On Minimal Inconsistent Signatures and their Application to Inconsistency Measurement

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Abstract
Minimal inconsistent sets have played an important role in the analysis and general handling of inconsistency in logical knowledge bases. We introduce a semantical counterpart of this notion we call *minimal inconsistent signature*, which is a minimal set of propositions such that projecting the knowledge base onto it still preserves the inconsistency. We analyse minimal inconsistent signatures and the corresponding dual notion of maximal consistent signatures in depth and show, among others, that the *hitting set duality* applies for them as well. We apply our new notions to the field of inconsistency measurement and derive a series of new inconsistency measures, which we analyse in terms of postulate satisfaction and general behaviour. Finally, we analyse the computational complexity of various problems within this new context.

1. Introduction

Reasoning with inconsistent information is a central issue for approaches to knowledge representation and reasoning [1, 2, 3, 4, 5, 6, 7, 8]. A standard approach to deal with inconsistency is to consider the *minimal inconsistent subsets* of the knowledge base. Given a (possibly inconsistent) knowledge base K consisting of (propositional) formulas, a *minimal inconsistent subset* K' is a set $K' \subseteq K$ that is inconsistent and every set K'' with $K'' \subsetneq K' \subseteq K$ is consistent (we will give formal definitions in Section 2). Minimal inconsistent subsets can directly be used for diagnosis and debugging [9], but also for inconsistency-tolerant reasoning by removing one formula from each minimal inconsistent subset [1, 7].

In this work, we define and analyse a new approach to analyse inconsistency, but defined in terms of signatures rather than subsets of the knowledge base. More precisely, we define a minimal inconsistent subsignature as a minimal set of propositions, such that *forgetting*[1] [10, 11] the remaining propositions from the knowledge base still retains its inconsistency. By considering both the notion of minimal inconsistent subsignatures and their counterpart, the maximal consistent subsignatures, we obtain a technical framework that is quite similar to the framework of minimal inconsistent subsets and maximal consistent subsets, but also features some additional interesting properties. We show that the classical *hitting set duality* [9] carries over to minimal inconsistent subsignatures as well, i. e., one can obtain maximal consistent subsignatures by removing a minimal hitting set of all minimal inconsistent subsignatures, and vice versa. We furthermore analyse one particular application area in detail, namely the area of *inconsistency measurement* [2, 12]. This area is concerned with developing measures that assess the *degree* of inconsistency in knowledge bases. Many of the existing measures are defined in terms of minimal inconsistent subsets and we analyse variants of these measures by using minimal inconsistent signatures instead of minimal inconsistent subsets. In order to complement our analysis, we also investigate the computational complexity of various problems pertaining to our approach.

To summarise, the contributions of this paper are as follows:

1. We revisit the notion of *forgetting* parts of the signature of a knowledge base for the purpose of

NMR'25: The 23rd International Workshop on Nonmonotonic Reasoning

[1]Note that we use a slightly non-standard interpretation of the term *forgetting*. In particular, we use forgetting to restore consistency. See Section 3 for details.

defining a semantical counterpart to minimal inconsistent subsets and make some new observations (Section 3).

2. We define *minimal inconsistent* and *maximal consistent* subsignatures and analyse their properties; in particular, we show that these structures also obey the *hitting set duality* (Section 4).
3. We define and analyse new inconsistency measures based on minimal inconsistent subsignatures and maximal consistent subsignatures (Section 5).
4. We analyse the computational complexity of various decision problems related to minimal inconsistent subsignatures (Section 6).

We will discuss necessary preliminaries in Section 2, discuss related work in Section 7, and conclude with a discussion in Section 8.

Proofs of technical results can be found in an extended version of this paper [13].

2. Preliminaries

Let At be some fixed propositional signature, i. e., a (possibly infinite) set of propositions, and let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the standard connectives $\wedge$ (*conjunction*), $\vee$ (*disjunction*), $\rightarrow$ (*implication*), and $\neg$ (*negation*). Let furthermore $\top, \bot \in \mathsf{At}$ be special propositions denoting *tautology* and *contradiction*, respectively.

Definition 1. A knowledge base K is a finite set of formulas $K \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases.

If Φ is a formula or a set of formulas we write $\mathsf{At}(\Phi)$ to denote the set of propositions appearing in Φ. For a set $\Phi = \{\phi_1, \ldots, \phi_n\}$ let $\bigwedge \Phi = \phi_1 \wedge \ldots \wedge \phi_n$ and $\neg\Phi = \{\neg\phi \mid \phi \in \Phi\}$.

Semantics to a propositional language are given by *interpretations* where an *interpretation* ω on At is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At (with the convention that $\omega(\top) = \mathsf{true}$ and $\omega(\bot) = \mathsf{false}$). An interpretation ω *satisfies* (or is a *model* of) a proposition $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulas in the usual way. For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$.

In the following, let Φ, Φ_1, Φ_2 be formulas or sets of formulas. Define the set of models $\mathsf{Mod}(\Phi) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models \Phi\}$. We write $\Phi_1 \models \Phi_2$ if $\mathsf{Mod}(\Phi_1) \subseteq \mathsf{Mod}(\Phi_2)$. Φ_1, Φ_2 are *equivalent*, denoted by $\Phi_1 \equiv \Phi_2$, if and only if $\mathsf{Mod}(\Phi_1) = \mathsf{Mod}(\Phi_2)$. If $\mathsf{Mod}(\Phi) = \emptyset$ we also write $\Phi \models\bot$ and say that Φ is *inconsistent* (or *unsatisfiable*).

Definition 2. Let K be a knowledge base.

1. $K' \subseteq K$ is called a *minimal inconsistent subset* of K if
   a) $K' \models\bot$ and
   b) for all K'' with $K'' \subsetneq K'$, $K'' \not\models\bot$.
2. $K' \subseteq K$ is called a *maximal consistent subset* of K if
   a) $K' \not\models\bot$ and
   b) for all K'' with $K' \subsetneq K'' \subseteq K$, $K'' \models\bot$.

Let $\mathsf{MIS}(K)$ and $\mathsf{MCS}(K)$ denote the set of all minimal inconsistent subsets of K and the set of all maximal consistent subsets of K, respectively.

Let furthermore $\mathsf{FREE}(K) = K \setminus \bigcup \mathsf{MIS}(K)$ denote the set of *free formulas* of K, i. e., those formulas of K that are not members of any minimal inconsistent subset of K. Moreover, a formula α is *safe* for a knowledge base K iff $\alpha \not\models\bot$ and $\mathsf{At}(\alpha) \cap \mathsf{At}(K \setminus \{\alpha\}) = \emptyset$. Let $\mathsf{SAFE}(K)$ denote the set of safe formulas of K and note that $\mathsf{SAFE}(K) \subseteq \mathsf{FREE}(K)$ [14].

3. Forgetting and Projecting

A *forgetting operator* is an operator that removes a given set of propositions from a signature of the knowledge base. Its initial motivation [10] was to be able to remove irrelevant parts of a knowledge base, while *retaining* previous inferences as much as possible. There exists certain properties that such an operator should satisfy [10, 11] and it makes sense (in the case of consistency) to identify *forgetting* with the *variable elimination operation*. Let $\phi[\psi \to \psi']$ denote the propositional formula that is obtained from ϕ by simultaneously replacing each occurrence of ψ in ϕ by ψ'.

Definition 3. For a formula ϕ and some $a \in \mathsf{At}(\phi)$ define the *elimination of a from ϕ*, denoted as $\phi \div a$, to be the formula $\phi \div a = \phi[a \to \top] \vee \phi[a \to \bot]$.

In other words, eliminating a from ϕ is equivalent to replacing a with $\top$ or $\bot$. A nice property of variable elimination is that inferences on the remaining part of the signature are retained [10]. We do not formalise this property here, but only show an example.

Example 1. Let $\phi = (a \wedge b) \vee (c \wedge \neg d)$. Forgetting a from ϕ gives us

$$\phi \div a = (\top \wedge b) \vee (c \wedge \neg d) \vee (\bot \wedge b) \vee (c \wedge \neg d) \equiv b \vee (c \wedge \neg d)$$

Note that, e. g., $\phi \models b \vee c$ and $\phi \div a \models b \vee c$.

Observe that variable elimination preserves inconsistency, i. e., if a formula is inconsistent then forgetting any proposition cannot restore consistency. For this to see, first observe that the order in which propositions are eliminated does not matter, so let $\phi \div S$ for a set $S \subseteq \mathsf{At}(\phi)$ denote the application of variable elimination in any order.

Proposition 1. $\phi \not\models \bot$ *if and only if* $\phi \div \mathsf{At}(\phi) \equiv \top$.

Our aim in the rest of this section is to devise a forgetting operation based on variable elimination that is able to *restore* consistency, i. e., by removing "conflicting" parts of the signature of the formula or knowledge base, we wish to end up with a consistent outcome. Note that restoring consistency will retract a lot of inferences, which is then not aligned with the initial motivation for forgetting from above. We illustrate our aim with a simple example.

Example 2. Consider the formula ϕ given by $\phi = a \wedge \neg a \wedge b$. Clearly $\phi \models \bot$. Intuitively, the proposition a (and the modelled information about it) is responsible for the inconsistency. We therefore expect that forgetting a leaves us with a formula $\phi' = b$, from which we can still derive meaningful information about b. Note, however, that $\phi \div a \equiv \bot$.

In order to define a forgetting operation with the above behaviour, we have to operate on the level of *proposition occurrences* rather than proposition. Since we do not wish to retain inferences by forgetting but only to remove propositions (and the information modelled for them), we allow proposition occurrences to be replaced by $\top$ or $\bot$ *individually*. For that, let

$$\phi[\psi \to \psi_1'/\psi_2'/\ldots/\psi_n']$$

denote the propositional formula that is obtained from ϕ by replacing the first occurrence of ψ in ϕ by ψ_1', the second occurrence of ψ in ϕ by ψ_2', and so on (the operation is undefined if the number of occurrences of ψ in ϕ is not equal to n).

Example 3. For the formula $\phi = a \wedge (b \vee a) \wedge \neg a$ we have $\phi[a \to \top/\bot/\bot] = \top \wedge (b \vee \bot) \wedge \neg\bot \equiv b$.

The above operation allows us to define a new variant of variable elimination as follows. Let $\#^{\phi} a$ denote the number of occurrences of $a \in \mathsf{At}(\phi)$ in ϕ.

Definition 4. For a formula ϕ and some $a \in \mathsf{At}(\phi)$ define

$$\phi \boxminus a = \bigvee_{x_1,\ldots,x_{\#\phi_a} \in \{\top,\bot\}} \phi[a \to x_1/\ldots/x_{\#\phi a}]$$

The operator $\boxminus$ allows the replacement of each occurrence of a with $\top$ or $\bot$ such that contradictions within a formula can be resolved. Let us consider again Example 2.

Example 4. Consider again

$$\phi = a \wedge \neg a \wedge b$$

Here we have

$$\phi \boxminus a = (\top \wedge \top \wedge b) \vee (\top \wedge \bot \wedge b) \vee (\bot \wedge \top \wedge b) \vee (\bot \wedge \bot \wedge b) \equiv \top \wedge \top \wedge b \equiv b$$

as desired.

Before we continue with an analysis of $\boxminus$ let us first give some intuitions and a simple syntactic characterisation of what $\boxminus$ does to a formula. It may not be apparent from the definition above, but what $\phi \boxminus a$ basically does is the following: it replaces every disjunction within ϕ that contains a or $\neg a$ by $\top$ and removes all occurrences of a and $\neg a$ from conjunctions. Recall that a formula ϕ is in *negation normal form (NNF)* if negations only appear right in front of propositions. For formulas in NNF we can characterise $\boxminus$ as follows.

Proposition 2. *For a formula ϕ in NNF and some $a \in \mathsf{At}(\phi)$ define $\phi \hat{\boxminus} a$ inductively on the structure of ϕ via*

$$\phi \hat{\boxminus} a = \begin{cases} \top & \text{if } \phi = a \text{ or } \phi = \neg a \\ \psi \hat{\boxminus} a \vee \psi' \hat{\boxminus} a & \text{if } \phi = \psi \vee \psi' \\ \psi \hat{\boxminus} a \wedge \psi' \hat{\boxminus} a & \text{if } \phi = \psi \wedge \psi' \end{cases}$$

If $a \notin \mathsf{At}(\phi)$ we define $\phi \hat{\boxminus} a = \phi$. Then

$$\phi \hat{\boxminus} a \equiv \phi \boxminus a$$

Since $\phi_1 \vee \ldots \vee \phi_n \vee \top \equiv \top$ and $\phi_1 \wedge \ldots \wedge \phi_n \wedge \top \equiv \phi_1 \wedge \ldots \wedge \phi_n$ for all $\phi_1, \ldots, \phi_n$, it should be clear that forgetting a from a formula ϕ in NNF means that we replace every disjunction within ϕ that contains a or $\neg a$ by $\top$ and remove all occurrences of a and $\neg a$ from conjunctions, as stated above.

Note that every formula can be translated into NNF with only a linear increase in size and that this translation yields an equivalent formula. Most of our examples are using formulas in NNF, so the above characterisation can be applied.

If multiple propositions are forgotten with $\boxminus$, it should be obvious that the order does not matter. So for a set $S = \{a_1, \ldots, a_n\} \subseteq \mathsf{At}(\phi)$ let

$$\phi \boxminus S = (\ldots((\phi \boxminus a_1) \boxminus a_2) \ldots) \boxminus a_n$$

with an arbitrary order among the propositions in S. Furthermore, for a knowledge base K and $S \subseteq \mathsf{At}(K)$ we write

$$K \boxminus S = \{\phi \boxminus S \cap \mathsf{At}(\phi) \mid \phi \in K\}$$

Example 5. Consider $K_1 = \{a, \neg a \wedge c\}$, we get

$$K_1' = K_1 \boxminus a = \{\top \vee \bot, (\neg\top \wedge c) \vee (\neg\bot \wedge c)\} \equiv \{c\}$$

Consider a syntactic variant of K_1, namely $K_2 = \{a \wedge \neg a, c\}$, we get

$$K_2' = K_2 \boxminus a = \{(\top \wedge \neg\top) \vee (\bot \wedge \neg\bot) \vee (\top \wedge \neg\bot) \vee (\bot \wedge \neg\top), c\} \equiv \{c\}$$

So this example shows that $\boxminus$ is (to some extent) not syntax-sensitive, even in the presence of inconsistency. We come back to this aspect later (in particular, see Proposition 10).

From the examples so far it should be clear that inferences are not necessarily retained (even on the remaining signature). In particular, in Example 4 we have $\phi \models \neg b$ (in fact ϕ entails everything), but $\phi \boxminus a \not\models \neg b$. In fact, we obtain the following observation.

Proposition 3. *Let ϕ be a formula such that $\phi \models \perp$. Then there is $S \subseteq \mathsf{At}(\phi)$ such that $\phi \boxminus S \not\models \perp$.*

The above observation shows that, from the perspective of inconsistency-tolerant reasoning, $\boxminus$ is a sensible choice for a forgetting operation, since it allows the restoration of consistency in any case. Moreover, $\boxminus$ does also not *introduce* inconsistencies.

Proposition 4. *Let K be a knowledge base and $S \subseteq \mathsf{At}(K)$. If K is consistent then $K \boxminus S$ is consistent and $K \models K \boxminus S$.*

Our forgetting operator $\boxminus$ allows us to project the signature of a knowledge base to a subset of its signature. We define this concept in a general manner as follows.

Definition 5. For a knowledge base K and $S \subseteq \mathsf{At}(K)$, the *projection* of K onto S, denoted $K|_S$, is defined as $K|_S = K \boxminus (\mathsf{At}(K) \setminus S)$.

Example 6. We consider again $K_1 = \{a, \neg a \wedge c\}$ and $K_2 = \{a \wedge \neg a, c\}$. We get $K_1|_{\{c\}} \equiv \{c\}$ and $K_2|_{\{c\}} \equiv \{c\}$.

4. Minimal inconsistent and maximal consistent subsignatures

The notion of projection allows us to define analogues to the concepts of minimally inconsistent subsets and maximally consistent subsets of a knowledge base K (see again Definition 2), based on a more semantical perspective. In general, we say that a set $S \subseteq \mathsf{At}(K)$ is a *consistent subsignature* of K iff $K|_S$ is consistent, otherwise it is called an *inconsistent subsignature*.

Definition 6. Let K be a knowledge base.

1. $S \subseteq \mathsf{At}(K)$ is called a *minimal inconsistent subsignature* of K if
   a) $K|_S \models \perp$ and
   b) for all S' with $S' \subsetneq S$, $K|_{S'} \not\models \perp$.
2. $S \subseteq \mathsf{At}(K)$ is called a *maximal consistent subsignature* of K if
   a) $K|_S \not\models \perp$ and
   b) for all S' with $S \subsetneq S' \subseteq \mathsf{At}(K)$, $K|_{S'} \models \perp$.

Let $\mathsf{MISig}(K)$ and $\mathsf{MCSig}(K)$ denote the set of all minimal inconsistent subsignatures and the set of all maximal consistent subsignatures, respectively.

We furthermore say that a proposition $a \in \mathsf{At}(K)$ is a *free proposition* in K iff $a \notin S$ for all $S \in \mathsf{MISig}(K)$.

Example 7. We consider again the knowledge base $K_1 = \{a, \neg a \wedge c\}$. Here we have

$$\mathsf{MISig}(K_1) = \{\{a\}\} \qquad \mathsf{MCSig}(K_1) = \{\{c\}\}$$

For $K_2 = \{a \wedge \neg a, c\}$ we get likewise

$$\mathsf{MISig}(K_2) = \{\{a\}\} \qquad \mathsf{MCSig}(K_2) = \{\{c\}\}$$

For both cases, c is also a free proposition.

Example 8. Consider

$$K_3 = \{a \wedge b \wedge d, \neg a \vee \neg b, b \wedge \neg c, (c \vee \neg b) \wedge d\}$$

Here we get

$$\mathsf{MISig}(K_3) = \{\{a, b\}, \{b, c\}\} \qquad \mathsf{MCSig}(K_3) = \{\{a, c, d\}, \{b, d\}\}$$

and d is a free proposition.

Some straightforward observations are as follows.

Proposition 5. *Let K be a knowledge base.*

1. *K is consistent iff MISig$(K) = \emptyset$ iff MCSig$(K) = \{\mathsf{At}(K)\}$.*
2. *MCSig$(K) \neq \emptyset$.*

Observe that item 2 above includes the case where the only consistent signature is empty, so we may have $\mathsf{MCSig}(K) = \{\emptyset\}$.

A particular property of the set of all minimal inconsistent subsets $\mathsf{MIS}(K)$ is its monotony wrt. expansions of K. More precisely, if $K \subseteq K'$ then $\mathsf{MIS}(K) \subseteq \mathsf{MIS}(K')$. For the corresponding semantical counterpart $\mathsf{MISig}(K)$, this is not generally true.

Example 9. Consider $K_4 = \{a \vee b, \neg a \wedge \neg b\}$. Here we have $\mathsf{MISig}(K_4) = \{\{a, b\}\}$. However, adding the formula a gives us $\mathsf{MISig}(K_4 \cup \{a\}) = \{\{a\}\}$ and therefore $\mathsf{MISig}(K_4) \not\subseteq \mathsf{MISig}(K_4 \cup \{a\})$.

But $\mathsf{MISig}(K)$ behaves monotonically when it comes to expansions of the signature.

Proposition 6. *Let K be a knowledge base and $S \subseteq \mathsf{At}(K)$. Then MISig$(K \boxminus S) \subseteq$ MISig(K).*

Another particularly interesting property of the sets of minimal inconsistent subsets and the set of maximal consistent subsets of a knowledge base K is the *hitting set duality* [9]. For that let us recall the definition of a hitting set.

Definition 7. A *hitting set* of a set of sets $M = \{M_1, \ldots, M_n\}$ is a set $H \subseteq M_1 \cup \ldots \cup M_n$ such that $H \cap M_i \neq \emptyset$ for all $i = 1, \ldots, n$. A hitting set H is *minimal* if there is no other hitting set H' with $H' \subsetneq H$.

The hitting set duality for $\mathsf{MIS}(K)$ and $\mathsf{MCS}(K)$ says that H is a minimal hitting set of $\mathsf{MIS}(K)$ iff $K \setminus H \in \mathsf{MCS}(K)$ [9]. Interestingly, we obtain the same duality for $\mathsf{MISig}(K)$ and $\mathsf{MCSig}(K)$.

Theorem 1. *Let K be a knowledge base. H is a minimal hitting set of MISig(K) iff $\mathsf{At}(K) \setminus H \in$ MCSig(K).*

A corollary of the above result is that free propositions can also be characterised as those propositions that appear in all maximal consistent subsignatures (as it is the case with free formulas and maximal consistent subsets).

Corollary 1. *Let K a knowledge base. A proposition $a \in \mathsf{At}(K)$ is a free proposition in K iff $a \in S$ for all $S \in$ MCSig(K).*

We continue with a more detailed analysis and comparison of the behaviours of minimal inconsistent subsets and signatures. As for the former, removing free propositions from a signature does not influence the structure of the minimal inconsistent subsignatures, as the following proposition shows.

Proposition 7. *Let K be a knowledge base and $a \in \mathsf{At}(K)$ a free proposition of K. Then MISig$(K) =$ MISig$(K \boxminus a)$.*

Minimal inconsistent subsignatures are not only robust against the removal of free propositions from the signature (as the above proposition showed) but also against the removal of free formulas from the knowledge base (as the next proposition shows).

Proposition 8. *Let K be a knowledge base and α a free formula of K. Then MISig*$(K) =$ *MISig*$(K \setminus \{\alpha\})$.

The previous two propositions show that our notion of a minimal inconsistent subsignature is quite suitable for capturing the essence of the reasons why a knowledge base is inconsistent, since removal of "independent" syntactic (i. e., formulas) or semantic (i. e., propositions) information does not influence it. On the other hand, the next proposition shows that removing semantic information that is involved in inconsistency indeed has an influence.

Proposition 9. *Let K be a knowledge base and $a \in \mathsf{At}(K)$ not a free proposition of K. Then MISig*$(K \boxminus a) \subsetneq$ *MISig*(K).

Note that the syntactic counterpart of the previous observation, i. e., that the removal of non-free syntactic information changes the structure of minimal inconsistent subsignatures, does not hold in general.

Example 10. Consider $K_5 = \{a, \neg a, a \wedge \neg a\}$ with $\mathsf{MISig}(K_5) = \{\{a\}\}$. Note that $a \wedge \neg a$ is obviously not a free formula of K_5, but $\mathsf{MISig}(K_5 \setminus \{a \wedge \neg a\}) = \{\{a\}\} = \mathsf{MISig}(K_5)$.

However, the notion of minimal inconsistent subsignature still behaves as expected in the previous example. The formula $a \wedge \neg a$ actually describes redundant semantical information and its removal does not impact which parts of the signature are responsible for producing the inconsistency. As a matter of fact, the set of minimal inconsistent subsignatures is, to some extent, robust against syntactic variations, even in the presence of inconsistency.

Proposition 10. *Let K be a knowledge base and α, β formulas. Then MISig*$(K \cup \{\alpha, \beta\}) =$ *MISig*$(K \cup \{\alpha \wedge \beta\})$.

The observation made in the previous proposition is quite remarkable. It says that in terms of analysing inconsistency through the signature, it does not matter whether a knowledge base is defined as a set of formulas or a single conjunction of these formulas. While this is obvious when reasoning with consistent knowledge bases, the case of inconsistency usually requires a distinction between using the logical conjunction and the "comma" operator, see [15] for an excellent discussion on this topic. In particular, note that, in general, $\mathsf{MIS}(K \cup \{\alpha, \beta\}) \neq \mathsf{MIS}(K \cup \{\alpha \wedge \beta\})$ (e. g. obviously $\mathsf{MIS}(\{a, \neg a\}) \neq \mathsf{MIS}(\{a \wedge \neg a\})$). However, our framework allows for an equal treatment of these syntactic variations.

5. Application to inconsistency measurement

We now consider the application of our framework of minimal inconsistent subsignatures and maximal consistent subsignatures for *inconsistency measurement*. In general, an inconsistency measure [2, 12] is a quantitative means to assess the severity of inconsistencies in knowledge bases. Let $\mathbb{R}^{\geq 0}$ denote the set of non-negative real numbers.

Definition 8. An inconsistency measure I is any function $I : 2^{\mathcal{L}(\mathsf{At})} \to \mathbb{R}^{\geq 0}$ with $I(K) = 0$ iff K is consistent.

Many existing inconsistency measures are based on minimal inconsistent and maximal consistent subsets of K, see [16] for a survey. We here consider the measures I_{MI} and $I_{\text{MI-C}}$, defined via

$$I_{\mathrm{MI}}(K) = |\mathsf{MIS}(K)|$$
$$I_{\text{MI-C}}(K) = \sum_{M \in \mathsf{MIS}(K)} 1/|M|$$

for any knowledge base K, both introduced by Hunter and Konieczny [17], as well as the measures I_{MC} and I_{P}, defined via

$$I_{\mathrm{MC}}(K) = |\mathsf{MCS}(K)| + |\mathsf{SC}(K)| - 1$$
$$I_{\mathrm{P}}(K) = |\bigcup_{M \in \mathsf{MIS}(K)} M|$$

both by Grant and Hunter [18], where $\mathsf{SC}(K) = \{\phi \in K \mid \phi \models \perp\}$ is the set of self-contradicting formulas of K.

We can use minimal inconsistent and maximal consistent subsignatures in a similar manner as minimal inconsistent and maximal consistent subsets are being used in the above measures.

Definition 9. Let K be a knowledge base. Define functions I_{MISig}, $I_{\mathrm{MISig\text{-}C}}$, I_{MCSig} and I_{PSig} via

$$I_{\mathrm{MISig}}(K) = |\mathsf{MISig}(K)|$$
$$I_{\mathrm{MISig\text{-}C}}(K) = \sum_{M \in \mathsf{MISig}(K)} \frac{1}{|M|}$$
$$I_{\mathrm{MCSig}}(K) = |\mathsf{MCSig}(K)| + |\mathsf{SCSig}(K)| - 1$$
$$I_{\mathrm{PSig}}(K) = \left| \bigcup_{M \in \mathsf{MISig}(K)} M \right|$$

with

$$\mathsf{SCSig}(K) = \{a \in \mathsf{At}(K) \mid K\,|_{\{a\}} \models \perp\}$$

is the set of *self-contradicting* propositions.

In other words, I_{MISig} returns the number of minimal inconsistent subsignatures as a measure of inconsistency. $I_{\mathrm{MISig\text{-}C}}$ is a refinement of this idea and weighs each minimal inconsistent subsignature by its inverse size (with the intuition that larger minimal inconsistent subsignatures constitute a less obvious reason for inconsistency than smaller subsignatures). I_{MCSig} uses maximal consistent subsignatures instead of minimal inconsistent subsignatures. The intuition is that the more maximal consistent subsignatures there are, the more possible ways to resolve the inconsistency exist, and, therefore, the larger the inconsistency. We include the set of self-contradicting propositions here in order to ensure that the value 0 is only attained for consistent knowledge bases (if, e. g., we have $\mathsf{MISig}(K) = \{\{a\}\}$ then there is also just one maximal consistent subsignature and without adding $|\mathsf{SCSig}(K)|$ the inconsistency value would be 0). Finally, the measure I_{PSig} takes the number of propositions appearing in at least one minimal inconsistent subsignature as a measure of inconsistency.

Example 11. We consider again K_3 from Example 8 with

$$K_3 = \{a \wedge b \wedge d, \neg a \vee \neg b, b \wedge \neg c, (c \vee \neg b) \wedge d\}$$

and

$$\mathsf{MISig}(K_3) = \{\{a, b\}, \{b, c\}\} \qquad \mathsf{MCSig}(K_3) = \{\{a, c, d\}, \{b, d\}\} \qquad \mathsf{SCSig}(K_3) = \emptyset$$

Here we get

$$I_{\mathrm{MISig}}(K_3) = 2 \qquad I_{\mathrm{MISig\text{-}C}}(K_3) = 1$$
$$I_{\mathrm{MCSig}}(K_3) = 1 \qquad I_{\mathrm{PSig}}(K_3) = 3$$

We can first observe that all new measures are indeed inconsistency measures (Definition 8), i. e., they return the value 0 in the case of consistency (and only in this case).

	MO	IN	DO	SI	PY	AI	SM	PI	PP
I_{MISig}	✗	✓	✗	✓	✗	✓	✓	✓	✓
$I_{\text{MISig-C}}$	✗	✓	✗	✓	✗	✓	✓	✓	✓
I_{MCSig}	✗	✓	✗	✓	✗	✓	✓	✓	✓
I_{PSig}	✗	✓	✗	✓	✗	✓	✓	✓	✓

Table 1
Compliance of our new measures wrt. rationality postulates.

Proposition 11. *The functions I_{MISig}, $I_{MISig\text{-}C}$, I_{MCSig} and I_{PSig} are inconsistency measures.*

Inconsistency measures are usually evaluated wrt. *rationality postulates* [16]. Due to space limitations, we do not consider all postulates from [16], but focus on the most prominent ones. Let I be any function $I : 2^{\mathcal{L}(\mathsf{At})} \to \mathbb{R}^{\geq 0}$.

Monotony (MO) If $K \subseteq K'$ then $I(K) \leq I(K')$.

Free-formula Independence (IN) If $\alpha \in \mathsf{FREE}(K)$ then $I(K) = I(K \setminus \{\alpha\})$.

Safe-formula Independence (SI) If α is safe for K then $I(K) = I(K \setminus \{\alpha\})$.

Dominance (DO) If $\alpha \notin K$, $\alpha \not\models \perp$ and $\alpha \models \beta$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$.

Penalty (PY) If $\alpha \notin \mathsf{FREE}(K)$ then $I(K) > I(K \setminus \{\alpha\})$.

MO states that adding formulas cannot decrease the degree of inconsistency. IN and SI state that removing free (resp. safe) formulas does not change the degree of inconsistency. DO requires that replacing formulas with semantically stronger information cannot decrease the degree of inconsistency. PY is the complement of IN and states that removing non-free formulas decreases the degree of inconsistency. We will consider one further postulate from [19] that is concerned with syntax irrelevance and is rarely satisfied by existing inconsistency measures [16].

Adjunction Invariance (AI) $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$.

As we will see below, our measures (naturally) do not comply with the postulates MO, DO, and PY, since these are particularly concerned with the role of formulas in inconsistency. Due to Proposition 10 (which also directly leads all our measures to satisfy AI) all our measures are insensitive to the exact structure of the formulas. However, the introduction of minimal inconsistent subsignatures brings us into the position to introduce semantical counterparts of these postulates, which are particularly well suited to describe our new measures:

Signature-monotony (SM) For $S \subseteq \mathsf{At}(K)$ it is $I(K \boxminus S) \leq I(K)$.

Free-proposition independence (PI) If a is a free proposition in K, then $I(K) = I(K \boxminus a)$.

Proposition-penalty (PP) If $a \in \mathsf{At}(K)$ is not a free proposition in K, then $I(K) > I(K \boxminus a)$.

In other words, SM states that forgetting parts of the signature of a knowledge base cannot increase the degree of inconsistency. PI states that removing free propositions cannot change the degree of inconsistency. Conversely, PP states that removing non-free propositions decreases the degree of inconsistency.

Naturally, our new measures satisfy the newly introduced postulates. In summary, we can make the following statement on the compliance of our new measures with all the considered postulates.

Theorem 2. *The compliance of the measures I_{MISig}, $I_{MISig\text{-}C}$, I_{MCSig} and I_{PSig} to the rationality postulates is as shown in Table 1.*

As it can be seen from Table 1, all our new measures behave similarly with respect to the considered postulates. However, in the next section we will see that they behave differently in terms of complexity.

6. Computational complexity

We assume familiarity with the standard complexity classes P, NP and coNP, see [20] for an introduction. We also require knowledge of the complexity class DP, which is defined as $\mathsf{DP} = \{L_1 \cap L_2 \mid L_1 \in \mathsf{NP}, L_2 \in \mathsf{coNP}\}$. In other words, DP is the class of problems that are the intersection of a problem in NP and a problem in coNP. We also use complexity classes of the polynomial hierarchy that can be defined (using oracle machines) via $\Sigma_1^{\mathsf{P}} = \mathsf{NP}$, $\Pi_1^{\mathsf{P}} = \mathsf{coNP}$, and

$$\Sigma_i^{\mathsf{P}} = \mathsf{NP}^{\Sigma_{i-1}^{\mathsf{P}}} \qquad \Pi_i^{\mathsf{P}} = \mathsf{coNP}^{\Sigma_{i-1}^{\mathsf{P}}}$$

for all $i > 1$, where $\mathcal{C}^{\mathcal{D}}$ denotes the class of decision problems solvable in class $\mathcal{C}$ with access to an oracle that can solve problems that are complete for $\mathcal{D}$. In analogy to DP, we define DP_2 via $\mathsf{DP}_2 = \{L_1 \cap L_2 \mid L_1 \in \Sigma_2^{\mathsf{P}}, L_2 \in \Pi_2^{\mathsf{P}}\}$. We also consider classes of the counting polynomial hierarchy [21]. In particular, the class CNP is the class of counting decision problems where the corresponding decision problem is in NP. More precisely, let $H(x, y)$ be a predicate, where it can be decided in non-deterministic polynomial time if $H(x, y)$ is true. Given x and a natural number $k \in \mathbb{N}$, the decision problem of deciding whether there are at least k instances of y, such that $H(x, y)$ is true, is then in CNP (the class $\mathsf{C_=NP}$ is defined analogously by replacing "at least" with "exactly"). Similarly, the class $\# \cdot \mathsf{coNP}$ is a counting complexity class [22] that contains problems that upon input x return the number k of instances y such that $H(x, y)$ is true, which itself is a problem in coNP. Finally, FP is the class of *functional problems* that can be computed in deterministic polynomial time.

Complexity results regarding some basic decision problems are as follows.

Theorem 3. *Let K be a knowledge base and $S \subseteq \mathsf{At}(K)$.*

1. *Deciding whether S is a consistent subsignature of K is NP-complete.*
2. *Deciding whether S is a minimal inconsistent subsignature of K is DP-complete.*
3. *Deciding whether S is a maximal consistent subsignature of K is DP-complete.*

We consider now problems related to our new inconsistency measures from Section 5. As in [23], we consider the following problems (for a given inconsistency measure I):

EXACT$_I$	**Input**:	$K, x \in \mathbb{R}$
	Output:	TRUE iff $I(K) = x$
UPPER$_I$	**Input**:	$K, x \in \mathbb{R}$
	Output:	TRUE iff $I(K) \leq x$
LOWER$_I$	**Input**:	$K, x \in \mathbb{R}$
	Output:	TRUE iff $I(K) \geq x$
VALUE$_I$	**Input**:	K
	Output:	The value of $I(K)$

Due to Theorem 3, decision problems related to (in-)consistency of signatures have the same complexity as the corresponding problems on formulas (e. g., deciding whether a set of formulas is consistent is NP-complete as is the problem of deciding whether a set of propositions is a consistent subsignature). The observations made in [23] about the above problems for the corresponding measures defined on the formula level then also extend to our new measures quite easily. More precisely, we get the following characterisations regarding computational complexity.

Theorem 4. *The computational complexity of the problems EXACT$_I$, UPPER$_I$, LOWER$_I$, VALUE$_I$ wrt. the measures I_{MISig}, $I_{MISig\text{-}C}$, I_{MCSig}, and I_{PSig} is as shown in Table 2.*

7. Related work

Our approach has some connections to previous works, in particular inconsistency-tolerant reasoning with paraconsistent logics, which we will discuss in Section 7.1. Further related work will be discussed in Section 7.2.

	$\textsc{Exact}_I$	$\textsc{Upper}_I$	$\textsc{Lower}_I$	$\textsc{Value}_I$
I_{MISig}	$\mathsf{C_{=}NP}$-h	CNP-c	CNP-c	$\# \cdot$ coNP-c
$I_{\mathsf{MISig\text{-}C}}$	$\mathsf{C_{=}NP}$-h	CNP-h	CNP-h	$\mathsf{FP}^{\# \cdot \mathsf{coNP}}$
I_{MCSig}	$\mathsf{C_{=}NP}$-h	CNP-c	CNP-c	$\# \cdot$ coNP-c
I_{PSig}	DP_2	Π_2^P-c	Σ_2^P-c	$\mathsf{FP}^{\Sigma_2^P[\log]}$

Table 2
Computational complexity of problems related to our new measures; all statements are membership statements, except statements with an additional "-c" (which are completeness statements) or "-h" (which are hardness statements)

α	β	$\upsilon(\alpha \wedge \beta)$	$\upsilon(\alpha \vee \beta)$
T	T	T	T
T	B	B	T
T	F	F	T
B	T	B	T
B	B	B	B
B	F	F	B
F	T	F	T
F	B	F	B
F	F	F	F

α	$\upsilon(\neg\alpha)$
T	F
B	B
F	T

Table 3
Truth tables for propositional three-valued logic.

7.1. Relationships with paraconsistent reasoning

We briefly recall Priest's 3-valued logic for paraconsistent reasoning [24]. A *three-valued interpretation* υ on At is a function $\upsilon : \mathsf{At} \to \{T, F, B\}$ where the values T and F correspond to the classical true and false, respectively. The additional truth value B stands for *both* and is meant to represent a conflicting truth value for a proposition. The function υ is extended to arbitrary formulas as shown in Table 3. An interpretation υ satisfies a formula α (or is a 3-valued model of that formula), denoted by $\upsilon \models^3 \alpha$ if either $\upsilon(\alpha) = T$ or $\upsilon(\alpha) = B$. Define $\upsilon \models^3 K$ for a knowledge base K accordingly. Let $\mathsf{Mod}^3(K)$ denote the set of all 3-valued models of K. Note that the interpretation υ_0 defined via $\upsilon_0(a) = B$ for all $a \in \mathsf{At}$ is a model of every formula, so it makes sense to consider *minimal* models wrt. the usage of the paraconsistent truth value B. A model υ of a knowledge base K is a *minimal model* of K if it is a model and there is no other model υ' of K with $(\upsilon')^{-1}(B) \subsetneq (\upsilon)^{-1}(B)$. Let $\mathsf{MinMod}^3(K)$ denote the set of minimal models of K.

We can define an inference relation on $\mathsf{MinMod}^3(K)$ by considering all minimal models. More formally, define $\mid\!\sim^3$ via $K \mid\!\sim^3 \alpha$ iff $\upsilon \models^3 \alpha$ for all $\upsilon \in \mathsf{MinMod}^3(K)$ For an in-depth discussion of the properties of this inference relation and a refined version of it see [25].

For a three-valued interpretation υ define its *two-valued* projection $\omega_\upsilon : \upsilon^{-1}(\{T, F\}) \to \{\mathsf{true}, \mathsf{false}\}$ via $\omega_\upsilon(a) = \mathsf{true}$ iff $\upsilon(a) = T$ and $\omega_\upsilon(a) = \mathsf{false}$ iff $\upsilon(a) = F$, for all $a \in \upsilon^{-1}(\{T, F\})$. In other words, ω_υ is a two-valued interpretation that is only defined on those propositions, where υ gives a classical truth value, and the truth value assigned by ω_υ agrees with υ. We can capture the relationship between three-valued models and inconsistent signatures as follows.

Proposition 12. *Let υ be a three-valued interpretation. Then $\upsilon \models^3 \alpha$ iff $\omega_\upsilon \models (\alpha \boxminus \upsilon^{-1}(B))$ for every formula α.*

So a three-valued interpretation υ is a model of α, if and only if the classical part of υ is a model of the formula obtained by forgetting those propositions assigned to B.

Proposition 13. *Let K be a knowledge base.*

1. *If $\upsilon \in \mathsf{MinMod}^3(K)$ then $\upsilon^{-1}(B) \in \mathit{MISig}(K)$.*

2. *If* $S \in \mathit{MISig}(K)$ *then there is* $\upsilon \in \mathsf{MinMod}^3(K)$ *with* $\upsilon^{-1}(B) = S$.

In other words, S is a minimal inconsistent subsignature if and only if there is a minimal 3-valued model that assigns B to exactly those propositions in S. Note that while works such as [25] analyse the inferential capabilities of (refined versions of) $\mid\!\sim^3$, the properties of minimal inconsistent subsignatures have (in the form as we did in the preceding section) not been analysed in that line of research before.

7.2. Further related work

Lang and Marquis [4, 5] also considered forgetting as a means to restore consistency and to reason under inconsistency. However, they also did not consider notions such as minimal inconsistent and maximal consistent subsignatures nor the application to inconsistency measurement. In fact, our approach could be used as a pre-processing step for that work to identify propositions that need to be forgotten in order to restore consistency. A further particular related work is then [26], which proposes an inconsistency measure I_F that is based on forgetting. More precisely, $I_F(K)$ (for a knowledge base K) is defined as the minimal number of proposition occurrences (across all propositions) that have to be replaced by either $\top$ or $\perp$ such that the resulting knowledge base is consistent. Note that neither of our measures coincides with I_F, in particular because I_F allows that only some of the occurrences of a proposition are forgotten. In our approach, although proposition occurrences may be replaced differently (by either $\top$ or $\perp$), we always forget a proposition completely. Only this allowed to derive our notions of minimal inconsistent and maximal consistent subsignatures. As such, other then the general used method, there is no direct relationship between I_F and our framework. However, one can also note that I_F is one of the other few existing measures that also satisfies AI (invariance of $\{\alpha, \beta\}$ and $\{\alpha \wedge \beta\}$).

Brewka et. al [27] consider a generalisation of the concept of inconsistency called *strong inconsistency*. A subset $S \subseteq K$ of formulas of a knowledge base K, is strongly inconsistent if every S' with $S \subseteq S' \subseteq K$ is inconsistent. In classical propositional logic, a set S is strongly inconsistent if and only if it is inconsistent, but the two concepts differ when considering non-monotonic formalisms, such as *answer set programming* (ASP) [28, 29]. Strong inconsistency and minimal inconsistent subsignatures are, in general, two orthogonal concepts that address different aspects of inconsistency handling. However, it is conceivable to combine both of them in non-monotonic formalisms such as ASP, and obtain *minimal strongly inconsistent subsignatures*. For that, we basically have to substitute requirements pertaining to *inconsistency* by *strong inconsistency* (such as in Definition 6). This would open up applications of our inconsistency measures in those formalisms as well, see also [30, 31].

8. Discussion and conclusion

We considered an approach to analyse inconsistency in a knowledge base through forgetting parts of the signature such that the remaining knowledge base is consistent. In particular, we considered the notions of minimal inconsistent and maximal consistent subsignatures as counterparts to minimal inconsistent and maximal consistent subsets. Structurally, minimal inconsistent and maximal consistent subsignatures behave similarly as their subset-based counterparts, in particular, we showed that the hitting set duality is also satisfied by those notions. We analysed the application of these notions to the field of inconsistency measurement and devised several novel and interesting new inconsistency measures. Finally, we studied several problems in this context wrt. their computational complexity.

A possible venue for future work is to develop signature-based variants of inconsistency-tolerant reasoning methods based on maximal consistent subsets such as the one by Rescher and Manor [1] or Konieczny et al. [7]. The latter work proposes inference relations that only consider *some* of the maximal consistent subsets of a knowledge base, where the consideration of maximal consistent subsets is determined by a scoring function. Adapting those scoring functions for maximal consistent subsignatures will therefore give rise to further inference relations. Moreover, the reasoning approach of Brewka [32], who considers *stratified knowledge bases*—i. e. knowledge bases where formulas are ranked according to their preference—, could also be cast into our framework by considering *stratified signatures*.

Finally, one could generalise our approach from propositional logic to more practical formalisms such as description logics [33] and databases [34, 35].

Acknowledgments

The research reported here was partially supported by the Deutsche Forschungsgemeinschaft (project "Explainable Belief Merging", grant 465447331).

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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