then the *combined set of possible worlds* $\Omega_1 \otimes \Omega_2 \subseteq \Omega(\Sigma_1 \cup \Sigma_2)$ is defined by:

$$(\Omega_1 \otimes \Omega_2) := \{\omega_1\omega_2 \mid (\omega_1, \omega_2) \in (\Omega_1 \times \Omega_2)\}\ . \tag{17}$$

Let $\Psi_1, \ldots, \Psi_n$ be epistemic states represented by total preorders over sets of possible worlds $\Omega_1, \ldots, \Omega_n$, which are respectively defined over pairwise disjoint signatures $\Sigma_1, \ldots, \Sigma_n$. As the result of merging $\Psi_1, \ldots, \Psi_n$ we henceforth expect an epistemic state represented by a total preorder over the combined possible worlds. More precisely, $(\Psi_1 \vartriangle \ldots \vartriangle \Psi_n) := (\Sigma^{\vartriangle}, \Omega^{\vartriangle}, \preceq^{\vartriangle})$ such that $\Sigma^{\vartriangle} = \bigcup_{i=1}^{n} \Sigma_i$, $\Omega^{\vartriangle} = \Omega_1 \otimes \ldots \otimes \Omega_n$, and $\preceq^{\vartriangle} \subseteq \Omega^{\vartriangle} \times \Omega^{\vartriangle}$. We can now define revision via merging of localized revision results as follows.

Definition 5 ($\circledast^{\vartriangle}$). Let Ψ be an epistemic state, let $\{\Sigma_1, \ldots, \Sigma_n\}$ be a splitting of $\preceq_\Psi$, and let $\mathcal{C} = \{C_1, \ldots, C_n\}$ with $C_i \in \mathcal{L}(\Sigma_i)$ for $i \in \{1, \ldots, n\}$. Let $*$ be a revision operator, and let $\vartriangle$ be a merging operator. Then the revision operator $\circledast^{\vartriangle}$ is defined by

$$\Psi \circledast^{\vartriangle} \mathcal{C} = (\Psi_{\downarrow\Sigma_1} * C_1) \vartriangle \ldots \vartriangle (\Psi_{\downarrow\Sigma_n} * C_n). \tag{18}$$

When merging $\Psi_1, \ldots, \Psi_n$, there should be no information conflicts since every epistemic state is concerned with a domain that is different from all others. Therefore, we expect that the merging is performed such that the individual Ψ_i can be recovered via marginalization. This is expressed by the following property:

(ReMarg) $(\Psi_1 \vartriangle \ldots \vartriangle \Psi_n)_{\downarrow\Sigma_i} = \Psi_i$

Moreover, the merging should not introduce any dependencies between atoms from different Σ_i. Hence, the merged epistemic state should split over the original signatures.

(MSplit) $(\Psi_1 \vartriangle \ldots \vartriangle \Psi_n)$ splits over $(\Sigma_1, \ldots \Sigma_n)$.

Just like (MR) and (P^{it}), these two merging postulates do not imply each other, i.e. they indeed specify different requirements of merging operators. Counterexamples would be very similar to the ones provided in [3].

Theorem 5. *Let $*$ be a revision operator, let $\triangle$ be a merging operator, and let $\circledast^{\triangle}$ be the revision operator constructed from $(*, \triangle)$ via Definition 5. Then the following propositions holds:*

- *If $\triangle$ satisfies (ReMarg), then $\circledast^{\triangle}$ satisfies (MR).*
- *If $\triangle$ satisfies (MSplit), then $\circledast^{\triangle}$ satisfies (P^{it}).*

Proof. Let Ψ be an epistemic state that splits over $\{\Sigma_1, \ldots, \Sigma_n\}$. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ with $C_i \in \mathcal{L}(\Sigma_i)$. For ($\mathrm{P}^{it}$), assume that $\triangle$ satisfies (MSplit). Then $\Psi \circledast^{\triangle} \mathcal{C}$ splits over $\{\Sigma_1, \ldots, \Sigma_n\}$, i.e. ($\mathrm{P}^{it}$) holds. For (MR), let $\triangle$ satisfy (ReMarg) and observe that

$$(\Psi \circledast^{\triangle} \mathcal{C})_{\downarrow\Sigma_i} = \big((\Psi_{\downarrow\Sigma_1} * C_1) \triangle \ldots \triangle (\Psi_{\downarrow\Sigma_n} * C_n)\big)_{\downarrow\Sigma_i} . \tag{19}$$

With (ReMarg), it follows that $(\Psi \circledast^{\triangle} \mathcal{C})_{\downarrow\Sigma_i} = (\Psi_{\downarrow\Sigma_i} * C_i)$. The state $\Psi_{\downarrow\Sigma_i}$ trivially splits over $\{\Sigma_i\}$. In this case $\circledast^{\triangle}$ and $*$ coincide. Thus $(\Psi \circledast^{\triangle} \mathcal{C})_{\downarrow\Sigma_i} = \Psi_{\downarrow\Sigma_i} \circledast^{\triangle} C_i$ and we are done. □

As an easy consequence of Theorem 5, the revision operator $\circledast^{\triangle}$ from Definition 5 satisfies (P^{MR}) if the utilized merging operator $\triangle$ satisfies (ReMarg) and (MSplit). Next we are going to investigate how such a merging operator $\triangle$ can be designed.

6.2. Product Combinations and Faithful Extensions

The first basic step towards merging total preorders, which are defined over disjoint subsignatures, is to define how relationships between possible worlds from the individual $\preceq_i$ should be adopted into the merged relation $\preceq^{\triangle}$. The following definition gives us the core of the merged relation, i.e., it captures the relationships which we want to hold in any version of $\preceq^{\triangle}$.

Definition 6 (Product Combination of TPOs)**.** Let $\preceq_1, \ldots, \preceq_n$ be total preorders over $\Omega_1, \ldots, \Omega_n$, respectively, with $\Omega_i \subseteq \Omega(\Sigma_i)$ for all $i \in \{1, \ldots, n\}$, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$. The *product combination* of these TPOs, denoted as $\preceq^{\otimes} = (\preceq_1 \otimes \ldots \otimes \preceq_n)$, is a relation over $\Omega_1 \otimes \ldots \otimes \Omega_n$, such that for all $(\omega_1, \ldots, \omega_n), (\omega_1', \ldots, \omega_n') \in (\Omega_1 \times \ldots \times \Omega_n)$:

$$\omega_1 \ldots \omega_n \preceq^{\otimes} \omega_1' \ldots \omega_n' \quad \text{iff} \quad \omega_i \preceq_i \omega_i' \text{ for all } i \in \{1, \ldots, n\}. \tag{20}$$

In essence, the product combination is what is called a *product order* in order theory (see e.g. [15]), except that it is defined on $\Omega_1 \otimes \ldots \otimes \Omega_n$ instead of the Cartesian product $\Omega_1 \times \ldots \times \Omega_n$. The resulting order $\preceq^{\otimes}$ is a preorder, i.e. a reflexive and transitive relation, but it is usually not total: if there are $\omega_i, \omega_i' \in \Omega_i$ and $\omega_j, \omega_j' \in \Omega_j$ ($i, j \in \{1, \ldots, n\}$) such that $\omega_i \prec_i \omega_i'$ and $\omega_j \prec_j \omega_j'$, then $\omega_i\omega_j'$ and $\omega_i'\omega_j$ are incomparable with respect to $\preceq^{\otimes}$. Since we assume in this paper that epistemic states are uniquely identified by TPOs, we will also talk about the "product combination of epistemic states Ψ_1, Ψ_2" when we mean the product combination of $\preceq_{\Psi_1}$ and $\preceq_{\Psi_2}$.

Example 2. Let Ψ_1, Ψ_2 be defined by the following total preorders:

$$\begin{aligned} \Psi_1 :&\quad ab \prec_1 a\overline{b} \prec_1 \overline{a}b \prec_1 \overline{a}\overline{b}\,, \\ \Psi_2 :&\quad cd \prec_2 c\overline{d} \prec_2 \overline{c}d \prec_2 \overline{c}\overline{d}\,. \end{aligned} \tag{21}$$

The *product combination* of Ψ_1 and Ψ_2 as given by Definition 6, i.e. the preorder $\preceq^{\otimes} = (\preceq_1 \otimes \preceq_2)$, is visualized in Figure 1. Note that the elements along the diagonals in "antidiagonal" direction, e.g. $abc\overline{d}$ and $a\overline{b}cd$, are incomparable with respect to $\preceq^{\otimes}$.

Since $\preceq^{\otimes}$ is not necessarily a total order, we need to extend this relation in order to represent a full epistemic state.

Definition 7 (faithful extension)**.** Let $\preceq$ be a preorder, i.e. a reflexive and transitive relation, over some set M. Let $\preceq'$ be a preorder over M that extends $\preceq$, i.e. $\preceq \subseteq \preceq'$. We call this extension *faithful* if for all $x, y \in M$, it holds that: $x \prec y$ implies $x \prec' y$.

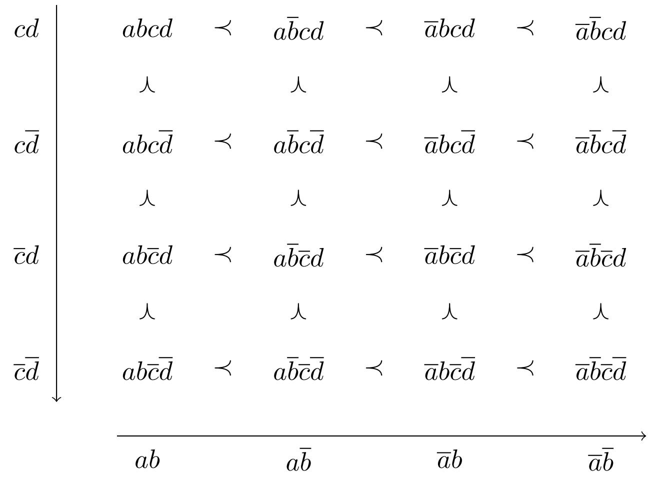

Figure 1: Product combination of two total preorders (defined over $\Sigma_1 = \{a, b\}$ resp. $\Sigma_2 = \{c, d\}$) from Example 2. The symbol $\prec$ stands for $\prec^{\otimes}$. The arrows indicate the influence of $\preceq_1$ and $\preceq_2$, respectively, on the resulting preorder $\preceq^{\otimes}$.

In other words, a faithful extension of a preorder preserves all strict preferences and all equalities. This means that the only additional preferences introduced by a faithful extension are preferences among elements which were incomparable before. Clearly, every preorder is a faithful extension of itself. Moreover, all faithful extensions of a preorder have the same minimal (and maximal) worlds as the original preorder they extend. For product combinations, this property can be generalized to the Lemma below.

Lemma 6. *Let $\Psi_1, \ldots, \Psi_n$ be epistemic states with $\Psi_i = (\Sigma_i, \Omega_i, \preceq_i)$ for all $i \in \{1, \ldots, n\}$, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$. Let $\preceq^{\triangle}$ be a faithful extension of $\preceq^{\otimes} = (\preceq_1 \otimes \ldots \otimes \preceq_n)$. Then for all $i \in \{1, \ldots, n\}$ and $\omega_i \in \Omega_i$, the following holds:*

$$\min(\mathrm{Mod}(\omega_i), \preceq^{\triangle}) = \{\omega_1 \ldots \omega_{(i-1)}\omega_i\omega_{(i+1)} \ldots \omega_n \mid \omega_j \in \min(\mathrm{Mod}(\Omega_j), \preceq_j) \text{ for all } j \neq i\} \,. \quad (22)$$

Proof. First we show the direction "$\subseteq$": Let $\omega \in \min(\mathrm{Mod}(\omega_i), \preceq^{\triangle})$. By construction, ω has the form $\omega = \omega_1 \ldots \omega_{(i-1)}\omega_i\omega_{(i+1)} \ldots \omega_n$ with $\omega_j \in \Omega_j$ for all $j \in \{1, \ldots, n\}$. Suppose that there was some ω_j such that $\omega_j \notin \min(\mathrm{Mod}(\Omega_j), \preceq_j)$. Then there is $\omega'_j \in \Omega_j$ with $\omega'_j \prec_j \omega_j$. Let ω' be the world obtained by replacing ω_j with ω'_j in ω. Then $\omega' \prec^{\otimes} \omega$ according to Definition 6, which implies $\omega' \prec^{\triangle} \omega$ due to Definition 7. This means that ω cannot be a minimal model of ω_i in $\preceq^{\triangle}$, which contradicts the initial assumption that $\omega \in \min(\mathrm{Mod}(\omega_i), \preceq^{\triangle})$.

"$\supseteq$": Let $\omega_j \in \min(\mathrm{Mod}(\Omega_j), \preceq_j)$ for all $j \neq i$, and let $\omega = \omega_1 \ldots \omega_{(i-1)}\omega_i\omega_{(i+1)} \ldots \omega_n$. Suppose that there was some $\omega' \models \omega_i$ such that $\omega' \prec^{\triangle} \omega$. Then, due to Definition 7, either $\omega' \prec^{\otimes} \omega$ or ω, ω' are incomparable w.r.t. $\preceq^{\otimes}$.

Case 1: $\omega' \prec^{\otimes} \omega$. Then $\omega'_j \preceq_j \omega_j$ for all $\omega_j, \omega'_j \in \Omega_j$ such that $\omega \models \omega_j$ and $\omega' \models \omega'_j$. Moreover, there must be at least one such pair $\omega_k, \omega'_k \in \Omega_k$ with $\omega \models \omega_k$ and $\omega' \models \omega'_k$ such that $\omega'_k \prec_k \omega_k$. However, that means $\omega_k \notin \min(\mathrm{Mod}(\Omega_k), \preceq_k)$, contradicting the initial assumption that $\omega_j \in \min(\mathrm{Mod}(\Omega_j), \preceq_j)$ for all $j \neq i$.

Case 2: ω, ω' are incomparable w.r.t. $\preceq^{\otimes}$. Then there must be $\omega_j, \omega'_j \in \Omega_j$ and $\omega_k, \omega'_k \in \Omega_k$ $(j, k \in \{1, \ldots, n\})$ with such that $\omega \models \omega_j\omega_k$ and $\omega' \models \omega'_j\omega'_k$ such that $\omega_j \prec_j \omega'_j$ and $\omega'_k \prec_j \omega_k$. However, that means again $\omega_k \notin \min(\mathrm{Mod}(\Omega_k), \preceq_k)$, causing a contradiction. □

Lemma 6 essentially states that the minimal models of every $\omega_i \in \Omega_i$ according to $\preceq^{\triangle}$ are those constructed from ω_i and minimal worlds in all other Ψ_j with $j \neq i$. As a consequence, marginalization is preserved.

Proposition 7. *Let $\Psi_1, \ldots, \Psi_n$ be epistemic states with $\Psi_i = (\Sigma_i, \Omega_i, \preceq_i)$ for all i, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$. Let $\preceq^{\triangle}$ be a faithful extension of $\preceq^{\otimes} = (\preceq_1 \otimes \ldots \otimes \preceq_n)$. Then for all $i \in \{1, \ldots, n\}$ the following holds:*

$$\preceq^{\triangle}_{\downarrow\Sigma_i} = \preceq^{\otimes}_{\downarrow\Sigma_i} = \preceq_i . \tag{23}$$

Proof. Lemma 6 directly implies that the minimal models of ω_i are the same in $\preceq^{\triangle}$ and $\preceq^{\otimes}$, because $\preceq^{\otimes}$ is a faithful extension of itself. Hence, the equality $\preceq^{\triangle}_{\downarrow\Sigma_i} = \preceq^{\otimes}_{\downarrow\Sigma_i}$ holds.

Let $\omega_i, \omega_i' \in \Omega_i$. According to Lemma 7, the minimal models of ω_i and ω_i' in $\preceq^{\otimes}$ have the form $\omega_i\mu$ and $\omega_i'\mu$, respectively, with $\mu \in (\Omega_1 \otimes \ldots \otimes \Omega_{(i-1)} \otimes \Omega_{(i+1)} \otimes \ldots \otimes \Omega_n)$. It follows with Definition 6 that $\omega_i\mu \preceq^{\otimes} \omega_i'\mu$ iff $\omega_i \preceq_i \omega_i'$. Therefore, the equality $\preceq^{\otimes}_{\downarrow\Sigma_i} = \preceq_i$ holds. ☐

Besides the marginalizations, faithful extension also preserve the splitting over the original signatures.

Proposition 8. *Let $\Psi_1, \ldots, \Psi_n$ be epistemic states with $\Psi_i = (\Sigma_i, \Omega_i, \preceq_i)$ for all i, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$. Let $\preceq^{\triangle}$ be a faithful extension of $\preceq^{\otimes} = (\preceq_1 \otimes \ldots \otimes \preceq_n)$. Then $\preceq^{\triangle}$ splits over $\{\Sigma_1, \ldots, \Sigma_n\}$.*

Proof. Let $\omega, \omega' \in \Omega^{\triangle}$ and let $i \in \{1, \ldots, n\}$. Note that $\omega^{\Sigma_i} = \omega_i$ and $(\omega')^{\Sigma_i} = \omega_i'$ for some $\omega_i, \omega_i' \in \Omega_i$ due to Definition 6. Therefore, we have to show that $\omega_i\mu \preceq^{\triangle} \omega_i'\mu$ iff $\omega_i \preceq^{\triangle}_{\downarrow\Sigma_i} \omega_i'$ holds for all $\mu \in (\Omega_1 \otimes \ldots \otimes \Omega_{(i-1)} \otimes \Omega_{(i+1)} \otimes \ldots \otimes \Omega_n)$.

"$\Rightarrow$": Assume w.l.o.g. that $\omega_i\mu \preceq^{\triangle} \omega_i'\mu$. Then $\omega_i'\mu \not\prec^{\triangle} \omega_i\mu$. Due to Definition 7, this means $\omega_i'\mu \not\prec^{\otimes} \omega_i\mu$. Hence $\omega_i' \not\prec_i \omega_i$, which is equivalent to $\omega_i \preceq_i \omega_i'$. Therefore, $\omega_i \preceq^{\triangle}_{\downarrow\Sigma_i} \omega_i'$ follows with Proposition 7.

"$\Leftarrow$": Assume w.l.o.g. that $\omega_i \preceq^{\triangle}_{\downarrow\Sigma_i} \omega_i'$. Due to Proposition 7, this is equivalent to $\omega_i \preceq_i \omega_i'$. Then by Definition 6, $\omega_i\mu \preceq^{\otimes} \omega_i'\mu$ holds for all $\mu \in (\Omega_1 \otimes \ldots \otimes \Omega_{(i-1)} \otimes \Omega_{(i+1)} \otimes \ldots \otimes \Omega_n)$, which implies $\omega_i\mu \preceq^{\triangle} \omega_i'\mu$ since $\preceq^{\triangle}$ extends $\preceq^{\otimes}$. ☐

Propositions 7 and 8 together directly imply the following theorem.

Theorem 9. *Let $\Psi_1, \ldots, \Psi_n$ be epistemic states with $\Psi_i = (\Sigma_i, \Omega_i, \preceq_i)$ for all $i \in \{1, \ldots, n\}$, and $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i \neq j$. Let $\Psi^{\triangle} = (\Sigma^{\triangle}, \Omega^{\triangle}, \preceq^{\triangle}) = \Psi_1 \triangle \ldots \triangle \Psi_n$ be the result of merging these epistemic states via a merging operator $\triangle$. If $\preceq^{\triangle}$ is a faithful extension of $\preceq^{\otimes} = (\preceq_1 \otimes \ldots \otimes \preceq_n)$, then $\triangle$ satisfies both (ReMarg) and (MSplit).*

6.3. Example Merging Operators

Theorem 9 provides us with a blueprint for TPO merging operators that satisfy (ReMarg) and (MSplit). We will now define a concrete merging operator that satisfies this property, i.e., one that merges TPOs by creating a faithful extension of their product combination.

Definition 8 (level-order merging operator)**.** Let $\Psi_1 = (\Sigma_1, \Omega_1, \preceq_1)$, $\Psi_2 = (\Sigma_2, \Omega_2, \preceq_2)$ be epistemic states with $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let $\kappa_1 = \rho(\preceq_1)$ and $\kappa_2 = \rho(\preceq_2)$. Then the *level-order merging operator* $\triangle^{\mathrm{lvl}}$ is defined by

$$\omega_1\omega_2 \preceq_{\Psi_1 \triangle^{\mathrm{lvl}} \Psi_2} \omega_1'\omega_2' \quad \text{iff} \quad \kappa_1(\omega_1) + \kappa_2(\omega_2) \leq \kappa_1(\omega_1') + \kappa_2(\omega_2') \tag{24}$$

for all $\omega_1, \omega_1' \in \Omega_1$ and $\omega_2, \omega_2' \in \Omega_2$.

The level-order[2] merging operator combines worlds into one layer[3] based on their distance from the lowermost layer in the product combination. To illustrate this, look again at Figure 1: each incomparable diagonal (see Example 2) becomes one layer in the merging result. Therefore, all strict preferences and equalities from the product combination are respected in the level-order merging result. Since level-order merging is based on addition of natural numbers, it is clear that it is both commutative and associative. Hence, Definition 8 can easily be generalized to merging an arbitrary number of epistemic states.

[2]The name is inspired by level-order traversal on graphs (breadth-first search).

[3]The *layers* of a TPO $\preceq_\Psi$ are the equivalence classes of worlds with respect to $\approx_\Psi$.

Proposition 10. *The level-order merging operator satisfies both (ReMarg) and (MSplit).*

Proof. We prove the proposition via Theorem 9 by showing that $\preceq^{\Delta} = \preceq_{\Psi_1 \Delta^{\text{lvl}} \Psi_2}$ is a faithful extension of the product combination $\preceq^{\otimes} = (\preceq_1 \otimes \preceq_2)$. First observe that $\omega_1 \preceq_1 \omega_1'$ and $\omega_2 \preceq_2 \omega_2'$ together imply $\kappa_1(\omega_1) + \kappa_2(\omega_2) \leq \kappa_1(\omega_1') + \kappa_2(\omega_2')$. Hence $\preceq^{\otimes} \subseteq \preceq^{\Delta}$. Moreover, $\omega_1 \prec_1 \omega_1'$ and $\omega_2 \preceq_2 \omega_2'$ (w.l.o.g.) together imply $\kappa_1(\omega_1) + \kappa_2(\omega_2) < \kappa_1(\omega_1') + \kappa_2(\omega_2')$. Therefore, $\omega_1\omega_2 \prec^{\otimes} \omega_1'\omega_2'$ implies $\omega_1\omega_2 \prec^{\Delta} \omega_1'\omega_2'$. This means that $\preceq^{\Delta}$ is a faithful extension of $\preceq^{\otimes}$, and the proposition follows via Theorem 9. □

Of course, level-order merging is not the only possible merging operator based on faithful extensions. A classic way of extending a product order to a total order is to construct a so-called *lexicographic order* [15], which could also be used for merging. However, note that the merging operator defined below is not commutative, i.e., the order of the epistemic states matters for the merging result.

Definition 9 (lexicographic merging operator)**.** Let $\Psi_1 = (\Sigma_1, \Omega_1, \preceq_1), \Psi_2 = (\Sigma_2, \Omega_2, \preceq_2)$ be epistemic states with $\Sigma_1 \cap \Sigma_2 = \emptyset$. The *lexicographic merging operator* Δ^{lex} is defined by

$$\omega_1\omega_2 \preceq_{\Psi_1 \Delta^{\text{lex}} \Psi_2} \omega_1'\omega_2' \quad \text{iff} \quad (\omega_1 \prec_1 \omega_1') \text{ or } (\omega_1 \approx_1 \omega_1' \text{ and } \omega_2 \preceq_2 \omega_2') \tag{25}$$

for all $\omega_1, \omega_1' \in \Omega_1$ and $\omega_2, \omega_2' \in \Omega_2$.

Despite not being commutative, the lexicographic merging operator defines a faithful extension of the product combination, and thus also preserves marginalization and splitting of the marginalized total preorders.

Proposition 11. *The lexicographic merging operator satisfies both (ReMarg) and (MSplit).*

Proof. Similar to the previous result, we prove the proposition by showing that $\preceq^{\Delta} = \preceq_{\Psi_1 \Delta^{\text{lex}} \Psi_2}$ is a faithful extension of $\preceq^{\otimes} = (\preceq_1 \otimes \preceq_2)$. Observe that $\omega_1 \preceq_1 \omega_1'$ and $\omega_2 \preceq_2 \omega_2'$ together imply $\omega_1\omega_2 \preceq_{\Psi_1 \Delta^{\text{lex}} \Psi_2} \omega_1'\omega_2'$, i.e. $\preceq^{\otimes} \subseteq \preceq^{\Delta}$. If $\omega_1\omega_2 \prec^{\otimes} \omega_1'\omega_2'$, then $(\omega_1 \prec_1 \omega_1'$ and $\omega_2 \preceq_2 \omega_2')$, or $(\omega_1 \preceq_1 \omega_1'$ and $\omega_2 \prec_2 \omega_2')$. If $\omega_1 \prec_1 \omega_1'$, then $\omega_1\omega_2 \prec^{\Delta} \omega_1'\omega_2'$ follows directly from Definition 9. Otherwise, if $(\omega_1 \preceq_1 \omega_1'$ and $\omega_2 \prec_2 \omega_2')$ still holds, we must have $\omega_1 \approx_1 \omega_1'$. Then $\omega_1\omega_2 \prec^{\Delta} \omega_1'\omega_2'$ also follows via Definition 9. Therefore, $\preceq^{\Delta}$ is a faithful extension of $\preceq^{\otimes}$, and the proposition follows via Theorem 9. □

Illustrating lexicographic merging again with Figure 1, the operator essentially proceeds "column-wise" from left to right, by appending the relative order between the $a\bar{b}$-worlds to the relative order between the ab-worlds and so on. This makes the first epistemic state take priority over the second one, which might make sense in some scenarios. For example, if the TPOs represent plausibilities on different scales, i.e., if ab is vastly more plausible than $a\bar{b}$, but cd is only slightly more plausible than $c\bar{d}$, then one might desire to have all ab-worlds be more plausible than all $a\bar{b}$-worlds.

In terms of propositional beliefs, however, the order in which the epistemic states are merged does not matter even when using non-commutative merging operators like lexicographic merging, as long as they are based on faithful extensions of product combinations. Since propositional beliefs only depend on the minimal worlds in the final merging result, and all faithful extensions have the same minimal worlds, they must share the same propositional beliefs as well.

We finish this section with the following example, which shows how the two merging operators Δ^{lvl} and Δ^{lex} may be used in a belief revision scenario in order to implement syntax splitting.

Example 3. Consider again the total preorder $\preceq_\Psi$ from Example 1, Equation (9). Instead of revising the complete TPO, we now want to revise $\Psi_{\downarrow\Sigma_1}$ and $\Psi_{\downarrow\Sigma_2}$ with $\Sigma_1 = \{a, b\}, \Sigma_2 = \{c\}$ independently, and merge the results together afterwards. These marginalizations of Ψ are given as follows:

$$\Psi_{\downarrow\Sigma_1}: \quad ab \quad \prec_{\Psi_1^*} \quad \begin{matrix} \bar{a}b \\ a\bar{b} \end{matrix} \quad \prec_{\Psi_1^*} \quad \bar{a}\bar{b}\,. \qquad \Psi_{\downarrow\Sigma_2}: \quad c \quad \prec_{\Psi_2^*} \quad \bar{c}\,. \tag{26}$$

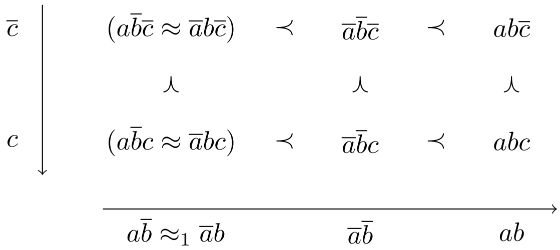

Figure 2: Product combination of Ψ_1^* and Ψ_2^* for Example 3. The symbol $\prec$ stands for $\prec^{\otimes}$. The arrows indicate the influence of $\preceq_{\Psi_1^*}$ and $\preceq_{\Psi_1^*}$, respectively, on the resulting preorder $\preceq^{\otimes}$.

Now we revise these marginalized TPOs with $C_1 = (\neg a \vee \neg b)$ resp. $C_2 = \neg c$ using the MSL-revision operator[4]. Let $\Psi_1^* = \Psi_{\downarrow\Sigma_1} *_\ell C_1$ and $\Psi_2^* = \Psi_{\downarrow\Sigma_2} *_\ell C_2$. Then the revisions result in the following TPOs:

$$\Psi_1^* : \quad \begin{matrix} \overline{a}b \\ a\overline{b} \end{matrix} \quad \prec_{\Psi_1^*} \quad \overline{a}\overline{b} \quad \prec_{\Psi_1^*} \quad ab\,. \qquad \Psi_2^* : \quad \overline{c} \quad \prec_{\Psi_2^*} \quad c\,. \tag{27}$$

Next, we use the merging operators presented in this section in order to obtain global revision results by $\mathcal{C} = \{C_1, C_2\}$ (in the spirit of Definition 5). The product combination of Ψ_1^* and Ψ_2^* is shown in Figure 2. First with Δ^{lvl}, merging Ψ_1^* and Ψ_2^* leads to the following result:

$$\Psi_1^* \; \Delta^{\text{lvl}} \; \Psi_2^* : \quad \begin{matrix} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{matrix} \quad \prec^{\Delta} \quad \begin{matrix} \overline{a}\overline{b}\overline{c} \\ \overline{a}bc \\ a\overline{b}c \end{matrix} \quad \prec^{\Delta} \quad \begin{matrix} ab\overline{c} \\ \overline{a}\overline{b}c \end{matrix} \quad \prec^{\Delta} \quad abc\,. \tag{28}$$

For the use of Δ^{lex}, there are two possible results depending on which epistemic state is given priority:

$$\Psi_1^* \; \Delta^{\text{lex}} \; \Psi_2^* : \quad \begin{matrix} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{matrix} \quad \prec^{\Delta} \quad \overline{a}\overline{b}\overline{c} \quad \prec^{\Delta} \quad ab\overline{c} \quad \prec^{\Delta} \quad \begin{matrix} \overline{a}bc \\ a\overline{b}c \end{matrix} \quad \prec^{\Delta} \quad \overline{a}\overline{b}c \quad \prec^{\Delta} \quad abc\,, \tag{29}$$

$$\Psi_2^* \; \Delta^{\text{lex}} \; \Psi_1^* : \quad \begin{matrix} \overline{a}b\overline{c} \\ a\overline{b}\overline{c} \end{matrix} \quad \prec^{\Delta} \quad \begin{matrix} \overline{a}bc \\ a\overline{b}c \end{matrix} \quad \prec^{\Delta} \quad \overline{a}\overline{b}\overline{c} \quad \prec^{\Delta} \quad \overline{a}\overline{b}c \quad \prec^{\Delta} \quad ab\overline{c} \quad \prec^{\Delta} \quad abc\,. \tag{30}$$

Clearly, all three global results shown above are different from each other. Depending on the specific use case, one may prefer one of them, or accept any result (e.g. if only propositional beliefs are relevant). As expected (because of Theorem 5 and Theorem 9), all three revision results adhere to the syntax splitting postulate (P^{MR}). As a further remark, none of these results coincides with the MSL-revision result from Example 1, Equation (10). This is also expected since the MSL-revision operator $*_\ell$ uses some information from Ψ which is lost during marginalization. Nevertheless, the TPO given in Equation (10) is also a faithful extension of the product combination of Ψ_1^* and Ψ_2^*.

7. Conclusions and Future Work

In this paper, we have shown how total preorders defined over disjoint signatures can be combined while preserving desirable properties which we have introduced via the postulates (ReMarg) and (MSplit). These postulates are satisfied if the merged total preorder is a faithful extension of the product combination of the original TPOs. Performing localized revision on marginalized epistemic states, and merging afterwards with an appropriate merging operator satisfying (ReMarg) and (MSplit), yields a global revision operator that satisfies the iterated syntax splitting postulates (P^{it}) and (MR). We have investigated two such merging operators, namely the level-order and the lexicographic merging operator, and have shown that they satisfy (ReMarg) and (MSplit). In future work we will investigate merging in cases where the signatures of the marginalized total preorders are not disjoint but may

[4]Technically, MSL revision with a single formula coincides with regular lexicographic revision [2].

share some elements which is the case in, e.g., [16]. We also plan to extend our results and postulates to revision with not just propositions but also conditionals such as in the case of c-revisions [17]. Finally we plan to investigate connections between the postulates (P^{it}) respectively (MR) and their OCF-counterparts (P^{ocf}) respectively (MR^{ocf}) as introduced in [3].

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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