Splitting Assumption-Based Argumentation Frameworks

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Abstract

Assumption-Based Argumentation (ABA) is a well-established rule-based formalism for modelling and reasoning in non-monotonic settings, with a wide range of applications. However, the high computational complexity of core reasoning tasks in ABA poses a significant challenge for its applicability in practice. This issue is further exacerbated when ABA frameworks (ABAFs) are instantiated into graph-based argumentation formalisms, such as Dung's Argumentation Frameworks (AFs) and Argumentation Frameworks with Collective Attacks (SETAFs). In the context of non-monotonic reasoning, a key strategy to address computational intractability is to optimise reasoning over a given knowledge base through divide-and-conquer algorithms. A paradigmatic example of this approach is splitting, where extensions of a given framework are computed incrementally, i.e. restricting the search space to sub-frameworks only, and then combining the obtained results. This approach has been successfully applied to SETAFs in the literature. Furthermore, a parametrised version has been introduced for AFs under stable semantics. However, the exponential growth produced by the instantiation process might undermine the usefulness of splitting on the argument graphs induced by ABAFs. For this reason, there is a need for splitting-based algorithms tailored for ABA. To address this issue, our work investigates the concept of splitting for ABAFs under common semantics. Furthermore, we generalise splitting to its parametrised version both for SETAFs and ABAFs.

Keywords

Assumption-Based Argumentation, Splitting, Collective Attacks

1. Introduction

Computational models of argumentation in AI [1] offer formal approaches to represent and reason over situations where contradicting or uncertain information is present. Among these, Assumption-Based Argumentation (ABA) [2] captures argumentative scenarios by means of so-called ABA frameworks (ABAFs), consisting of a set of defeasible sentences (assumptions) and inference rules. Argumentative reasoning is then performed in a two-step process: first an argument graph comprising arguments and their relations is generated from the ABAF, by means of the so-called *instantiation* procedure; then, argumentation *semantics* are applied to the obtained graph in order to find acceptable sets of arguments.

Although ABA is a well-established formalism to perform non-monotonic reasoning, with applications in medical decision-making, explainable AI and, more recently, causal discovery [3, 4, 5], the high computational complexity of core reasoning tasks in ABA poses a significant challenge for its deployment in practice. This issue is further exacerbated when ABA frameworks are instantiated into abstract argumentation formalisms [6], such as Dung's Argumentation Frameworks (AFs) [7] and Argumentation Frameworks with Collective Attacks (SETAFs) [8].

In the context of non-monotonic reasoning, one prominent strategy to address computational intractability is to optimise reasoning over a given knowledge base through divide-and-conquer algorithms. A paradigmatic example of this approach is *splitting*, originally developed for answer-set programming [9] and later adapted to other nonmonotonic formalisms, e.g. default theories [10] and recently Abstract Argumentation [11, 12, 13, 14]. This approach focuses on incrementally computing the extensions of a given abstract argumentation framework by means of the extension of its sub-frameworks, thereby avoiding to consider the entire solution-space of the original framework. Nonetheless, when applied to argument graphs derived from ABAFs, the exponential blow-up caused by the instantiation process

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can invalidate the usefulness of splitting. This motivates the need for splitting techniques that operate directly on ABAFs. To this end, this paper makes the following contributions:

- We begin by reviewing existing notions of splitting for AFs (Section 2) and SETAFs (Section 3).
- We then introduce a novel notion of ABA splitting in Section 4, along with the syntactic adjustments required to establish a splitting theorem, which we prove under standard argumentation semantics.
- In Section 5, we extend our results to the more general framework of parameterised splitting [12], showing that under the stable semantics, a splitting theorem holds for both ABAFs and SETAFs.
- Finally, Section 6 concludes with a summary and outlines directions for future research.

2. Preliminaries

Assumption-Based Argumentation We recall here the basic concepts of assumption-based argumentation (ABA) [15]. Debates are represented by means of so-called ABA Frameworks (ABAFs), which consist of a deductive system $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L}$ is a set of sentences, and $\mathcal{R}$ is a set of rules over $\mathcal{L}$. A rule $r \in \mathcal{R}$ has the form $a_0 \leftarrow a_1, \ldots, a_n$ with $a_i \in \mathcal{L}$, $body(r) = \{a_1, \ldots, a_n\}$ and $head(r) = a_0$.

Definition 1. *An ABAF is a tuple $(\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$, where $(\mathcal{L}, \mathcal{R})$ is a deductive system, $\mathcal{A} \subseteq \mathcal{L}$ a set of assumptions, and $^- : \mathcal{A} \to \mathcal{L}$ is a total mapping, called contrary function.*

For a set of assumptions, $S \subseteq \mathcal{A}$ we use $\overline{S}$ to indicate the set of contraries of S. Conversely, we define the partial function $\alpha : \mathcal{L} \mapsto \mathcal{A}$ assigning an assumption to its contrary $b \in \overline{\mathcal{A}}$ such that $\alpha(b) = a$ if $b = \overline{a}$. This generalises to sets of contraries as before. For a set of rules R, we fix $head(R) = \{head(r) \mid r \in R\}$, $body(R) = \{body(r) \mid r \in R\}$. Further we use $atom(S) = \{p \in \mathcal{L} \mid p \in S,\ \alpha(p) \in S \text{ or } \overline{p} \in S\}$. In what follows, we read $atom(p)$ as $atom(\{p\})$. For a rule $r \in \mathcal{R}$, we say that r is: a fact if $body(r) = \emptyset$; a loop-rule is $\overline{a} = head(r)$ and $a \in body(r)$. A sentence $q \in \mathcal{L}$ is tree-derivable from $S \subseteq \mathcal{A}$ and rules $R \subseteq \mathcal{R}$, denoted by $S \vdash^R q$, if there is a finite rooted labelled tree T where: the root of T is labelled with q; the set of labels for the leaves of T is equal to S or $S \cup \{\top\}$; and for every inner node v of T there is a rule $r \in R$ such that v is labelled with $head(r)$, and every successor of v is labelled with $a \in body(r)$ or $\top$ if $body(r) = \emptyset$. We sometimes write $S \vdash q$ instead of $S \vdash^R q$ if it does not cause confusion. Moreover, we call $Th_D(S) = \{p \in \mathcal{L} \mid S \vdash p\}$ the theory of S w.r.t. the ABAF D. Throughout the paper, we assume that ABAFs do not contain *dummy rules*, whose body is not derivable from any set of assumptions.

Definition 2. *Let $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ be an ABAF. A set $S \subseteq \mathcal{A}$ attacks $T \subseteq \mathcal{A}$ if $S' \vdash a$ for some $S' \subseteq S$ and $a \in T$. A set S is conflict-free in an ABAF D ($S \in cf(D)$) if it does not attack itself; S defends T iff it attacks each attacker of T ; S is admissible ($S \in adm(D)$) if it is conflict-free and defends itself.*

We say a set S of assumptions attacks an assumption a if S attacks the singleton $\{a\}$. In this paper, we assume ABAFs to be flat, unless specified otherwise. We call an ABAF flat if every set S of assumptions is closed (i.e. $S \vdash a$ implies $a \in S$) and *non-flat* otherwise. We next recall definitions for grounded, complete, preferred, and stable ABA semantics (abbr. *grd*, *com*, *pref*, *stb*).

Definition 3. *Let D be an ABAF and let $S \in adm(D)$. $S \in com(D)$ iff S contains every assumption set it defends; $S \in grd(D)$ iff S is $\subseteq$-minimal in $com(D)$; $S \in pref(D)$ iff S is $\subseteq$-maximal in $com(D)$; $S \in stb(D)$ iff S attacks each $\{x\} \subseteq \mathcal{A} \setminus S$. We call $\sigma(D)$ the set of σ-extensions of the ABAF D.*

SETAF Instantiation König et al. [16] have shown that flat ABAFs naturally correspond to argumentation frameworks with collective attacks (SETAFs) [8].

Definition 4. *A SETAF is a pair $SF = (A, R)$ where A is a finite set of* arguments*, and $R \subseteq 2^A \times A$ is the* attack relation*. For an attack $(T, h) \in R$ we call T the* tail *and h the* head *of the attack. We write (t, h) to denote the set-attack $(\{t\}, h)$. For $S \subseteq A$, we say S* attacks *an argument $a \in A$ if there is an attack $(T, a) \in R$ with $T \subseteq S$. Moreover, for a set $B \subseteq A$ we say that S attacks B if S attacks some $b \in B$. We use $S^+_R = \{a \mid S \text{ attacks } a\}$ and define the* range *of S w.r.t. R as $S^\oplus_R = S \cup S^+_R$.*

Every ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ can be instantiated as the SETAF $SF_D = (A_D, R_D)$ by setting $A_D = \mathcal{A}$ and $(S, a) \in R_D$ iff $S \vdash \overline{a}$ [16].

Example 1. *Consider an ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ with assumptions $\mathcal{A} = \{a, b, c, d\}$, $\mathcal{L} = \mathcal{A} \cup \overline{\mathcal{A}} \cup \{p\}$ and rules $\mathcal{R} = \{\overline{a} \leftarrow b, p;\ \ p \leftarrow c;\ \ \overline{b} \leftarrow a;\ \ \overline{d} \leftarrow b\}$. The induced SETAF is $SF_D = (\{a, b, c, d\}, \{(\{b, c\}, a), (a, b), (b, d)\})$.*

Notice that such mapping is many-to-one. Indeed, we lose p when instantiating the first two rules into $(\{b, c\}, a)$. For this reason, SETAFs can be seen – syntactically – as a fragment of flat ABAFs.

Splitting We now recall Baumann's splitting approach for AFs [11]. A splitting identifies two sub-frameworks F_1 and F_2 separated by a set of attacks going from F_1 to F_2. Then, the information contained in an extension of F_1 is propagated, computing the so-called *reduct* of F_2 accordingly.

Definition 5. *Let $F = (A, R)$ be an AF, $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ two sub-frameworks of F s.t. $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$ and $R = R_1 \cup R_2 \cup R_3$ with $R_3 \subseteq A_1 \times A_2$. The triple (F_1, F_2, R_3) is called a* splitting *of F. For such a splitting and a set $E \subseteq A_1$, the (E, R_3)-*reduct *is the AF $AF' = (A', R')$ with $A' = A_2 \setminus E^{+}_{R_3}$ and $R' = R_2 \cap (A' \times A')$. Moreover, the set of* undecided arguments *w.r.t. $E \subseteq A_1$ is $U_E = A_1 \setminus E^{\oplus}_{R_1}$.*

The reduct is designed to take care of arguments attacked by the extension E. Further, to account for the propagation of undecided arguments w.r.t. E, a further *modification* is needed: self-attacks are propagated from F_1 to arguments in F_2.

Definition 6. *Let (F_1, F_2, R_3) be a splitting for an AF F and E an extension of F_1. Moreover, take $F'_2 = (A'_2, R'_2)$ as the (E, R_3)-reduct of F_2 and U_E as the set of undecided arguments w.r.t. E. The (U_E, R_3)-modification of F_2 is defined as $mod_{U_E,R_3}(F'_2) = (A'_2, R'_2 \cup \{(b, b) \mid \exists a \in U_E : (a, b) \in R_3\})$.*

Using these definitions, Baumann [11] has shown that it is possible to split the AF and compute the extensions for each sub-framework incrementally such that their combination yields extensions of the original framework.

Theorem 1 **([11]).** *Let (F_1, F_2, R_3) be a splitting for an AF $F = (A, R)$ with $F_i = (A_i, R_i)$ and $\sigma \in \{cf, adm, stb, com, pref, grd\}$.*

1. *If $E_1 \in \sigma(F_1)$ and $E_2 \in \sigma(mod_{U_E,R_3}(F'_2))$, then $E_1 \cup E_2 \in \sigma(F)$.*
2. *If $E \in \sigma(F)$, then $E \cap A_1 \in \sigma(F_1)$ and $E \cap A_2 \in \sigma(mod_{U_E,R_3}(F'_2))$.*

Later, this idea has been generalised by relaxing the strict separation requirement, which significantly narrows the applicability of splitting, introducing so-called *parametrised splitting* [12]. Instead of demanding that the first part is completely unaffected by the second, it allows some forms of interaction. This generalisation is captured by the notion of *quasi-splitting*, where arguments in F_1 may be externally attacked by arguments in F_2. The goal is to preserve correctness while broadening the applicability of splitting. This is achieved by enriching F_1 with meta-information that encodes facts about potential influences (e.g. attacks) from the second sub-framework. In particular, for each externally attacked argument a, a fresh argument a' is added to F_1 along with a symmetric attack on a, enforcing a choice between a and a' in F_1. Then, F_2 is modified accordingly: the previous choices are propagated in the second sub-framework via the reduct as well as additional nodes and attacks. Stable extensions of the entire AF are then recovered by composing compatible solutions from the two modified sub-frameworks.

3. Splitting Argumentation Frameworks with Collective Attacks

In this section, we recall fundamentals regarding splitting in the presence of collective attacks [14]. The notion of splitting for SETAFs generalises the one for Dung-style AFs.

Definition 7. *Let $SF = (A, R)$ be a SETAF, $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$ two sub-frameworks of SF such that $A_1 \cap A_2 = \emptyset$, $A = A_1 \cup A_2$ and $R = R_1 \cup R_2 \cup R_3$ with $R_3 \subseteq \big((2^{A_1} \setminus \{\emptyset\}) \cup 2^{A_2}\big) \times A_2$. We call a* splitting *of SF the triple (SF_1, SF_2, R_3). Moreover, we call R_3 the set of* links *wrt (SF_1, SF_2, R_3) and say that a link is* undecided *if no argument in its tail is defeated, but at least one is undecided.*

As for AFs, the general idea is to compute extensions of SF as a combination of extensions of SF_1 and SF_2. Due to the links from SF_1 to SF_2 we have to modify SF_2 according to the extension(s) of SF_1 to account for the prior accepted and rejected arguments. Following Baumann [11], we introduce the notions of *reduct* and *modification*, in application to the second part (that is, SF_2) of the original SETAFs. Intuitively, the reduct takes care of the arguments in SF_2 that are already defeated by E_1 by removing them, and modifies the links by leaving the remaining part of the attack in the reduct.

Definition 8 (Reduct)**.** *Let (SF_1, SF_2, R_3) be a splitting for a SETAF SF. We define the (E_1, R_3)-*reduct *(or simply reduct) of SF_2 for some extension E_1 of SF_1 as the SETAF $SF_2' = (A_2', R_2')$ where, $A_2' = \{a \in A_2 \mid a \notin (E_1)^+_{R_3}\}$ and*

$$R_2' = \{(T, h) \in R_2 \mid T \subseteq A_2', h \in A_2'\} \cup \\ \{(T \cap A_2', h) \mid (T, h) \in R_3, T \cap A_2' \neq \emptyset, h \in A_2', T \cap A_1 \subseteq E_1, T \cap (E_1)^+_{R_3} = \emptyset\}.$$

When dealing with undecidedness, what guides our intuition towards a certain modification is not the status of the arguments in SF_1, but rather the status of the *links*. Hence, we decide to slightly tweak the original definition and base our notion solely on the undecided *links*.

Definition 9 (Undecided Links)**.** *Given a splitting (SF_1, SF_2, R_3) for a SETAF SF and an extension $E_1 \in SF_1$ we define the* set of undecided links *w.r.t. E_1 as:*

$$U^{E_1}_{R_3} = \{(T, h) \in R_3 \mid T \cap (E_1)^+_{R_1 \cup R_3} = \emptyset \text{ and } \exists t \in T : t \in A_1 \setminus (E_1)^{\oplus}_{R_1}\}.$$

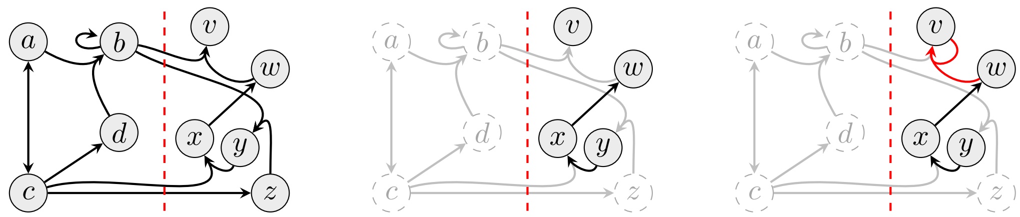
In what follows, we define the *modification*, which is applied on the reduct, and accounts for the effects of the undecided links. In particular, we add to SF_2 one self-attacking argument which also partially attacks the target for each *undecided* attack in R_3.

Definition 10 (Modification)**.** *Let (SF_1, SF_2, R_3) be a splitting for a SETAF SF and E_1 an extension of SF_1. Take SF_2' as the (E_1, R_3)-reduct of SF_2 and $U^{E_1}_{R_3}$ as the set of undecided links w.r.t. E_1. We denote with $mod^{E_1}_{R_3}(SF_2')$ the $U^{E_1}_{R_3}$-modification (or simply modification) of SF_2' s.t.:*

$$mod^{E_1}_{R_3}(SF_2') = (A_2', R_2' \cup \{((T \cap A_2') \cup \{h\}, h) \mid (T, h) \in U^{E_1}_{R_3},\ h \in A_2'\}).$$

Before we present the splitting theorem we illustrate Definitions 8–10 in the following example.

Example 2. *In (a) we have a SETAF SF with a splitting that separates the arguments $A_1 = \{a, b, c, d\}$ from $A_2 = \{v, w, x, y, z\}$. We see that $E_1 = \{c\}$ is admissible in the left part of the splitting. In (b) we see the reduct w.r.t. the set $\{c\}$, where a and d are defeated by c (as $\{c\}^+_{R_1} = \{a, d\}$) and b is undecided. This reduct contains from the right part all arguments except z, which is defeated by c (as $\{c\}^+_{R_3} = \{z\}$). We see that most attacks are removed from the right part, but (x, w) persists (since it is in R_2 and all involved arguments remain), and the attack $(\{c, y\}, x)$ is changed to (y, x). The attack $(\{b, z\}, y)$ is removed since z is defeated. The attack $(\{b, w\}, v)$ is also removed, as b is undecided (i.e., $\{b, w\} \cap A_1 \not\subseteq E_1$). However, in (c) we see that the latter case is important for the modification: the attack $(\{b, w\}, v)$ is an* undecided link*, which means in the modification we introduce the attack $(\{v, w\}, v)$. Now, since $\{y, w\}$ is admissible, we obtain $\{c, y, w\}$ as an admissible set for SF.*



(a) SETAF SF *(b) $(\{c\}, R_3)$-reduct* *(c) $U^{\{c\}}_{R_3}$-modification*

Having these notions at hand, we now establish the adequacy of the splitting technique for SETAFs. We start by establishing that (a) conflict-freeness of the sub-frameworks SF_1 and SF_2 carries over to the whole SETAF SF, and (b) conflict-free sets of SF induce conflict-free subsets in SF_1 and SF_2'.

Proposition 1 (Buraglio et al. [14]). *Let (SF_1, SF_2, R_3) be a splitting for a SETAF $SF = (A, R)$ with $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$. Let $SF_2^\star = mod_{R_3}^{E_1}(SF_2')$.*

1. *If $E_1 \in cf(SF_1)$ and $E_2 \in cf(SF_2^\star)$, then $E_1 \cup E_2 \in cf(SF)$.*
2. *If $E \in cf(SF)$, then $E \cap A_1 \in cf(SF_1)$ and $E \cap A_2 \in cf(SF_2')$.*

Finally, we are ready to characterize the splitting algorithm by generalising the splitting theorem for SETAFs under the standard Dung semantics.

Theorem 2 (Buraglio et al. [14]). *Let (SF_1, SF_2, R_3) be a splitting for a SETAF $SF = (A, R)$ with $SF_1 = (A_1, R_1)$, $SF_2 = (A_2, R_2)$, and $\sigma \in \{stb, adm, com, pref, grd\}$.*

1. *If $E_1 \in \sigma(SF_1)$ and $E_2 \in \sigma\big(mod_{R_3}^{E_1}(SF_2')\big)$, then $E_1 \cup E_2 \in \sigma(SF)$.*
2. *If $E \in \sigma(SF)$, then $E \cap A_1 \in \sigma(SF_1)$ and $E \cap A_2 \in \sigma\big(mod_{R_3}^{E\cap A_1}(SF_2')\big)$.*

While the existing instantiation procedure from ABA frameworks to SETAFs provides a foundation for defining splitting, attempting to directly replicate the SETAF-style idea of splitting among assumptions fails to yield a natural notion of splitting. This disconnect stems from a fundamental structural difference: in SETAFs, attacks are primitive, whereas in ABA, they are derived from the underlying deductive system $(\mathcal{L}, \mathcal{R})$. As a result, naively mimicking SETAF-style splitting in ABA would require (i) arbitrarily partitioning the assumption set into $\mathcal{A}_1$ and $\mathcal{A}_2$, and (ii) computing attacks as derivations from assumptions in $\mathcal{A}_1$ to those in $\mathcal{A}_2$. However, splitting should be possible solely by inspecting the knowledge base at hand. Moreover, while instantiating ABAFs into SETAFs has been shown useful in specific contexts [5, 17], this approach comes with a critical drawback: it can yield an exponential growth in the number of collective attacks generated, thus increasing in input size. This inefficiency motivates many ABA solvers to operate directly on ABAFs rather than relying on their abstract representations. Therefore, to enable an efficient form of splitting, we propose a dedicated splitting algorithm tailored to the syntactic structure of ABAFs.

4. Splitting in Assumption-Based Argumentation

In this section we present splitting results for ABAFs. The rule-set of an ABAF is split into a bottom and a top part whenever no assumption occurs in the bottom part whose contrary is derived by some rule in the top. This ensures that the assumptions in the bottom can be evaluated independently of what can be deduced by inspecting the top part. We capture this intuition via the notion of splitting set:

Definition 11. *Given an ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$, a set $S \subseteq \mathcal{L}$ is a splitting set (or simply a splitting) of D if $S = atom(S)$ and for all $r \in \mathcal{R}$, $head(r) \in S$ implies $body(r) \subseteq S$.*

A splitting set partitions the deductive system into two sub-systems $(\mathcal{L}_1,\mathcal{R}_1)$ and $(\mathcal{L}_2,\mathcal{R}_2)$, called the 'bottom' and 'top'. In particular, we have (i) $\mathcal{L}_1 = S$ and $\mathcal{R}_1 = \{r \in \mathcal{R} \mid head(r) \in S\}$ and (ii) $\mathcal{L}_2 = \mathcal{L} \setminus S$ and $\mathcal{R}_2 = \{r \in \mathcal{R} \mid head(r) \notin S\}$. These induce respectively two sub-frameworks $D_1 = (\mathcal{L}_1, \mathcal{R}_1, \mathcal{A}_1, ^{-1})$ and $D_2 = (\mathcal{L}_2, \mathcal{R}_2, \mathcal{A}_2, ^{-2})$ with $\mathcal{A}_i = \mathcal{L}_i \cap \mathcal{A}$ and the contrary function $^{-i}$ defined over $\mathcal{A}_i$.

Example 3. *Consider the ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ corresponding to the SETAF of Example 2, where $\mathcal{A} = \{a, b, c, d, v, w, x, y, z\}$, $\mathcal{L} = \mathcal{A} \cup \overline{\mathcal{A}} \cup \{p, q\}$, and the rule-set $\mathcal{R}$ consists of the following:*

$$\overline{w} \leftarrow q \qquad q \leftarrow x \qquad \overline{x} \leftarrow c, y \qquad \overline{y} \leftarrow z, p \qquad \overline{z} \leftarrow \overline{d} \qquad \overline{v} \leftarrow p, w$$
$$p \leftarrow b \qquad \overline{b} \leftarrow b \qquad \overline{b} \leftarrow a, d \qquad \overline{d} \leftarrow c \qquad \overline{a} \leftarrow c \qquad \overline{c} \leftarrow a$$

Take the set $S = \{a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d}, p\}$. It can be easily checked that S is a splitting set of D, through which we obtain two sub-systems $(\mathcal{L}_1, \mathcal{R}_1)$ and $(\mathcal{L}_2,\mathcal{R}_2)$ with $\mathcal{R}_1$ (bottom) and $\mathcal{R}_2$ (top) are exactly the second line and the first line of rules in $\mathcal{R}$. Moreover, $\mathcal{L}_1 = S$ and $\mathcal{L}_2 = \mathcal{L} \setminus S$.

Notice that some atoms contained in $\mathcal{L}_1$ (but not in $\mathcal{L}_2$) may occur in the body of some rule in $\mathcal{R}_2$ (c and p in Example 3). This intermediate mismatch will be resolved later by the notion of *reduct*. Moreover, their occurrence in the top rules does not affect the acceptance status of such atoms. In fact, a first sanity check, we observe that our notion of splitting prevents building attacks from assumptions of D_2 towards assumptions of D_1 using top-rules in $\mathcal{R}_2$. This is ensured by the fact that contraries of assumptions occurring in the bottom part are not derived via rules in the top part (via construction of $\mathcal{R}_2$). As a result, assumptions in $\mathcal{A}_1$ are attacked only via rules in $\mathcal{R}_1$ by assumptions in $\mathcal{A}_1$. Thus, no attack generated from $\mathcal{A}_2$ (by means of rules in $\mathcal{R}_2$) is directed towards $\mathcal{A}_1$.

Proposition 2. *Let D be an ABAF and S a set of literals that splits D into D_1 and D_2. For every derivation $T \vdash^R \overline{a}$ with $a \in \mathcal{A}_1$, it holds that $R \subseteq \mathcal{R}_1$ and $T \subseteq \mathcal{A}_1$.*

The attacks of the bottom part can be extended in a conservative way: whatever happens in the second sub-framework does not affect the acceptability status of assumptions in D_1. Thus, to compute incrementally an extension of an ABAF D, we can first select an extension E of D_1 and later modify D_2 according to the information contained in E. Consequently, we can evaluate the modified framework D_2 and augment its extensions with E. Again, we follow the approach of Baumann and appeal to the notions of *reduct* and *modification* to realise the modification of D_2 in a two-step process. First, we propagate all the information we get from a σ-extension E of D_1 to ensure that rules which are in contrast with E are removed. The outcome is called the E-reduct of D_2.

Definition 12. *Let $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ be an ABAF, S a set that splits D into two sub-frameworks D_1 and D_2 and E a σ-extension of D_1. We call $D_2^E = (\mathcal{L}_2, \mathcal{R}_2^E, \mathcal{A}_2, \overline{}^2)$ the E-reduct (or simply reduct) of D_2, where $\mathcal{R}_2^E$ is obtained by deleting:*

- *each rule $r \in \mathcal{R}_2$ with $body(r) \cap S \not\subseteq Th_{D_1}(E)$;*
- *all literals in $Th_{D_1}(E)$ from the remaining rules.*

As we anticipated, all and only the atoms occurring in the rule-set of the reduct are contained in $\mathcal{L}_2$. Therefore, the reduct can be evaluated in complete isolation from D_1. In the second step, we modify the reduct to propagate the information about assumptions (or their contraries) which are not contained in $Th_{D_1}(E)$. We call these assumptions undecided, as they are not in E nor their contrary is derivable from it (i.e. are not attacked by E). Then, the set of undecided assumptions of D_1 w.r.t. E is $\mathsf{UA}_{D_1}(E) = \{a \in \mathcal{A}_1 \mid a \notin E \text{ and } \overline{a} \notin Th_{D_1}(E)\}$. Since their status can be transmitted to other assumptions via rules, we need to introduce the concept of *undecided theory* of D_1, capturing all statements derivable from a set of undecided (and not defeated) assumptions.

Definition 13. *Let $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ be an ABAF and $E \in \sigma(D)$. The* undecided theory *of D w.r.t. E is*

$$\mathsf{UT}_D(E) = \{p \in \mathcal{L} \mid \exists T \subseteq \mathcal{A} \text{ s.t. } T \vdash p,\ \ T \cap \mathsf{UA}_D(E) \neq \emptyset,\ \overline{T} \cap Th_D(E) = \emptyset\}.$$

Rules in D_2 whose body contain elements of $\mathsf{UT}_{D_1}(E)$ might carry over undecidedness from D_1. However, this scenario could be overwritten by the presence of *incompatible sentences* w.r.t E, captured by $\mathsf{IS}_{D_1}(E) = Th_{D_1}(E^+_{\mathcal{R}_1}) \cup \overline{E}$, where $E^+_{\mathcal{R}_1} = \{a \in \mathcal{A}_1 \mid E \vdash^R \overline{a}, R \subseteq \mathcal{R}_1\}$. Hence, a set of sentences from D_1 will carry undecidedness to sentences in D_2 if and only if (i) none of its elements is incompatible and (ii) at least one of its elements is in the undecided theory w.r.t. the previously selected extension. This concept mirrors the notion of undecided links for SETAFs.

We are now in the position to formally define the *modification* of D_2^E. First, we expand the set of sentences with a fresh assumption x_u and corresponding contrary. Further, we introduce (i) a loop-rule for x_u and (ii) a modified version of every rule with some undecided (but no incompatible) sentence in the body. In particular, we expand their body with x_u, after projecting to $\mathcal{L}_2$.

Definition 14. *Let D be an ABAF, S a set that splits D into two sub-frameworks D_1 and D_2 and E an extension of D_1. Let D_2' be the E-reduct of D_2. We use $mod^E_{D_1}(D_2') = D_2^\star = (\mathcal{L}_2^\star, \mathcal{R}_2^\star, \mathcal{A}_2^\star, \overline{}^\star)$ to denote the E-modification (or simply modification) of D_2' such that $D_2^\star = D_2'$ if $\mathsf{UA}_{D_1}(E) = \emptyset$, and otherwise:*

$$\mathcal{L}_2^\star = \mathcal{L}_2 \cup \{x_u, \overline{x_u}\};$$

$$\mathcal{R}_2^\star = \mathcal{R}_2' \cup \{\overline{x_u} \leftarrow x_u\} \cup \{head(r) \leftarrow (body(r) \cap \mathcal{L}_2) \cup \{x_u\} \mid$$
$$r \in \mathcal{R}_2,\ body(r) \cap \mathsf{IS}_{D_1}(E) = \emptyset,\ body(r) \cap \mathsf{UT}_{D_1}(E) \neq \emptyset\}.$$

Example 4. *Consider again the ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ from Example 3 and splitting set $S = \{a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d}, p\}$. We know that $\{c\} \in pref(D_1)$. Therefore, the $\{c\}$-reduct of D_2 is $D_2^{\{c\}} = (\mathcal{L}_2, \mathcal{R}_2^{\{c\}}, \mathcal{A}_2, \overline{}^2)$, where the rule-set $\mathcal{R}_2^{\{c\}}$ is:*

$$\overline{w} \leftarrow q \qquad q \leftarrow x \qquad \overline{x} \leftarrow y, c \qquad \overline{y} \leftarrow z, p \qquad \overline{z} \leftarrow \overline{d} \qquad \overline{v} \leftarrow p, w$$

Moreover, the set of undecided assumptions is $\mathsf{UA}_{D_1}(\{c\}) = \{b\}$ and $\mathsf{UT}_{D_1}(\{c\}) = \{b, \overline{b}, p\}$. We then compute the modification by expanding the set of sentences with $\{x_u, \overline{x_u}\}$ and $\mathcal{R}_2^{\{c\}}$ such that:

$$\overline{w} \leftarrow q \qquad q \leftarrow x \qquad \overline{x} \leftarrow y \qquad \overline{y} \leftarrow z, x_u \qquad \overline{z} \leftarrow \qquad \overline{v} \leftarrow w, x_u \qquad \overline{x_u} \leftarrow x_u$$

It is easy to see that $\{y, w\} \in pref(mod_{D_1}^E(D_2'))$, and retrieve $\{c, y, w\}$, as for Example 2.

We can now prove that our procedure preserves conflict-free sets under incremental computation as well as projection to sub-frameworks, similarly to Section 3.

Proposition 3. *Let S be a splitting set for an ABAF D into D_1 and D_2.*

1. *if $E_1 \in cf(D_1)$ and $E_2 \in cf(mod_{D_1}^{E_1}(D_2^{E_1}))$, then $E_1 \cup E_2 \in cf(D)$.*
2. *if $E \in cf(D)$, then $E_1 = E \cap \mathcal{A}_1 \in cf(D_1)$ and $E_2 = E \cap \mathcal{A}_2 \in cf(D_2^{E_1})$.*

Proof. For notational convenience, let $E = E_1 \cup E_2$ and let $D_2' = (\mathcal{L}_2, \mathcal{R}_2', \mathcal{A}_2, \overline{}^2)$ be the reduct of D_2 w.r.t. $E_1 = E \cap \mathcal{A}_1$. (1.) To prove the statement we need to show that there is no $a \in E_1 \cup E_2$ and and $R \in \mathcal{R}$ such that $E_1 \cup E_2 \vdash^R \overline{a}$. Towards contradiction, assume there is indeed such an a. Thus either (i) $a \in E_1$ or (ii) $a \in E_2$. Assume (i) is true, that is $\exists a \in E_1$ such that $E_1 \cup E_2 \vdash^R \overline{a}$ and $R \in \mathcal{R}$. From Proposition 2, we know that $E_2 = \emptyset$ and $R \subseteq \mathcal{R}_1$. Thus, $E_1 \vdash^R \overline{a}$, in contradiction with $E_1 \in cf(D_1)$. Assume now that (ii) is true, i.e. $\exists a \in E_2$ and $R \in \mathcal{R}$ such that $E_1 \cup E_2 \vdash^R \overline{a}$. Hence, there is a tree-derivation τ from $E_1 \cup E_2 \cup \{\top\}$ rooted in $\overline{a}$ and a non-empty set of rules $R_2 = R \cap \mathcal{R}_2$. For each rule $r \in R_2$, there are three possible outcomes when computing $mod_{D_1}^{E_1}(D_2^{E_1}) = D_2^\star$: (a) r does not get removed when computing the reduct; (b) r gets removed and later added in the modification; (c) r gets removed for good. Assume (a) is the case. If a rule r is not removed when computing the reduct, it is modified into a rule $r' \in \mathcal{R}_2'$ such that $body(r') = body(r) \setminus Th_{D_1}(E_1)$ and $head(r') = head(r)$. Thus, $body(r')$ consists of elements of E_2 or atoms derivable from it. Therefore, $E_2 \vdash^{\mathcal{R}_2^{E_1}} \overline{a}$ and consequently $E_2 \vdash^{\mathcal{R}_2^\star} \overline{a}$ (more rules). Finally, we get $E \notin cf(D_2^\star)$, contradicting our hypothesis. Assume now (b) is the case. By definition of derivation, this means that E_2 derives $\overline{a}$ in $D_2^\star$ only if $x_u \in E_2$. However, this contradicts conflict-freeness of E_2 in the modification. Finally, consider case (c). Since r gets removed, but not added in the modification, we infer that $body(r) \cap \mathsf{IS}_{D_1}(E_1) \neq \emptyset$. Hence, either $\overline{E_1} \cap body(r) \neq \emptyset$ or $Th_{D_1}((E_1)^+_{\mathcal{R}_1}) \cap body(r) \neq \emptyset$. However, since $E_1 \in cf(D_1)$, this means that either r is a dummy rule or that $\exists b \in body(r) \cap \mathcal{A}_1 \not\subseteq E_1$. Thus, in both cases $E_1 \cup E_2 \not\vdash^R \overline{a}$, contradicting our assumption.

(2.) Suppose now that $E \in cf(D)$. From this we derive that $E \cap \mathcal{A}_1 \in cf(D_1)$ (subset of a conflict-free set). We now show that $E \cap \mathcal{A}_2 \in cf(D_2')$. Towards contradiction, assume $E \cap \mathcal{A}_2 \notin cf(D_2')$. There is an $a \in E \cap \mathcal{A}_2$ such that $E \cap \mathcal{A}_2 \vdash^{\mathcal{R}_2'} \overline{a}$. By definition of reduct, we know that each $r' \in \mathcal{R}_2'$ is obtained from a corresponding rule $r \in \mathcal{R}_2$ such that $body(r) \subseteq body(r') \cup Th_{D_1}(E \cap \mathcal{A}_1)$. Therefore, $(E \cap \mathcal{A}_1) \cup (E \cap \mathcal{A}_2) \vdash^{\mathcal{R}_1 \cup \mathcal{R}_2} \overline{a}$. By definition of splitting, we know that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and $E = (E \cap \mathcal{A}_1) \cup (E \cap \mathcal{A}_2)$, deriving $E \vdash^{\mathcal{R}} \overline{a}$, and finally $E \notin cf(D)$. Contradiction. □

We prove our algorithm is adequate with respect to most common semantics. Due to space constraints we present proof details only for stable and admissible semantics, which are prototypical for the others.

Theorem 3. *Let S be a splitting set for an ABAF D into D_1 and D_2 and $\sigma = \{stb, adm, com, pref, grd\}$.*

1. *if* $E_1 \in \sigma(D_1)$ *and* $E_2 \in \sigma(mod_{D_1}^{E_1}(D_2^{E_1}))$*, then* $E_1 \cup E_2 \in \sigma(D)$.
2. *if* $E \in \sigma(D)$*, then* $E_1 = E \cap \mathcal{A}_1 \in \sigma(D_1)$ *and* $E_2 = E \cap \mathcal{A}_2 \in \sigma(mod_{D_1}^{E_1}(D_2^{E_1}))$.

Proof. **(stable).** First notice that from $E_1 \in \mathit{stb}(D_1)$, we get $U_{D_1}(E_1) = \emptyset$, and consequently $\mathcal{A}'_2 = \mathcal{A}^\star_2$.

(1.) From Proposition 3 together with the hypotheses that $E_1 \in \mathit{stb}(D_1)$ and $E_2 \in \mathit{stb}(D^\star_2)$, we know that $E_1 \cup E_2 \in \mathit{cf}(D)$. Thus, for any $a \in \mathcal{A} \setminus E$, we show that $a \in E^+_R$, i.e. $E \vdash^R \overline{a}$ for some $R \subseteq \mathcal{R}$. We proceed by cases. Let $a \in \mathcal{A}_1$. From hypothesis we know that $E_1 \vdash^{R_1} \overline{a}$ for some $R_1 \subseteq \mathcal{R}_1$ which immediately implies $a \in E^+_{R_1}$. Let $a \in \mathcal{A}_2$. From hypothesis, we know that $E_2 \vdash^{R_2} \overline{a}$ for some $R_2 \subseteq \mathcal{R}'_2$. Thus, for each rule $r' \in \mathcal{R}'_2$ there is a rule $r \in \mathcal{R}_2$ such that $body(r) \subseteq body(r) \cup Th_{D_1}(E_1)$. Hence, it follows directly that $E_1 \cup E_2 = E \vdash^R \overline{a}$ for some $R \subseteq \mathcal{R}_1 \cup \mathcal{R}_2 = \mathcal{R}$.

(2.) Assume $E \in \mathit{stb}(D)$. From this we know that $E \cup E^+_{\mathcal{R}} = E^\oplus_{\mathcal{R}} = \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. We first prove that $E_1 = E \cap \mathcal{A}_1 \in \mathit{stb}(D_1)$. From Proposition 3 we know $E \cap \mathcal{A}_1 \in \mathit{cf}(D_1)$. Moreover, from Proposition 2, we know that any set of assumptions which is not entirely contained in $\mathcal{A}_1$ attacks $a \in \mathcal{A}_1$ via rules in $\mathcal{R}_1$, therefore we get $E \cap \mathcal{A}_1 \vdash^{R_1} \overline{a}$ for all $a \in \mathcal{A}_1 \setminus E$ for some $R_1 \subseteq \mathcal{R}_1$. Hence, $E_1 \in \mathit{stb}(D_1)$. We know turn to prove $E_2 = E \cap \mathcal{A}_2 \in \mathit{stb}(D'_2)$. We know conflict-freeness holds from Proposition 3. Hence, we only need to show that for every $a \in \mathcal{A}'_2 \setminus E_2$, $E_2 \vdash^{R'_2} \overline{a}$ for some $R'_2 \subseteq \mathcal{R}'_2$. Since $E \vdash^R \overline{a}$ in D, we have two possibilities: (a) $E_1 = \emptyset$ or (b) $E_1 \neq \emptyset$. If (a) holds, we get $R \subseteq \mathcal{R}_2$ and $E = E_2 \vdash^R \overline{a}$ where $E_2 \subseteq \mathcal{A}'_2$ and $a \in \mathcal{A}'_2$. Thus, $E_2 \vdash^R \overline{a}$ holds for some $R \subseteq \mathcal{R}'_2$. If (b) holds, $E_1 \cup E_2 \vdash^R \overline{a}$ in D. Therefore, each rule $r \in R \cap \mathcal{R}_2$ has a corresponding rule $r' \in \mathcal{R}'_2$ such that $body(r') = body(r) \setminus Th_{D_1}(E_1)$. Since $E_1 \cup E_2 \in \mathit{cf}(D)$ by hypothesis, we know that $Th_{D_1}(E_1) \cap \overline{E_2} = \emptyset$. Hence, $(E \setminus E_1) \vdash^{R'_2} \overline{a}$ where $R'_2 \subseteq \mathcal{R}'_2$. In both cases we have $E_2 \cup (E_2)^+_{\mathcal{R}'_2} = \mathcal{A}'_2$, concluding $E_2 \in \mathit{stb}(D'_2)$.

(admissible). (1.) Since admissibility implies confict-freeness from Proposition 3, we know that $E = E_1 \cup E_2 \in \mathit{cf}(D)$. Thus we only need to show that E defends itself in D, i.e. for all $a \in E$, if $T \vdash \overline{a}$, then $T' \vdash \overline{t}$ for some $t \in T$ and $T' \subseteq E$. If $a \in E_1$, we know that a is defended by E_1 in $\mathcal{A}_1$ from hypothesis. Thus, from Proposition 2, we can deduce that $E_1 \in \mathit{adm}(D)$. Consider now an assumption $a \in E_2$ and some $T \subseteq \mathcal{A}$ such that $T \vdash^R \overline{a}$ and $R \subseteq \mathcal{R}$. If $\overline{T} \cap Th_{D_1}(E_1) \neq \emptyset$, then E_1 defends a against T in D. If $\overline{T} \cap Th_{D_1}(E_1) = \emptyset$, this means that $T \subseteq \mathcal{A}_2$ and $T \vdash \overline{a}$ in D'_2 (a is attacked in the reduct) or $T \cup x_u \vdash^R \overline{a}$ in $D^\star_2$ (a is attacked in the modification). In both cases, since E_2 is conflict-free and defends a in $D^\star_2$, there is a $T' \subseteq E_2$ such that $T' \vdash \overline{t}$ with $t \in T$. We distinguish two cases: either (i) $T' \vdash \overline{t}$ already in D_2, in which case a is defended by E in D, or (ii) there is some $T'' \supset T'$ such that $T'' \vdash \overline{t}$ in D and $T'' \cap \mathcal{A}_1 \subseteq E_1$. Thus, since $T \subseteq E_1 \cup E_2$, a is defended by $E_1 \cup E_2$ in D. In any case a is defended in D by E, i.e. $E \in \mathit{adm}(D)$.

(2.) By Proposition 3, we get $E_1 \in \mathit{cf}(D_1)$ and $E_2 \in \mathit{cf}(D'_2)$. First, we know that $E_1 \in \mathit{adm}(D_1)$ because E defends itself in D and E_1 is not attacked by a subset of $\mathcal{A}_2$ (Proposition 2). It remains to prove that $E_2 \in \mathit{adm}(D^\star_2)$. Take an assumption $a \in E_2$ such that $T \vdash \overline{a}$ in $D^\star_2$. Each such derivation corresponds to exactly one derivation $T' \vdash^R \overline{a}$ with $R \subseteq \mathcal{R}$. There are two cases: either (i) $T' = T \subseteq \mathcal{A}_2$ and $R \subseteq \mathcal{R}_2$ or (ii) $T' \supset T \setminus \{x_u\}$ where $T' \setminus T \subseteq E_1$ (assumptions deleted from simplified rules in the reduct). From both (i) and (ii) we deduce that $\overline{T'} \cap Th_{D_1}(E_1) = \emptyset$: for (i) because it would entail $T \not\subseteq \mathcal{A}_2$; for (ii) because otherwise $T \not\vdash \overline{a}$ in $D^\star_2$. Nonetheless, since E defends a in D, in case (i) there is a counter-attack $T'' \vdash^{\mathcal{R}_2} \overline{t}$ such that $T'' \subseteq E$ and $t \in (T \setminus \{a\})$. In case (ii), the same holds but $t \in (T' \setminus \{a, x_u\})$. If $T'' \cap \mathcal{A}_1 = \emptyset$, we know that $\{t\} \subseteq \mathcal{A}'_2$ and together with the fact that $T'' \subseteq E$, we derive $T'' \subseteq E \cap \mathcal{A}'_2 = E_2$. Hence, T'' defends a from T in $D^\star_2$. If $T'' \cap \mathcal{A}_1 \neq \emptyset$, then $T'' \cap \mathcal{A}_1 \subseteq E_1$. Therefore, from $T'' \subseteq E$ we get $T'' \cap \mathcal{A}'_2 \vdash^{\mathcal{R}'_2} \overline{t}$, which defends a against T in $D^\star_2$. Thus a is always defended in $D^\star_2$, as desired. □

5. Parametrised Splitting

We now introduce a more general version of splitting for ABAFs and SETAFs, called *parametrised splitting* [12]. This relaxes the structural constraint for the application of splitting, allowing assumptions (resp. arguments) in the bottom part to be attacked from assumptions (resp. arguments) in the top.

The number of these assumptions/arguments represents a measure of how far we are from obtaining a splitting.

5.1. Assumption-Based Argumentation

We first introduce a parametrised version of splitting for ABA. In contrast with the previous notion, we allow some contraries of assumptions occurring in bodies of $\mathcal{R}_1$ to appear as the heads of rules in $\mathcal{R}_2$. The concept of a splitting set is then generalised accordingly in the following way:

Definition 15. *For any ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$, a set $S \subseteq \mathcal{L}$ is a called quasi-splitting of D if $S = atom(S)$ and for all $r \in \mathcal{R}$, $head(r) \in S$ implies $body(r) \setminus \mathcal{A} \subseteq S$. Let $V_S^{\leftarrow} = \{b \in \mathcal{A} \setminus S \mid \exists r, r' \in \mathcal{R} : b \in body(r) \cap \mathcal{A}, head(r) \in S, head(r') = \overline{b}, r \neq r'\}$. We call S:*

- *k-splitting of D, if $|V_S^{\leftarrow}| = k$;*
- *(proper) splitting of D, if $|V_S^{\leftarrow}| = 0$.*

As before, the rule-set is split into a bottom and top part, depending on the rule-head respectively being or not in S. As a result, $V_S^{\leftarrow}$ is the set of assumptions in the bottom whose contrary is derived in the top. We call $V_S^{\leftarrow}$ the set of *vulnerabilities* with respect to S, since it contains assumptions that are attacked by S. Whenever $|V_S^{\leftarrow}| \neq 0$, there are some heads in $\mathcal{R}_2$ whose corresponding assumption may appear in bodies of $\mathcal{R}_1$. Therefore, the notion of splitting of Definition 11 corresponds to a 0-splitting.

To account for elements of $V_S^{\leftarrow}$, the ABAFs D_1 and D_2 induced by the chosen splitting set are constructed in a slightly different way than before. In particular, we fix D_1 and D_2 as before, but let $\mathcal{L}_1 = S \cup V_S^{\leftarrow} \cup \overline{V_S^{\leftarrow}}$. Moreover, since contraries in $\overline{V_S^{\leftarrow}}$ may be derived by top rules, the status of their corresponding assumptions in the bottom depends on rules in the top. Consequently D_1 cannot be evaluated in complete isolation from the rest, in contrast with proper splitting.

For computing extension of the sub-framework D_1, we first need to modify the ABAF. First, we modify the rules by removing body-atoms not in $\mathcal{L}_1$. Indeed, these atoms occur in $\mathcal{L}_2$ and are unattacked in D, therefore they can be disregarded when evaluating D_1. Further, we proceed by adding: (i) a fresh assumption b' (and its contrary $\overline{b'}$) for each $b \in V_S^{\leftarrow}$; (ii) rules which encode the choice for or against the presence of each assumption $b \in V_S^{\leftarrow}$ in the extension. In this way, we store at the object level the meta-information regarding our choices on each $b \in V_S^{\leftarrow}$.

Definition 16. *Let $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ be an ABAF, $S \subseteq \mathcal{L}$ be a quasi-splitting of D inducing the sub-frameworks D_1 and D_2. Moreover, let $V_S^{\leftarrow}$ be the set of vulnerabilities of D_1 with respect to S and $(\mathcal{R}_1)_{\downarrow \mathcal{L}_1} = \{head(r) \leftarrow body(r) \cap \mathcal{L}_1 \mid r \in \mathcal{R}_1\}$. We construct $\llcorner D_1 \lrcorner = (\llcorner \mathcal{L}_1 \lrcorner, \llcorner \mathcal{R}_1 \lrcorner, \llcorner \mathcal{A}_1 \lrcorner, ^-)$ as the ABAF obtained from D_1 by letting:*

- *$\llcorner \mathcal{L}_1 \lrcorner = \mathcal{L}_1 \cup \{b', \overline{b'} \mid b \in V_S^{\leftarrow}\}$;*
- *$\llcorner \mathcal{R}_1 \lrcorner = (\mathcal{R}_1)_{\downarrow \mathcal{L}_1} \cup \{\overline{b} \leftarrow b', \overline{b'} \leftarrow b \mid b \in V_S^{\leftarrow}\}$.*

Intuitively, the additional rules allow us to choose whether we want to accept an extension E of $\llcorner D_1 \lrcorner$ containing b or one that does not. After this choice, we can safely compute the E-reduct of D_2, as for proper splitting. In this way, we propagate the meta-information to which we committed by means of our choice. A further modification of D_2 is now needed to make sure that our hypothesis regarding b is ensured: we add a fact-rule $b \leftarrow$ or a loop-rule $\overline{b} \leftarrow b$, depending on whether the previously chosen extension E contains b or b'. These represent a form of (positive and negative) constraints in ABA.

Definition 17. *Let $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, ^-)$ be an ABAF, S a quasi-splitting of D into D_1 and D_2. Moreover, let $V_S^{\leftarrow}$ be the set of vulnerabilities with respect to S and D_2^E the E-reduct of D_2 for some $E \in \sigma(\llcorner D_1 \lrcorner)$. We denote with $\ulcorner D_2^E \urcorner = (\mathcal{L}_2, \ulcorner \mathcal{R}_2^E \urcorner, \mathcal{A}_2, ^{-2})$ the ABAF obtained augmenting $\mathcal{R}_2^E$ with:*

$$\{b \leftarrow \mid b \in E \cap V_S^{\leftarrow}\} \cup \{\overline{b} \leftarrow b \mid b' \in E\}.$$

Notice that such modification might make the ABAF $\ulcorner D_2^E \urcorner$ non-flat, as $cl(\emptyset) = \{b \mid b \in E \cap V_S^{\leftarrow}\}$. For stable semantics, however, this does not result in a higher complexity for the same reasoning tasks.

Example 5. *Consider the ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ where $\mathcal{A} = \{a, b, c, d\}$, $\mathcal{L} = \mathcal{A} \cup \overline{\mathcal{A}} \cup \{p\}$, and rule-set $\mathcal{R}$ as follows:*

$$\overline{b} \leftarrow a \qquad\qquad \overline{d} \leftarrow b \qquad\qquad \overline{a} \leftarrow p, c \qquad\qquad p \leftarrow b$$

First notice that $E = \{b, c\}$ and $E' = \{a, c, d\}$ are stable extensions in D. Now let $S = \{a, \overline{a}, d, \overline{d}, p\}$ be a quasi-splitting of D and $V_S^{\leftarrow} = \{b\}$ the set of vulnerabilities w.r.t. S. We get $\llcorner \mathcal{L}_1 \lrcorner = S \cup \{b\} \cup \{\overline{b}\} \cup \{b', \overline{b'}\}$ and $\llcorner \mathcal{R}_1 \lrcorner$ such that:

$$\overline{d} \leftarrow b \qquad\qquad \overline{a} \leftarrow p, c \qquad\qquad p \leftarrow b \qquad\qquad \overline{b'} \leftarrow b \qquad\qquad \overline{b} \leftarrow b'$$

We derive two stable extensions $E_1 = \{b\}$ and $E_1' = \{b', a, d\}$. Now consider D_2 with $\mathcal{L}_2 = \mathcal{L} \setminus S = \{b, c, \overline{b}, \overline{c}\}$. For the former we get $\ulcorner D_2^{E_1} \urcorner$ with $\ulcorner \mathcal{R}_2^{E_1} \urcorner = \emptyset \cup \{b \leftarrow\}$ from which we derive $E_2 = \{b, c\}$ as a stable extension. For the latter we get $\ulcorner D_2^{E_1'} \urcorner$ with $\ulcorner \mathcal{R}_2^{E_1} \urcorner = \{\overline{b} \leftarrow\} \cup \{\overline{b} \leftarrow b\}$ from which we derive $E_2' = \{c\}$ as a stable extension. We then obtain $E = (E_1 \cap S) \cup E_2$ and $E' = (E_1' \cap S) \cup E_2'$.

Theorem 4. *For an ABAF $D = (\mathcal{L}, \mathcal{R}, \mathcal{A}, \overline{})$ and a quasi-splitting $S \subseteq \mathcal{L}$ of D:*

1. *if $E_1 \in stb(\llcorner D_1 \lrcorner)$ and $E_2 \in stb(\ulcorner D_2^{E_1} \urcorner)$, then $(E_1 \cap S) \cup E_2 \in stb(D)$.*
2. *if $E \in stb(D)$, then there is a set $X \subseteq \{a' \mid a \in V_S^{\leftarrow}\}$ such that $E_1 = (E \cap S) \cup X \in stb(\llcorner D_1 \lrcorner)$ and $E_2 = E \cap \mathcal{A}_2 \in stb(\ulcorner D_2^{E_1} \urcorner)$.*

Proof. In what follows, for notational convenience, let $E = E_1 \cup E_2$ and let $D_2' = (\mathcal{L}_2, \mathcal{R}_2', \mathcal{A}_2, \overline{}^2)$ be the reduct of D_2 w.r.t. $E_1 = (E \cap S) \cup X$.

(1.) To prove the statement we need to show $(E_1 \cap S) \cup E_2 \in cf(D)$ and $((E_1 \cap S) \cup E_2)^{\oplus}_{\mathcal{R}} = \mathcal{A}$. We start with conflict-freeness. Since $E_1 \in cf(\llcorner D_1 \lrcorner)$, then $E_1 \cap S \in cf(\llcorner D_1 \lrcorner)$ (less assumptions) and $E_1 \cap S \in cf(D_1)$ (less attacks). Since $\mathcal{L}_1 = \mathcal{L} \cap S$, $\mathcal{R}_1 = \{r \in \mathcal{R} \mid head(r) \in S\}$, we can derive $E_1 \cap S \in cf(D)$. Consider now $E_2 \in stb(\ulcorner D_2' \urcorner)$. Since being stable implies conflict-freeness we immediately get $E_2 \in cf(\ulcorner D_2' \urcorner)$. Again, since $\mathcal{R}_2' \subseteq \ulcorner \mathcal{R}_2' \urcorner$, we obtain $E_2 \in cf(D_2')$. Furthermore, Proposition 3 for proper splittings, together with $E_1 \cap S \in cf(D_1)$ and $E_2 \in cf(D_2')$, entail $(E_1 \cap S) \cup E_2 \nvdash^R \overline{a}$ for any $a \in E_2$ and $R \subseteq \mathcal{R}$. It only remains to consider possible attacks from E_2 to $E_1 \cap S$ in D. Suppose that there are $T \subseteq E_2$ and $a \in E_1 \cap S$ such that $T \vdash^R \overline{a}$ for some $R \subseteq \mathcal{R}$. First, notice that since $T \subseteq E_2$, we get $body(R) \cap S = \emptyset \subseteq Th_{D_1}(E_1)$, and thus $R \subseteq \mathcal{R}_2'$. Moreover, $a \in V_S^{\leftarrow}$ so that $\ulcorner \mathcal{R}_2' \urcorner = \mathcal{R}_2' \cup \{a \leftarrow\}$. Therefore, $T \vdash^R a$ and $T \vdash^R \overline{a}$ for some $R \subseteq \ulcorner \mathcal{R}_2' \urcorner$, i.e. E_2 is either not conflict-free or not closed in $\ulcorner D_2' \urcorner$.

We now show that $((E_1 \cap S) \cup E_2)^{\oplus}_{\mathcal{R}} = \mathcal{A}$. Towards contradiction, consider an assumption $a \notin ((E_1 \cap S) \cup E_2)^{\oplus}_{\mathcal{R}}$. Assume $a \in S$. By hypothesis, $E_1 \in stb(\llcorner D_1 \lrcorner)$, i.e. either $a \in E_1$ or $E_1 \vdash^R \overline{a}$ for some $R \subseteq \llcorner \mathcal{R}_1 \lrcorner$. From our assumption, we get $a \notin (E_1 \cap S)^{\oplus}_{\mathcal{R}}$, that is (i) $a \notin E_1 \cap S$ and (ii) $E_1 \cap S \nvdash^R \overline{a}$ for any $R \subseteq \mathcal{R}$. If (i) holds, we immediately derive $E_1 \vdash^R \overline{a}$ for some $R \subseteq \llcorner \mathcal{R}_1 \lrcorner$. Consider now our assumption (ii). Because $a \in S$ we know that every rule of R is contained in $\mathcal{R}_1$. For the same reason such rules are in $\llcorner \mathcal{R}_1 \lrcorner$ ($b \notin V_S^{\leftarrow}$). Therefore, $E_1 \cap S \nvdash^R \overline{a}$ for any $R \subseteq \llcorner \mathcal{R}_1 \lrcorner$ in contradiction with our hypothesis. Assume now $a \in \mathcal{A} \setminus S$. By hypothesis we know either $a \in E_2$ or $E_2 \vdash^R \overline{a}$ for some $R \subseteq \ulcorner \mathcal{R}_2' \urcorner$. From the assumption, we get $a \notin (E_2)^{\oplus}_{\mathcal{R}}$, that is (i) $a \notin E_2$ and (ii) $E_2 \nvdash^R \overline{a}$ for any $R \subseteq \mathcal{R}$. As before, from (i) and our hypothesis we derive $E_2 \vdash^R \overline{a}$ must hold for some $R \subseteq \ulcorner \mathcal{R}_2' \urcorner$. If $a \in V_S^{\leftarrow}$, there are two possibilities: $a \in E_1 \setminus S$ or $a \notin E_1 \setminus S$. In the first scenario, $\ulcorner \mathcal{R}_2' \urcorner = \mathcal{R}_2' \cup \{a \leftarrow\}$. Again, $E_2 \vdash^R a$ and $E_2 \vdash^R \overline{a}$ for some $R \subseteq \ulcorner \mathcal{R}_2' \urcorner$, in contradiction with the fact that E_2 is a stable extension of $\ulcorner D_2' \urcorner$. If $a \notin E_1 \setminus S$, then $a' \in E_1$, which means $\ulcorner \mathcal{R}_2' \urcorner = \mathcal{R}_2' \cup \{\overline{a} \leftarrow a\}$. Since $a \notin E_2$, the loop-rule $\overline{a} \leftarrow a$ is not in R, therefore $R \subseteq \mathcal{R}_2'$. Thus, for each rule $r' \in \mathcal{R}_2'$ there is a rule $r \in \mathcal{R}_2$ such that $body(r) \subseteq body(r) \cup Th_{D_1}(E_1)$. Hence, it follows directly that $(E_1 \cap S) \cup E_2 \vdash^R \overline{a}$ for some $R \subseteq \mathcal{R}_1 \cup \mathcal{R}_2 = \mathcal{R}$. If $a \notin V_S^{\leftarrow}$, then $a \notin \mathcal{A}_1$. If $E_2 \vdash^R \overline{a}$ for some $R \subseteq \ulcorner \mathcal{R}_2' \urcorner$, it is not because $\{\overline{a} \leftarrow a\} \subseteq \ulcorner \mathcal{R}_2' \urcorner$. Thus $R \subseteq \mathcal{R}_2'$. As before, for each rule $r' \in \mathcal{R}_2'$ there is exactly one rule $r \in \mathcal{R}_2$ such that $body(r) \subseteq body(r') \cup Th_{D_1}(E_1 \cap S)$. As a result, in the entire rule-set $\mathcal{R}$ we obtain $(E_1 \cap S) \cup E_2 \vdash \overline{a}$ is a derivation in D. This contradicts our assumption.

(2.) First we get $E \in cf(D)$ and thus $E \cap S \in cf(D_1)$ (less attacks). Now let $B = \mathcal{A}_1 \setminus (E \cap S)^{\oplus}_{\mathcal{R}}$. Since $E \in stb(D)$, it attacks every other assumption. Hence, we can infer that assumptions in B are contained in $\mathcal{A}_1$ and attacked by $E \cap \mathcal{A}_2$ in D, that is $B \subseteq E^+_{\mathcal{R}} \setminus (E \cap S)^{\oplus}_{\mathcal{R}} = (E \cap \mathcal{A}_2)^+_{\mathcal{R}}$. Therefore there is a rule $r \in \mathcal{R}_2$ with $head(r) = \overline{b}$ for each $b \in B$, meaning that $B = V^{\leftarrow}_S$. Now let $X = \{b' \mid b \in B\}$. Thus $\ulcorner \mathcal{R}'_2 \urcorner$ contains a pair of rule $\{\overline{b} \leftarrow b', \overline{b'} \leftarrow b\}$ for each $b \in B$. Consequently, conflict-freeness of $(E \cap S) \cup X$ is ensured since $b \notin E \cap S$ for all $b \in B$. Moreover, X attacks every $b \in B$ in $\llcorner D_1 \lrcorner$, making $(E \cap S) \cup X$ stable.

It now remains to show $E_2 = E \cap \mathcal{A}_2 \in stb(\ulcorner D'_2 \urcorner)$. As before, we know that $E \cap \mathcal{A}_2 \in cf(D_2)$ since E is conflict-free in D (less assumptions), and $E \cap \mathcal{A}_2 \in cf(D'_2)$ because $Th_{D'_2}(E \cap \mathcal{A}_2) \subseteq Th_{D_2}(E \cap \mathcal{A}_2)$ (less rules and attacks). Consider now the modified framework $\ulcorner D'_2 \urcorner$ wrt $(E \cap S) \cup X$. By construction, $E \cap \mathcal{A}_2 \notin cf(\ulcorner D'_2 \urcorner)$ only if $b \in B \cap (E \cap \mathcal{A}_2)$. Recall that $B \subseteq (E \cap \mathcal{A}_2)^+_{\mathcal{R}}$. Thus, $E \cap \mathcal{A}_2 \notin cf(D)$. By contradiction, we derive that $E \cap \mathcal{A}_2$ is conflict free in $\ulcorner D'_2 \urcorner$. We now show that $E_2 \vdash^R \overline{a}$ for all $a \in \mathcal{A}_2 \setminus E_2$ and some $R \subseteq \ulcorner \mathcal{R}'_2 \urcorner$. Towards contradiction, we assume there is an $a \in \mathcal{A}_2 \setminus E_2$ such that $E_2 \not\vdash^R \overline{a}$, i.e. $\overline{a} \notin Th_{\ulcorner D'_2 \urcorner}(E_2)$. Therefore, since $a \notin E_2$, we get $\overline{a} \notin Th_{D'_2}(E_2)$. Hence, before the reduct is applied, it holds that $(E \cap S) \cup E_2 \not\vdash^R \overline{a}$ with $R \subseteq \mathcal{R}_2$. Since no rule $r \in \mathcal{R}_1$ is such that $head(r) = \overline{a}$, we derive $(E \cap S) \cup E_2 \not\vdash^R \overline{a}$ in D, in contradiction with out hypothesis. Finally, we ensure that $cl(E_2) = E_2$ in $\ulcorner D'_2 \urcorner$. Assume the contrary holds. Since D'_2 is flat, that means $\{a \leftarrow\} \subseteq \ulcorner \mathcal{R}'_2 \urcorner$ and $a \notin E_2$. These facts respectively entail $a \in E_1$ and $E_2 \vdash \overline{a}$ (from previous paragraph). This contradicts the conflict-freeness of E. Thus, E_2 is conflict-free, closed and attacks every other assumption. $\square$

5.2. Argumentation Frameworks with Collective Attacks

In this section we investigate a notion of parametrised splitting for SETAFs, which generalises the one for AFs [12]. First, we introduce the notion of quasi-splitting for SETAFs.

Definition 18. *Let $SF = (A, R)$ be a SETAF, $SF_1 = (A_1, R_1)$ and $SF_2 = (A_2, R_2)$ two sub-frameworks of SF such that $A_1 \cap A_2 = \emptyset$ and $A = A_1 \cup A_2$. We call a* quasi-splitting *of SF the tuple $(SF_1, SF_2, R^{\leftarrow}_3, R^{\rightarrow}_3)$ with $R^{\rightarrow}_3 \subseteq \left((2^{A_1} \setminus \{\emptyset\}) \cup 2^{A_2}\right) \times A_2$, $R^{\leftarrow}_3 \subseteq \left((2^{A_2} \setminus \{\emptyset\}) \cup 2^{A_1}\right) \times A_1$ and $R = R_1 \cup R_2 \cup R_3$. Moreover, we say that $R^{\leftarrow}_3$ and $R^{\rightarrow}_3$ are the set of* incoming *and* outgoing *links w.r.t. the splitting. The splitting $(SF_1, SF_2, R^{\leftarrow}_3, R^{\rightarrow}_3)$ is called:*

- *the k-splitting of SF, if $|R^{\leftarrow}_3| = k$;*
- *(proper) splitting of SF, if $|R^{\leftarrow}_3| = 0$.*

While the idea of quasi-splitting is carried out in a conceptually similar manner than for ABAFs, the concrete modifications that we require are fairly different. In particular, we start by augmenting SF_1 with fresh arguments that encode meta-information regarding incoming links. For each of these, we introduce symmetric attacks to force a choice between the target of the incoming link and the new one.

Definition 19. *Let $SF = (A, R)$ be a SETAF, $(SF_1, SF_2, R^{\leftarrow}_3, R^{\rightarrow}_3)$ be a quasi-splitting of SF inducing the sub-frameworks SF_1 and SF_2. We construct $\llcorner SF_1 \lrcorner = (\llcorner A_1 \lrcorner, \llcorner R_1 \lrcorner)$ as the SETAF obtained from SF_1 by letting:*

- *$\llcorner A_1 \lrcorner = A_1 \cup \{b' \mid b \in A^+_{R^{\leftarrow}_3}\}$;*
- *$\llcorner R_1 \lrcorner = R_1 \cup \{(\{b\}, b'), (\{b'\} \cup (T \cap \mathcal{A}_1), b) \mid (T, b) \in R^{\leftarrow}_3\}$.*

We call E a conditional extension *of SF_1 iff it is a stable extension of $\llcorner SF_1 \lrcorner$.*

As for proper splitting, E_1 is used to compute the reduct of SF_2. Further, in this setting the the meta-information in E_1 plays a role. In particular, if $a \notin E_1$ and is not attacked by $E_1 \cap A_1$, then it must be attacked externally.

Definition 20. *Let SF be a SETAF and $(SF_1, SF_2, R_3^{\leftarrow}, R_3^{\rightarrow})$ be a quasi-splitting of SF. Moreover, let E_1 be a conditional extension of SF_1. We call*

$$EA_1^{E_1} = \{a \in A_1 \setminus E_1 \mid a \notin (E_1 \cap A_1)_R^+\}$$

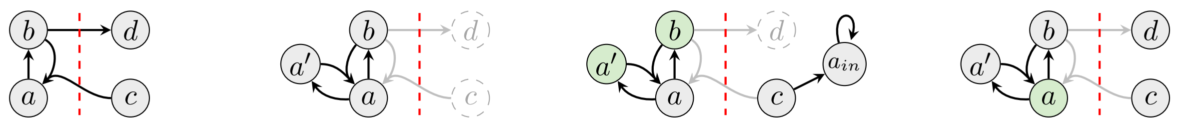
the set of externally attacked arguments in SF_1 w.r.t. E_1.

Next, we introduce a modification of $SF_2^{E_1}$ that takes into account information regarding incoming links. First, we add set-self-attacks to make coflicting those sets of arguments attacking E_1 via $R_3^{\leftarrow}$. Further, for each of these externally attacked arguments, we introduce a self-attacking argument a_{in} attacked by the remaining part of an incoming link.

Definition 21. *Let SF be a SETAF and $(SF_1, SF_2, R_3^{\leftarrow}, R_3^{\rightarrow})$ be a quasi-splitting of SF. Moreover, let E_1 be a conditional extension of SF_1 and $EA_1^{E_1} = \{a \in A_1 \setminus E_1 \mid a \notin (E_1 \cap A_1)_R^+\}$. We denote with $\ulcorner SF_2^{E_1} \urcorner = (\ulcorner A_2^{E_1} \urcorner, \ulcorner R_2^{E_1} \urcorner)$ the SETAF where:*

$$\begin{aligned}
\ulcorner A_2^{E_1} \urcorner =& A_2^{E_1} \cup \{a_{in} \mid a \in EA_1^{E_1}\};\\
\ulcorner R_2^{E_1} \urcorner =& R_2^{E_1} \cup \{(T, b) \mid b \in T \subseteq A_2^{E_1}, (T, E_1) \in R_3^{\leftarrow}\} \cup\\
& \{a_{in}, a_{in}), (T, a_{in}) \mid a \in EA_1^{E_1}, T \subseteq A_2^{E_1}, \exists T' \supseteq T \text{ s.t. } (T', a) \in R_3^{\leftarrow}\}.
\end{aligned}$$

Example 6. *Consider the SETAF $SF = (A, R)$ where $A = \{a, b, c, d\}$ and $R = \{(a, b), (\{b, c\}, a), (b, d)\}$ and its quasi-splitting as depicted below (a). We have two possible stable extensions $E = \{b, c\}$ and $E' = \{a, c, d\}$. After modification, the first sub-framework $\llcorner SF_1 \lrcorner$ (b) has two stable extensions: $E_1 = \{a', b\}$ and $E_1' = \{a\}$. These yield two different modifications $\ulcorner SF_2^{E_1} \urcorner$ (c) and $\ulcorner SF_2^{E_1'} \urcorner$ (d), with respect to $EA_1^{E_1} = \{a\}$ and $EA_1^{E_1'} = \emptyset$. Consequently, their only stable extensions are $E_2 = \{c\}$ and $E_2' = \{c, d\}$ respectively.*



(a) SF *(b)* $\llcorner SF_1 \lrcorner$ *(c)* $\ulcorner SF_2^{E_1} \urcorner$ *(d)* $\ulcorner SF_2^{E_1'} \urcorner$

Towards proving the splitting theorem, we adapt a useful lemma from [12] in the context of SETAFs.

Lemma 1. *Let $SF = (A, R)$ be a SETAF with $\mathcal{B}, \mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq 2^A$ sets of sets of arguments in SF. Moreover, let $D = \{d_1, \ldots, d_n\}$ be fresh arguments such that $D \cap A = \emptyset$. The stable extensions of*

$$SF' = (A \cup D, R \cup \{(B, b) \mid b \in B \in \mathcal{B}\} \cup \{(d_i, d_i), (C, d_i) \mid d_i \in D, C \in \mathcal{C}_i, 1 \leq i \leq n\})$$

are exactly the stable extensions E of SF such that (i) $B \not\subseteq E$ for any $B \in \mathcal{B}$ and (ii) $C \subseteq E$ for at least one $C \in \mathcal{C}_i$ and every $\mathcal{C}_i$ with $i \in \{1, \ldots, n\}$.

Proof. Suppose $E \in stb(SF)$ such that (i) and (ii) hold. Since $E \subseteq A$ and (i) holds, we get $E \in cf(SF')$. Moreover, from (ii) we derive that $E^{\oplus}_{R(SF')} = E^{\oplus}_{R(SF)} \cup D = A \cup D$, deriving $E \in stb(SF')$. Assume $E \in stb(SF')$. Since every d_i is self-attacking, we know that $E \subseteq A$. Thus $E \in cf(SF)$ (less attacks). Further, $E^{\oplus}_{R(SF')} = A \cup D$ from hypothesis, and $E^{\oplus}_{R(SF)} = E^{\oplus}_{R(SF')} \setminus D = (A \cup D) \setminus D = A$, proving that $E \in stb(SF)$. We now show (i) and (ii). For (i) notice that for all $B \subseteq \mathcal{B}_i$, we have $B \not\subseteq E$ because E is conflict-free in SF'. For (ii), each $\mathcal{C}_j$ is the set of sets attacking the corresponding $d_j \in D$. Therefore, at least one of such attacking sets $C \in \mathcal{C}_j$ is guaranteed to be in E since $E^{\oplus}_{R(SF')} = A \cup D$. □

Notice that the SETAFs SF and SF' in the lemma above corresponds exactly to $SF_2^{E_1}$ and $\ulcorner SF_2^{E_1} \urcorner$, where $\mathcal{B}$ is the set of sets attacking E_1 and each $\mathcal{C}_i$ the set of sets attacking each $a_{in} \in EA_1^{E_1}$. The lemma is thus utilised to show the following parametrised splitting theorem focusing on $SF_2^{E_1}$ only.

Theorem 5. *Let SF be a SETAF and $(SF_1, SF_2, R_3^{\leftarrow}, R_3^{\rightarrow})$ be a quasi-splitting of SF. Moreover, let $\llcorner SF_1 \lrcorner$ and $\ulcorner SF_2' \urcorner = \ulcorner SF_2^{E_1} \urcorner$ be as per Definitions 19 and 21.*

1. *If $E_1 \in stb(\llcorner SF_1 \lrcorner)$ and $E_2 \in stb(\ulcorner SF_2' \urcorner)$, then $(E_1 \cap A_1) \cup E_2 \in stb(SF)$.*
2. *If $E \in stb(SF)$, then there is a set $X \subseteq \{b' \mid b \in A^+_{R_3^{\leftarrow}}\}$ such that $E_1 = (E \cap A_1) \cup X \in stb(\llcorner SF_1 \lrcorner)$ and $E_2 = E \cap A_2 \in stb(\ulcorner SF_2' \urcorner)$.*

Proof. (1.) We first prove conflict-freeness. From hypothesis, $E_1 \in cf(\llcorner SF_1 \lrcorner)$ implies $E \cap A_1 \in cf(SF_1)$ since $E \cap A_1 \subseteq E_1$ and $R_1 \subseteq \llcorner R_1 \lrcorner$. Thus, $E \cap A_1 \in cf(SF)$ because $R_1 = R \cap (2^{A_1} \times A_1)$. From the fact that $E_2 \in stb(\ulcorner SF_2' \urcorner)$ together with Lemma 1, we know that $E_2 \in stb(SF_2')$, and thus $E_2 \in cf(SF)$. We now consider possible attacks from $E_1 \cap A_1$ to E_2 and viceversa. Clearly, $(E_1 \cap A_1, E_2) \notin R$ since $E_1 \cap A_1 \subseteq A_1$ and $E_2 \subseteq A_2'$ (recall $A_2' = A_2 \setminus (E_1)^+_{R_3^{\leftarrow}}$). Assume towards contradiction that $(E_2, E_1 \cap A_1) \in R$, i.e. $(T, a) \in R_3^{\leftarrow}$ for some $T \subseteq E_2 \subseteq A_2'$ and $a \in E_1 \cap A_1$. If this is the case, then by construction of $\ulcorner SF_2' \urcorner$ we have $(T, b) \in \ulcorner R_2' \urcorner$ for some $b \in T$, violating the coflict-freeness of E_2 in $\ulcorner SF_2' \urcorner$. Hence, $(E_1 \cap A_1) \cup E_2 \in cf(SF)$. We show that $((E_1 \cap A_1) \cup E_2)^{\oplus}_R = A_1 \cup A_2 = A$ by contradiction. Assume $a \notin ((E_1 \cap A_1) \cup E_2)^{\oplus}_R$. If $a \in A_1$, we deduce that $a \in EA_1^{E_1}$. As before, given that $E_2 \in stb(\ulcorner SF_2' \urcorner)$, it holds that $E_2 \in stb(SF_2')$ via Lemma 1, and $(E_2, a_{in}) \in \ulcorner R_2' \urcorner$. Therefore, E_2 attacks a via $R_3^{\leftarrow}$ in SF, contradicting our assumption. If $a \in A_2$, together with our assumption, we get $a \in A_2'$ (elements in $E_1 \setminus A_1$ do not attack arguments in A_2). Again, since $E_2 \in stb(\ulcorner SF_2' \urcorner)$, it holds that $E_2 \in stb(SF_2')$ via Lemma 1. Thus, $a \in (E_2)^{\oplus}_R$, contradicting our assumption. Therefore, $((E_1 \cap A_1) \cup E_2)^{\oplus}_R = A$ and $(E_1 \cap A_1) \cup E_2 \in stb(F)$.

(2.) We first show that $(E \cap A_1) \cup X \in stb(\llcorner SF_1 \lrcorner)$. Let $B = \{b_1, \dots, b_n\} = A_1 \setminus (E \cap A_1)^{\oplus}_R$ and $X = \{b_i' \mid b_i \in B\}$. Since $E \in stb(SF)$, it follows that $B \subseteq (E_2)^+_R$. Hence, by construction of $\llcorner R_1 \lrcorner$, we derive that $(E \cap A_1) \cup X \in stb(\llcorner SF_1 \lrcorner)$. Consider now E_2. From $E \in stb(SF)$, we get $E_2 \in stb(SF_2')$ because each $a \in A_2'$ is attacked by E_2 in SF. Moreover, since E is conflict-free in SF, there is no $T \subseteq E_2$ such that $(T, E_1) \in R_3^{\leftarrow}$ (E_2 satisfies (i)). Notice that $B = EA_1^{E_1}$, i.e. each $b_i \in B$ is in A_1 and externally attacked. Recall that $B \subseteq (E_2)^+_R$. Therefore, there is some $T' \subseteq E_2$ such that (T', b_i) for each $b_i \in EA_1^{E_1}$. By construction of $\ulcorner SF_2' \urcorner$, a fresh argument $(b_i)_{in} = d_i$ is introduced for each $b_i \in B$ along with (T', d_i) (E_2 satisfies (ii)). Thus Lemma 1 applies, concluding $E_2 \in stb(\ulcorner SF_2' \urcorner)$. ☐

6. Conclusion and Future Work

In this paper, we have presented a modification-based approach to splitting assumption-based argumentation frameworks. In particular, we have shown that 1. if one computes an extension E_1 in D_1, then applies the reduct and modification, and obtains an extension E_2 of the remaining sub-framework, their set-union is an extension of the whole framework. This characterises the incremental computation of the extension E by evaluating the two sub-frameworks. Conversely, we show that 2. if we project an arbitrary extension E of the whole framework to its sub-frameworks, we obtain extensions E_1 for D_1 and E_2 for the (E_1)-modified version of D_2. Since this is bound to specific structure of the underlying ABAF, we have considered a more general variant of splitting called parametrised splitting inspired by Baumann et al. [12]. Results in this setting have been presented for both ABAFs and SETAFs. Moreover, it is easy to see that each of the steps involved can be carried out efficiently and implemented on top of common ABA (or SETAF) solvers. Therefore, an obvious next step is that of implementing our algorithm and perform an experimental evaluation in the spirit of Baumann et al. [18]. In particular, we believe that parametrised splitting could be helpful in the context of the recently proposed Argumentative Causal Discovery, which faces a major challenge in terms of its scalability and exhibits suboptimal performance on larger instances [5].

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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