Kinematics Principles for Inductive Reasoning from Conditional Belief Bases

Alexander Hahn[1,*], Gabriele Kern-Isberner[1], Lars-Phillip Spiegel[2] and Christoph Beierle[2]

[1]Technische Universität Dortmund (TU Dortmund University), August-Schmidt-Straße 1, 44227 Dortmund, Germany
[2]FernUniversität in Hagen, Universitätsstraße 11, 58097 Hagen, Germany

Abstract

The kinematics principle, originating from probability theory, captures the idea that conditional beliefs should be independent from changes in the plausibility of facts. Furthermore, conditional information with respect to exclusive cases should be relevant only to the respective case, but not influence others. This principle was recently adapted to belief revision of ranking functions and total preorders. In this paper, we propose a kinematics principle for non-monotonic inference relations induced from conditional belief bases. We derive this principle from the connection between inductive reasoning and belief revision of both total preorders and ranking functions. Moreover, we evaluate several inference operators from the literature with respect to this new kinematics principle for inductive reasoning.

Keywords

inductive reasoning, kinematics principle, belief change, conditionals, total preorders, ranking functions

1. Introduction

Intelligent agents often reason with incomplete background knowledge while assuming that the available information is sufficient to draw reasonable inferences. For example, when communicating scientific results, we often provide examples with some limited context Δ (e.g. about penguins) and assume that unrelated information (e.g. about sparrows) can be left out without distorting the picture. More formally, if all information in Δ is conditional information based on a common premise A, we would expect that additional information Δ' about the case of $\neg A$ does not influence the inferences for the case of A. So our inferences about the case of A should be the same whether we provide Δ or $\Delta \cup \Delta'$ as background knowledge. Moreover, it should not matter whether A actually holds or not.

These ideas are very close to the *kinematics principle* which has been studied for belief revision. This principle originates from probability theory, where it captures the idea that changes in the probability of facts should not impact the conditional beliefs given those facts [1]. An extension of this core idea called *Subset Independence* has been formulated for probabilistic belief change [2], essentially stating that the conditional beliefs should not only be independent from changes in the probability of facts, but also independent from changes in other conditional beliefs, as long as the premises refer to exclusive cases. This property has recently been adapted as *Generalized Ranking Kinematics* for the revision of ranking functions [3], and as *Qualitative Kinematics* for the revision of total preorders [3].

In this paper, we are going to derive a kinematics principle for inductive reasoning from the above-mentioned principles for belief revision. We are going to study its relationship to similar postulates from the literature, and also evaluate well-known inference relations with respect to this adapted principle. In summary, the main contributions of this paper are:

- We provide a more detailed account of inductive inference operators by emphasizing the underlying two-step process of inducing an epistemic state first, and an inference relation afterwards.

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*Corresponding author.
alexander.hahn@tu-dortmund.de (A. Hahn); gabriele.kern-isberner@cs.tu-dortmund.de (G. Kern-Isberner); lars-phillip.spiegel@fernuni-hagen.de (L. Spiegel); christoph.beierle@fernuni-hagen.de (C. Beierle)
0009-0008-6114-2594 (A. Hahn); 0000-0001-8689-5391 (G. Kern-Isberner); 0009-0001-1962-752X (L. Spiegel); 0000-0002-0736-8516 (C. Beierle)

- We extend the notion of conditionalization to inference relations and inductive inference operators.
- We propose a kinematics principle for inductive reasoning called (IRK), and evaluate several approaches from the literature with respect to (IRK) in order to highlight its relevance.
- As a byproduct of our research, we show that for every ranking function κ, there exists a conditional belief base Δ such that κ is a c-representation of Δ, which implies that all ranking functions (and thus all total preorders) can be expressed via revisions of uniform epistemic states.

The remainder of this paper is structured as follows. In Section 2, we briefly discuss related work. In Section 3, we provide formal preliminaries and recall basic definitions which are relevant for this paper. In Section 4, we recall the kinematics principles for the revision of ranking functions and total preorders. In Section 5, we recall inductive inference operators and investigate their connection to belief revision. In Section 6, we lift the concept of conditionalization to both inference relations and inductive inference operators. In Section 7, we present the main result of this paper, which is the kinematics principle for inductive reasoning. Afterwards, we evaluate several inference operators with respect to this principle in Section 8. In Section 9 we discuss the relationship between kinematics and syntax splitting. We end this paper with conclusions and some pointers to future work in Section 10.

2. Related Work

Our work builds upon recent work done by Kern-Isberner, Sezgin, and Beierle [3, 4] which laid the foundation for the kinematics principle we propose for inductive reasoning. A different adoption of subset independence (which our kinematics principle is based on) in a semi-quantitative reasoning framework can be found in [5].

In [6], the relationship between several kinds of belief change and inductive reasoning has been investigated, which is highly relevant for this paper since it enables us to transfer techniques used in belief revision to non-monotonic reasoning.

Similarly, in [7] the concept of *syntax splitting* was carried over from belief revision to non-monotonic reasoning. Similar to kinematics, syntax splitting postulates that non-relevant information should not influence inference results. However, as the name implies, syntax splitting is more syntactical in nature (although there are obvious implications for the underlying semantics), since relevance in the case of syntax splitting refers to the need to use common symbols in the logical language to express beliefs, and is expressed with marginalization. Kinematics, on the other hand, focuses on reasoning about semantically exclusive cases, which corresponds to conditionalization.

3. Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet $\Sigma = \{a, b, c, \ldots\}$. Formulas $A, B, C, \ldots$ are formed using the standard connectives $\wedge, \vee, \neg$. For conciseness of notation, we will write AB instead of $A \wedge B$ for conjunctions, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. The symbol $\top$ denotes an arbitrary propositional tautology. The set of all *possible worlds* (propositional interpretations) over Σ is denoted by Ω, and $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $\mathrm{Mod}(A)$. Similarly, for sets of propositions $\mathcal{S} \subseteq \mathcal{L}$, $\mathrm{Mod}(\mathcal{S})$ denotes the set of possible worlds that satisfy all elements of $\mathcal{S}$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $\mathrm{Mod}(A) \subseteq \mathrm{Mod}(B)$, as usual. Analogously, for sets of propositions $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$, $\mathcal{A} \models \mathcal{B}$ holds iff $\mathrm{Mod}(\mathcal{A}) \subseteq \mathrm{Mod}(\mathcal{B})$. Logical equivalence between formulas is denoted by $\equiv$. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

We also consider *conditionals* $(B|A) \in (\mathcal{L}|\mathcal{L})$ which express statements like "If A then plausibly B". The formula A is called the *antecedent*, and B is called the *consequent* of the conditional $(B|A)$. A *conditional belief base* Δ is a finite set of conditionals. For every possible world ω, let $\mathrm{ver}_\Delta(\omega) =$

$\{(B|A) \in \Delta \mid \omega \models AB\}$ and $\mathrm{fal}_\Delta(\omega) = \{(B|A) \in \Delta \mid \omega \models A\overline{B}\}$ be the sets of conditionals from Δ which are *verified* resp. *falsified* by ω. Semantics for conditionals and conditional belief bases are provided by *epistemic states* Ψ via an acceptance relation $\models$. For formulas A, we have $\Psi \models A$ iff $\Psi \models (A|\top)$, meaning that A is accepted in Ψ. This allows us to subsume plausible propositional formulas in terms of conditionals, which supports a more coherent view on reasoning and revision.

In this paper, we consider two types of (representations for) epistemic states: total preorders and ranking functions over possible worlds. *Total preorders* (TPOs) $\preceq \subseteq \Omega \times \Omega$ are total and transitive relations. As usual, $\omega_1 \prec \omega_2$ if $\omega_1 \preceq \omega_2$, but not $\omega_2 \preceq \omega_1$, and $\omega_1 \approx \omega_2$ if both $\omega_1 \preceq \omega_2$ and $\omega_2 \preceq \omega_1$. Total preorders represent plausibility orderings, with the most plausible worlds being located in the lowermost layer of $\preceq$ which we denote by $\min(\Omega, \preceq)$. More generally, if $\Omega' \subseteq \Omega$ is a subset of possible worlds, $\min(\Omega', \preceq)$ denotes the set of minimal worlds in Ω' according to $\preceq$. The preorder $\preceq$ is lifted to a relation between propositions[1] in the usual way: $A \preceq B$ if there is $\omega \models A$ such that $\omega \preceq \omega'$ for all $\omega' \models B$. A conditional $(B|A)$ is accepted by $\preceq$, denoted by $\preceq \models (B|A)$, if $AB \prec A\overline{B}$.

Ordinal Conditional Functions (OCFs, also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$ [8] assign degrees of implausibility, or surprise, to possible worlds. The degree of (im)plausibility of a formula A is defined by $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. A proposition A is accepted by κ, denoted by $\kappa \models A$, if $\omega \models A$ for all ω such that $\kappa(\omega) = 0$; this is equivalent to saying that $\kappa(\overline{A}) > 0$. This notion can be extended in a natural way to assign ranks to sets of formulas $\mathcal{S} \subseteq \mathcal{L}$ via $\kappa(\mathcal{S}) = \min\{\kappa(\omega) \mid \omega \models \mathcal{S}\}$. Conditionals are accepted by κ, written as $\kappa \models (B|A)$, if $\kappa(AB) < \kappa(A\overline{B})$. Note that these definitions are in full compliance with corresponding definitions for total preorders.

An epistemic state Ψ is called *TPO-representable* iff its (qualitative) conditional beliefs can be modeled via a total preorder, i.e. there exists a total preorder $\preceq$ such that $\Psi \models (B|A)$ iff $\preceq \models (B|A)$ for all $(B|A) \in (\mathcal{L}|\mathcal{L})$. The corresponding total preorder is then denoted as $\preceq_\Psi$. Clearly, OCFs and TPOs themselves are TPO-representable. A *uniform* epistemic state Ψ_u accepts only trivial conditionals, i.e. $\Psi_u \models (B|A)$ iff $A \models B$. For uniform TPOs $\preceq_u$ this means $\omega \approx_u \omega'$ for all ω, ω'; and for uniform OCFs κ_u we have $\kappa_u(\omega) = 0$ for all ω.

For the rest of this paper, we assume that all formulas A are *consistent* (i.e. $\mathrm{Mod}(A) \neq \emptyset$), all conditionals $(B|A)$ are *contingent* (i.e. $\mathrm{Mod}(AB) \neq \emptyset$ and $\mathrm{Mod}(A\overline{B}) \neq \emptyset$), and all conditional belief bases Δ are *(strongly) consistent* (i.e. there exists a TPO $\preceq$ with $\preceq \models \Delta$) [9]. This allows us to present our approach in a straightforward way without worrying about technical intricacies of these limit cases.

For an OCF κ and $A \in \mathcal{L}$, the *conditionalization* of κ by A is an OCF $\kappa|A : \mathrm{Mod}(A) \to \mathbb{N}$ such that $\kappa|A(\omega) = \kappa(\omega) - \kappa(A)$ for all $\omega \in \Omega$ [10]. Qualitative conditionalization of TPOs was first introduced in [7] and later refined in [4], improving the compatibility with conditionalization of OCFs.

Definition 1 (Conditionalized TPO). Let $\preceq$ be a total preorder over Ω and let $A \in \mathcal{L}$. The *conditionalization of $\preceq$ by A* is defined as a total preorder $\preceq|^A$ over $\mathrm{Mod}(A)$ such that

$$\omega_1 \preceq|^A \omega_2 \quad \text{iff} \quad \omega_1 \preceq \omega_2 \quad \text{for all } \omega_1, \omega_2 \in \mathrm{Mod}(A)\ .$$

The acceptance relation of $\preceq|^A$ is only defined for formulas $B, C \models A$ since $\preceq|^A \models (C|B)$ iff there is a world $\omega \in \mathrm{Mod}(BC) \cap \mathrm{Mod}(A)$ (since $\preceq|^A$ is defined over $\mathrm{Mod}(A)$) such that $\omega \prec|^A \omega'$ for all $\omega' \in \mathrm{Mod}(B\overline{C}) \cap \mathrm{Mod}(A)$. Note that this is a restriction of B, but not an additional restriction of C, since we already assume that $(C|B)$ is contingent, i.e. $B \models A$ already implies $C \not\models \neg A$. Hence both the verification and the falsification of $(C|B)$ must imply A.

4. Kinematics Principle for Revision

The kinematics principle for belief revision originates from probability theory. The assumption of probability kinematics [1] states that conditional probabilities $P(\cdot|A)$ given some fact A should be preserved when the probability $P(A)$ of the fact itself changes.

[1]Note that this lifted relation over propositions is not necessarily a total preorder. Hence, when referring to TPOs, we always mean the underlying order over possible worlds.

This principle was adapted as *Generalized Ranking Kinematics (GRK)* to ranking theory by [3]. The definition of (GRK) relies on so-called *case splittings*, which are defined below.

Definition 2 (Case Splitting)**.** Let Δ be a conditional belief base Δ and let $A_1, \dots, A_n \in \mathcal{L}$ be exclusive and exhaustive formulas. Then $A_1, \dots, A_n$ are called a *case splitting* (or *premise splitting*) of Δ if there are subsets $\Delta_1, \dots, \Delta_n \subseteq \Delta$ such that $\Delta = \Delta_1 \cup \dots \cup \Delta_n$ and for every $1 \le i \le n$, the antecedents of the conditionals in Δ_i imply A_i.

Note that exclusiveness is the stronger requirement here, since exhaustiveness can always be achieved by adding an additional case. For instance, if the premises of $\Delta_1, \dots, \Delta_n \subseteq \Delta$ respectively imply non-exhaustive (but exclusive) cases $A_1, \dots, A_n$, we can add the remaining case $\neg(A_1 \vee \dots \vee A_n)$ representing the empty set of conditionals $\emptyset \subseteq \Delta$.[2]

Now the postulate of Generalized Ranking Kinematics for OCF-revision operators $*$ reads as follows.

(GRK) Let $\Delta = \Delta_1 \cup \dots \cup \Delta_n$ be a set of conditionals with a case splitting $A_1, \dots, A_n$. Let $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \dots, n\}$. Then for all OCFs κ and all cases A_i the following holds:

$$(\kappa * (\Delta \cup \{S\}))|A_i = (\kappa|A_i) * \Delta_i \tag{1}$$

The (GRK) principle combines two notions of relevance, which we illustrate by considering two special cases which yield crucial properties for revision operators:

(CaseRel) $(\kappa * \Delta)|A_i = (\kappa|A_i) * \Delta_i$.

(CaseIrr) $(\kappa * \{S\})|A_i = \kappa|A_i$.

The first property, *case relevance*[3], captures the idea that when focusing on the case A_i, only the conditionals talking about this case should be relevant. Therefore, when conditionalizing the revision result $\kappa * \Delta$ by A_i, we should obtain the same result as if we conditionalized first and then performed the revision with only the relevant information Δ_i locally. Note that the exclusiveness of the cases plays a crucial role here. The second postulate, *case irrelevance*, concerns itself with facts: When talking about what would happen in the case of A_i, it should not matter how plausible A_i actually is. Therefore, learning about the plausibility of any of the cases should not change the conditionalized revision result.

The following short proposition summarizes these implications of (GRK). In order to connect (IRK) to (CaseRel), we need the following basic postulate from [3].

(TI*) $\Psi * (\Delta \cup \{\top\}) = \Psi * \Delta$

Proposition 1. *If (TI*) holds, then (GRK) implies both (CaseRel) and (CaseIrr).*

Proof. (CaseRel) follows from (GRK) via $S \equiv \top$ and (TI*). (CaseIrr) follows from (GRK) via $\Delta = \emptyset$. □

Note that the proposition above only claims an implication, not equivalence, since (CaseRel) and (CaseIrr) together do not restrict how $*$ handles a revision where conditional and propositional information is provided at the same time.

The kinematics principle for the revision of ranking functions (GRK) was adapted as a qualitative kinematics principle by [4] for the revision of epistemic states represented by total preorders.

(QK) Let $\Delta = \Delta_1 \cup \dots \cup \Delta_n$ be a set of conditionals with a case splitting $A_1, \dots, A_n$. Let $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \dots, n\}$. Then for all total preorders $\preceq$ and all cases A_i the following holds:

$$(\preceq * (\Delta \cup \{S\}))|A_i = (\preceq|^{A_i}) * \Delta_i \tag{2}$$

[2]This also means that one can always achieve a trivial case splitting of any conditional belief base: one case is the disjunction of all premises appearing in Δ, and the other case is the negation of this disjunction.

[3]This property is called (GRK$^{\text{weak}}$) in [3].

We end this section with the following lemma, which formalizes an important property of conditional belief bases with case splittings: It is impossible for one possible world to verify (or falsify) conditionals from different subsets because of the exclusiveness of the premises.

Lemma 2. *Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ be a set of conditionals, and let $A_1, \ldots, A_n$ be a case splitting of Δ such that for all $1 \leq i \leq n$, the premises in Δ_i imply A_i. Then for every possible world $\omega \in \Omega$, there exists an $1 \leq j \leq n$ such that*

$$(\mathrm{ver}_\Delta(\omega) \cup \mathrm{fal}_\Delta(\omega)) \subseteq \Delta_j \ . \tag{3}$$

Proof. Let $\omega \in \Omega$. If $(\mathrm{ver}_\Delta(\omega) \cup \mathrm{fal}_\Delta(\omega)) = \emptyset$ then Equation (3) holds for all Δ_i. Otherwise, there exists a conditional $(C|B) \in \Delta$ such that $\omega \models B$. Since the cases $A_1, \ldots, A_n$ are exhaustive, there must be a set Δ_j such that $(C|B) \in \Delta_j$ and $B \models A_j$. Consequently, $\omega \models A_j$ as well. Because of the exclusiveness of the premises, we have $\omega \models \overline{A_k}$ for all $k \neq j$. Hence ω cannot verify or falsify any conditionals in $\Delta \setminus \Delta_j$. Therefore, Equation (3) holds. □

5. Inductive Inference Operators

Many non-monotonic inference relations from the literature are *inductive* in the sense that they depend on explicitly given background beliefs, which we will represent as a set of conditionals Δ. In [7] the following axioms expressing *Direct Inference* and *Trivial Vacuity* are given for inductive inference relations:

(DI) $(B|A) \in \Delta$ implies $A \mid\!\sim_\Delta B$.

(TV) If $\Delta = \emptyset$, then $A \mid\!\sim_\Delta B$ only if $A \models B$.

Definition 3 (Inductive Inference Operators $\mathbf{C}$)**.** An *inductive inference operator* from conditional belief bases on $\mathcal{L}$ is a mapping $\mathbf{C}$ that assigns to each conditional belief base $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ an inference relation $\mid\!\sim_\Delta$ on $\mathcal{L}$ such that (DI) and (TV) are satisfied: $\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta$.

When defining an inductive inference operator, we often assume that the operator first constructs an epistemic state from the belief base Δ, and then yields the inference relation induced by the epistemic state. For example, in [7], classes of *model-based inductive inference operators* are defined which map conditional belief bases to TPOs or OCFs.

In order to make this two-step process more explicit, we define two inductive operators in this section: *inductive epistemic operators*, which map conditional belief bases to epistemic states, and *epistemic inference operators*, which map epistemic states to inference relations. This distinction adds a formal structure to what is often done or assumed implicitly, and will enable us to investigate both steps independently.

5.1. Epistemic Inference Operators

We start with the definition of epistemic inference operators, which essentially link inference to acceptance of conditionals.

Definition 4 (Epistemic Inference Operator $\mathbf{I}$)**.** An *epistemic inference operator* over $\mathcal{L}$ and a class of epistemic states $\mathbb{S}$ is a mapping from epistemic states $\Psi \in \mathbb{S}$ to inference relations $\mid\!\sim_\Psi \subseteq \mathcal{L} \times \mathcal{L}$, $\mathbf{I} : \Psi \mapsto \mid\!\sim_\Psi$, such that $A \mid\!\sim_\Psi B$ iff $\Psi \models (B|A)$.

We choose the symbol $\mathbf{I}$ to denote epistemic inference operators since $\mathbf{I}(\Psi)$ can be read as "the **i**nference relation induced by Ψ". There are two specific epistemic inference operators which we will make use of in this paper.

- $\mathbf{I}^{\mathrm{tpo}} : \mathbb{TPO} \to 2^{\mathcal{L} \times \mathcal{L}}$ maps $\preceq \mapsto \mid\!\sim_\preceq$ such that $A \mid\!\sim_\preceq B$ iff $AB \prec A\overline{B}$.
- $\mathbf{I}^{\mathrm{ocf}} : \mathbb{OCF} \to 2^{\mathcal{L} \times \mathcal{L}}$ maps $\kappa \mapsto \mid\!\sim_\kappa$ such that $A \mid\!\sim_\kappa B$ iff $\kappa(AB) < \kappa(A\overline{B})$.

Note that we have $\mathbf{I}^{\mathrm{ocf}}(\kappa) = \mathbf{I}^{\mathrm{tpo}}(\preceq_\kappa)$.

5.2. Inductive Epistemic Operators

Next we define inductive epistemic operators mapping conditional knowledge bases to epistemic states.

Definition 5 (Inductive Epistemic Operator $\mathbf{E}$)**.** An *inductive epistemic operator* over $\mathcal{L}$ is a mapping from conditional belief bases $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ to epistemic states $\Psi_\Delta \in \mathbb{S}$, $\mathbf{E} : \Delta \mapsto \Psi_\Delta$, such that the following properties are satisfied:

(DI$^\Psi$) $\mathbf{E}(\Delta) \models \Delta$.

(TV$^\Psi$) $\mathbf{E}(\emptyset) = \Psi_u$.

The term $\mathbf{E}(\Psi)$ can be read as "the **e**pistemic state induced by Ψ", hence the choice of the symbol $\mathbf{E}$ to denote inductive epistemic operators.

Recently, a more general form of such mappings were investigated from a philosophical perspective in [6]. The authors of [6] claim that inductive reasoning, i.e., the completion of some $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ to a full-fledged epistemic state Ψ_Δ, can be considered a special case of belief revision. They introduced an operator ind, mapping Δ to the result of a belief revision process: $\mathrm{ind}_{\Psi_{bk}}(\Delta) = \Psi_{bk} * \Delta$, where Ψ_{bk} represents a prior epistemic state with background knowledge, and $*$ is some suitable revision operator. In the most simple scenario, assuming no (relevant) background beliefs, we have $\Psi_{bk} = \Psi_u$. In this paper, we restrict ourselves to this base case of inductive reasoning, i.e., we do not consider inductive inference with separate background knowledge states. Accordingly, we define the following inductive epistemic operator:

$$\mathbf{E}^* = \Psi_u * \Delta \;, \tag{4}$$

where $\Psi_u \in \{\kappa_u, \preceq_u\}$ and $*$ is a suitable revision operator such that (DI$^\Psi$) and (TV$^\Psi$) are satisfied.

By composing inductive epistemic operators and epistemic inference operators, we obtain inductive inference operators.

Proposition 3. *Let* $\mathbf{E} : 2^{(\mathcal{L}|\mathcal{L})} \to \mathbb{S}$ *be an inductive epistemic operator and let* $\mathbf{I} : \mathbb{S} \to \mathcal{L} \times \mathcal{L}$ *be an epistemic inference operator. Then* $\mathbf{C} = \mathbf{I} \circ \mathbf{E}$ *is an inductive inference operator.*

Proof. Let $\Delta \subseteq (\mathcal{L}|\mathcal{L})$. It is clear from the definition of $\mathbf{E}$ and $\mathbf{I}$ that $\mathbf{C}(\Delta)$ is well-defined and yields ahan inference relation $\mid\!\sim_\Delta$. We need to show that this inference relation satisfies (DI) and (TV).

For (DI), let $(B|A) \in \Delta$. Because of (DI$^\Psi$), $\mathbf{E}(\Delta) \models (B|A)$. Then it follows immediately from the definition of $\mathbf{I}$ that $A \mid\!\sim_\Delta B$ since $\mid\!\sim_\Delta = \mid\!\sim_{\mathbf{E}(\Delta)}$.

For (TV), we have $\mathbf{C}(\emptyset) = \mid\!\sim_{\Psi_u}$ because of (TV$^\Psi$), and $\Psi_u \models (B|A)$ iff $A \models B$ by definition. □

5.3. Inductive Inference via Belief Revision

With Proposition 3, we have now established that belief revision operators can be used to construct inductive inference operators. Next, we are going to show that all model-based inductive inference operators $\mathbf{C}^{\mathrm{ocf}}$ and $\mathbf{C}^{\mathrm{tpo}}$ as presented in [7] can be expressed in this way, i.e., it is possible to obtain any desired OCF or TPO from the uniform epistemic state via belief revision. Our proof for this uses Kern-Isberner's c-Revisions [11].

Definition 6 (c-Revision)**.** Let κ be an OCF and $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ a set of conditionals. Then a *c-revision of* κ *by* Δ is an OCF $\kappa^* = \kappa *^c \Delta$ of the form

$$\kappa^*(\omega) = \eta_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{5}$$

with *impact factors* $\eta_i \geq 0$ for each $(B_i|A_i)$, satisfying

$$\eta_i > \min_{\omega \models A_i B_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \eta_j \Big\} - \min_{\omega \models A_i \overline{B}_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \eta_j \Big\} \tag{6}$$

and a *normalization factor* $\eta_0 \in \mathbb{N}$ to ensure $\kappa^{-1}(0) \neq \emptyset$, guaranteeing that κ^* is again an OCF.

Observe that having κ_u as the prior OCF in Equation (5) simplifies the equation a lot, since $\kappa_u(\omega) = 0$ for all ω, i.e., the ranks in κ^* only depend on the interactions between the conditionals in Δ (together with their associated impact factors) and not on some prior epistemic state. Since we only consider consistent belief bases, also no normalization via η_0 is needed. Such OCFs, which can be represented as a sum of impact factors, are called *c-representations* (of the respective conditional belief base Δ).

Using c-revisions, we can show the following proposition. The quite technical proof has been omitted due to space constraints.

Proposition 4. *Let κ be an OCF. Then there are a conditional belief base Δ and a c-revision operator $*^c$ such that $\kappa = \kappa_u *^c \Delta$.*

Note that Proposition 4 has far-reaching consequences. It essentially says that every OCF κ is a c-representation (of some conditional belief base Δ), i.e., whenever we want to prove a property for all OCFs, it suffices to show that it holds for c-representations (as long as we may choose Δ freely).

Moreover, the following proposition shows that all model-based inductive inference operators $\mathbf{C}^{\mathrm{ocf}}$ and $\mathbf{C}^{\mathrm{tpo}}$ as presented in [7] can be expressed using belief revision as the induction mechanism.

Proposition 5. *Every model-based inductive inference operator $\mathbf{C}^{\mathrm{ocf}}$ can be represented as $\mathbf{C}^{\mathrm{ocf}} = \mathbf{I}^{\mathrm{ocf}} \circ \mathbf{E}^*$ for some revision-based inductive inference operator $\mathbf{E}^*$.*

Proof. Let $\mathbf{C}^{\mathrm{ocf}}$ be an inductive inference operator such that for every Δ, we have $\mathbf{C}^{\mathrm{ocf}}(\Delta) = \mathrel{|\!\sim}_\kappa$ for some OCF κ. Then we can define $\mathbf{E}^*$ such that $\mathbf{E}^*(\Delta) = \kappa_u * \Delta$ with a revision operator $*$ such that $\kappa_u * \Delta = \kappa_u *^c \Delta' = \kappa$ for some suitable $*^c$ and Δ'. The existence of $*$ follows from Proposition 4. □

Using qualitative c-revisions from [4], it is straight-forward to prove analogous results to Propositions 4 and 5 for total preorders, i.e. every TPO (and hence every $\mathbf{C}^{\mathrm{tpo}}$) can be constructed via belief revision. In that sense, the proposition above supports the claim of [6] that inductive reasoning can be understood as a special case of belief revision for TPOs and OCFs.

6. Conditionalization of Inference and Induction

Conditionalization is an important operation on epistemic states, enabling an agent to (temporarily) focus on some specific case A, resulting in an epistemic state where $\neg A$ is considered as impossible. This is crucial for efficient reasoning as well as for reasoning about hypothetical scenarios.

In this section we are going to lift the concept of conditionalization to both inference relations and inductive inference operators, with the goal of capturing an agent's inference behavior under the assumption that a specific proposition A holds.

6.1. Conditionalized Inference Relations

When conditionalizing a ranking function (or a total preorder) by a formula A, the result is a ranking function (resp. total preorder) over the models of A, while all models of $\neg A$ are excluded. Since inference relations are defined over formulas, we need to restrict the language accordingly. Therefore, we introduce the *scope* of a formula A, which is the set of all formulas that imply A.

Definition 7 (Scope)**.** The *scope* of a formula $A \in \mathcal{L}$ is defined as $\mathrm{Sc}(A) := \{B \in \mathcal{L} \mid B \models A\}$.

The conditionalization of an inference relation by a formula A should focus on the inferences that can be drawn in the case of A being true, considering all other cases as impossible.

Definition 8 (Conditionalized Inference Relation $\mathrel{|\!\sim}|^A$)**.** The *conditionalization* of an inference relation $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ by $A \in \mathcal{L}$ is an inference relation $\mathrel{|\!\sim}|^A \subseteq \mathrm{Sc}(A) \times \mathrm{Sc}(A)$ such that for all $B, C \in \mathrm{Sc}(A)$:

$$B \mathrel{|\!\sim}|^A C \quad \text{iff} \quad B \mathrel{|\!\sim} C\,.$$

In other words, $\mathrel{|\!\sim}|^A = \mathrel{|\!\sim} \cap (\mathrm{Sc}(A) \times \mathrm{Sc}(A))$. The idea behind defining $\mathrel{|\!\sim}|^A$ over the scope of A is that for evaluating inferences based on models, only models of A should be relevant.

The following proposition shows that conditionalization of inference relations induced by total preorders is compatible with conditionalization of the total preorders themselves.

Proposition 6. *If $\mathrel{|\!\sim}_{\preceq}$ is an inference relation induced by a total preorder $\preceq$ and $A \in \mathcal{L}$, then for all $B, C \in \mathrm{Sc}(A)$, it holds that $B \mathrel{|\!\sim}_{\preceq}|^A\, C$ iff $B \mathrel{|\!\sim}_{\preceq|^A} C$.*

Proof. Let $A \in \mathcal{L}$ and $B, C \in \mathrm{Sc}(A)$. We have $B \mathrel{|\!\sim}_{\preceq}|^A\, C$ iff $B \mathrel{|\!\sim}_{\preceq} C$. This is equivalent to $BC \prec B\overline{C}$, which holds iff there exists $\omega \in \mathrm{Mod}(BC)$ such that $\omega \prec \omega'$ for all $\omega' \in \mathrm{Mod}(B\overline{C})$. Since $\mathrm{Mod}(BC) \subseteq \mathrm{Mod}(A)$, $BC \prec B\overline{C}$ holds iff $BC \prec|^A\, B\overline{C}$. This is equivalent to $B \mathrel{|\!\sim}_{\preceq|^A} C$. □

Corollary 7. *If $\mathrel{|\!\sim}_{\kappa}$ is an inference relation induced by an OCF κ, and $A \in \mathcal{L}$, then for all $B, C \in \mathrm{Sc}(A)$, it holds that $B \mathrel{|\!\sim}_{\kappa}|^A\, C$ iff $B \mathrel{|\!\sim}_{\kappa|^A} C$.*

Proof. Observe that $\mathrel{|\!\sim}_{\kappa} = \mathrel{|\!\sim}_{\preceq_{\kappa}}$. The corollary then follows immediately from Proposition 6. □

6.2. Conditionalized Inductive Inference

After defining conditionalization for inference relations, we are now ready to apply the concept of conditionalization to inductive inference operators as well.

Definition 9 (Conditionalized Inductive Inference Operator $\mathbf{C}|^A$)**.** Let $\mathbf{C}$ be an inductive inference operator on $\mathcal{L}$ and let $A \in \mathcal{L}$. Then the *conditionalization of* $\mathbf{C}$ *by* A is an inductive inference operator $\mathbf{C}|^A$ on $\mathrm{Sc}(A)$ such that for all $\Delta \subseteq (\mathrm{Sc}(A)|\,\mathrm{Sc}(A))$ it holds that $\mathbf{C}|^A(\Delta) = \mathbf{C}(\Delta)|^A$.

The following proposition shows that the conditionalization of inductive inference operators is well-defined. The proof is immediate from Definition 8.

Proposition 8. *If $\mathbf{C}$ is an inductive inference operator, then for every $A \in \mathcal{L}$, $\mathbf{C}|^A$ satisfies (DI) and (TV).*

7. A Kinematics Principle for Inductive Reasoning

The core ideas behind (GRK) is that for revision with case-specific information, the plausibility of the case and additional information about other cases should not influence the local revision result. As a result of (GRK), conditionalization and revision are interchangeable as long as the new information applies to exclusive cases. We can apply similar ideas to inductive inference: When we reason based on conditional information concerning exclusive cases, then case-specific information about one case should not influence reasoning about the other cases.

In order to work towards a version of (GRK) for inductive reasoning, let us first examine the consequences of (GRK) for applying the ind-operator. Let κ be a ranking function, let Δ be a conditional knowledge base, and let $A_1, \ldots A_n$ be a premise splitting of $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ (with the premises in Δ_i implying A_i for all $1 \le i \le n$). Moreover, let $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \ldots, n\}$. Then (GRK) implies the following:

$$\mathrm{ind}_{\kappa}(\Delta \cup \{S\})|A_i = (\kappa * (\Delta \cup \{S\}))|A_i = (\kappa|A_i) * \Delta_i = \mathrm{ind}_{\kappa|A_i}(\Delta_i)\,. \tag{7}$$

More intuitively, (GRK) implies that conditionalization of an epistemic state induced via belief revision and conditionalization of the background knowledge κ are interchangeable as long as there is an appropriate case splitting in Δ. This is a first interesting result, but we can go one step further. Applying (GRK) again yields:

$$\mathrm{ind}_{\kappa}(\Delta \cup \{S\})|A_i = (\kappa|A_i) * \Delta_i = (\kappa * \Delta_i)|A_i = \mathrm{ind}_{\kappa}(\Delta_i)|A_i\,. \tag{8}$$

Since inductive epistemic operators are a special case of ind, Equations (7) and (8) above directly imply

$$\mathbf{E}^*(\Delta \cup \{S\})|A_i = \mathbf{E}^*(\Delta_i)|A_i \tag{9}$$

for inductive epistemic operators $\mathbf{E}^*$ based on OCF-revision operators $*$ satisfying (GRK). Since $\mathbf{I}^{\mathrm{ocf}}$ is compatible with conditionalization according to Corollary 7, we obtain

$$(\mathbf{I}^{\mathrm{ocf}} \circ \mathbf{E}^*)(\Delta \cup \{S\})|^{A_i} = (\mathbf{I}^{\mathrm{ocf}} \circ \mathbf{E}^*)(\Delta_i)|^{A_i} \; . \tag{10}$$

Analogously, we can derive Equations (9) and (10) also for TPO-revision operators satisfying (QK).

Recall that $(\mathbf{I}^{\mathrm{ocf}} \circ \mathbf{E}^*)$ is an inductive reasoning operator according to Proposition 3. By generalizing Equation (10) to arbitrary inductive inference operators, we arrive at the following postulate of *Inductive Reasoning Kinematics.*

(IRK) Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ be a set of conditionals, and let $A_1, \ldots, A_n$ be a case splitting of Δ. Let $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \ldots, n\}$. Then for all cases A_i, the following holds:

$$\mathbf{C}(\Delta \cup \{S\})|^{A_i} = \mathbf{C}|^{A_i}(\Delta_i) \tag{11}$$

Observe that the conditionalization on the right-hand side of Equation (11) is applied to the inductive reasoning operator instead of the output inference relation. This emphasizes that the whole inductive reasoning process may happen within a limited scope. As long as (IRK) is fulfilled, we can inductively reason in closed local semantic contexts without worrying about the influence of non-relevant information like the plausibility of facts or conditional information regarding excluded cases.

Example 1. Suppose that we were writing a scientific paper and wanted to provide an example about the properties of a bird in the case that it was a penguin. In order to convince the reader that we were not hiding any important information, we could construct our whole inference relation including inferences about eagles, owls, sparrows, and other types of birds. Afterwards, we could focus only on the relevant parts about penguins.

However, assuming that (IRK) holds, we could significantly shorten the reasoning process: We would not have to construct the whole inference relation (which could require constructing a representation of our complete epistemic state about birds), but stay comfortably within the scope about penguins and only consider the relevant conditional information.

Moreover, we would expect the reader of our example not to care about whether the bird in question actually happens to be a penguin or not, but to recognize that the example is hypothetical and that the plausibility of the bird being a penguin is irrelevant.

On the other hand, from a more technical perspective, applying the conditionalization to the inference operator instead of the induced inference relation in Equation (11) is not an additional restriction (or less of a restriction) of the induced inference relation, since (IRK) can equivalently be formulated as follows by expanding the definition of the inference operator in question: for all A_i and all $B, C \in \mathrm{Sc}(A_i)$ $(1 \leq i \leq n)$, it should hold that $B \mathrel{|\!\sim}_{\Delta \cup \{S\}}|^{A_i} C$ iff $B \mathrel{|\!\sim}_{\Delta_i}|^{A_i} C$.

We formalize the connection between the kinematics principles for revision—(GRK) and (QK)—and (IRK) with the following proposition. The proof is very similar to the initial derivation of the principle starting from Equation (7) above.

Proposition 9. *Let $*$ be a revision operator for OCFs or TPOs that satisfies (GRK) or (QK), respectively. Then the inductive inference operator $\mathbf{C}^* = (\mathbf{I} \circ \mathbf{E}^*)$ (with $\mathbf{I} \in \{\mathbf{I}^{\mathrm{ocf}}, \mathbf{I}^{\mathrm{tpo}}\}$ chosen suitably) satisfies (IRK).*

Similar to (GRK), we can split (IRK) into two notions of (ir)relevance. Let $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$ be a set of conditionals, and let $A_1, \ldots, A_n$ be a case splitting of Δ. Let $S = \bigvee_{j \in J} A_j$ with $\emptyset \neq J \subseteq \{1, \ldots, n\}$.

(CaseRel$^{\mathsf{IR}}$) $\mathbf{C}(\Delta)|^{A_i} = \mathbf{C}|^{A_i}(\Delta_i)$ for all cases A_i.

(CaseIrr$^{\text{IR}}$) $\mathbf{C}(\{S\})|^{A_i} = \mathbf{C}|^{A_i}(\emptyset)$ for all cases A_i.

Just like (CaseRel), the postulate (CaseRel$^{\text{IR}}$) states that when reasoning about one of the cases A_i, considering the set Δ_i is sufficient. The postulate (CaseIrr$^{\text{IR}}$) states that the plausibility of the cases should be truly irrelevant for conditional reasoning, since we obtain the same inferences as if the belief base was empty. Together with (TV), this amounts to obtaining only classical consequences. In order for (IRK) to imply these sub-postulates, we again need to assume that tautologies do not influence the inductive reasoning mechanism:

(TI$^{\text{IR}}$) $\mathbf{C}(\Delta \cup \{\top\}) = \mathbf{C}(\Delta)$

Proposition 10. *If (TI^{IR}) holds, then (IRK) implies both (CaseRel^{IR}) and (CaseIrr^{IR}).*

Proof. From (IRK), (CaseRel$^{\text{IR}}$) follows via $S \equiv \top$ and (TI$^{\text{IR}}$), and (CaseIrr) follows via $\Delta = \emptyset$. □

We conclude this section by showing that (IRK) is not self-evident. In particular, when $\mathbf{E}(\Delta)$ and $\mathbf{E}(\Delta_i)$ are chosen arbitrarily, it is easy to violate (IRK), as the following example shows.

Example 2. Let $\Delta = \{(b|a), (\overline{b}|\overline{a})\}$. Clearly $A_1 = a, A_2 = \overline{a}$ is a case splitting of Δ with $\Delta_1 = \{(b|a)\}$ and $\Delta_2 = \{(\overline{b}|\overline{a})\}$. Now let $\mathbf{C} = \mathbf{I}^{\text{tpo}} \circ \mathbf{E}$ be an inductive inference operator such that

$$\begin{aligned} \mathbf{E}(\Delta) : &\quad abc \prec_\Delta ab\overline{c} \prec_\Delta \overline{a}\overline{b}c \prec_\Delta \dots , \\ \mathbf{E}(\Delta_1) : &\quad ab\overline{c} \prec_{\Delta_1} abc \prec_{\Delta_1} \dots , \end{aligned}$$

where the dots "..." above represent an arbitrary order over all remaining worlds in Ω and in $\text{Mod}(a)$, respectively. Then we have $ab \mid\!\sim_\Delta ac$, but $ab \mid\!\sim_{\Delta_1} a\overline{c}$. Therefore, (IRK) is not fulfilled.

The problem in the example above essentially amounts to the atom c not being restricted by any conditional, allowing $\mathbf{E}$ to choose the relative plausibility of the worlds abc and $ab\overline{c}$ freely. A similar counterexample for inference relations induced from OCFs can be constructed analogously.

8. Evaluation of Inference Relations from the Literature

In this section, we are going to evaluate inference relations from the literature with respect to (IRK). The proofs of the propositions in this section are straight-forward with the help of Lemma 2 resp. Proposition 9 and thus have been omitted due to limited space.

System Z System Z [12] is a well-known method to construct the minimal ranking model for a conditional belief base. It is based on a notion of *tolerance*. A conditional $(B|A)$ is *tolerated* by a set of conditionals $\Delta \subseteq (\mathcal{L}|\mathcal{L})$ iff there exists a possible world ω such that $\omega \models AB$ and $\omega \not\models C\overline{D}$ for all $(D|C) \in \Delta$; in other words, a conditional is tolerated iff it can be verified without falsifying any conditional in Δ. A *tolerance partition* of Δ is a partition $(\Delta_0, \dots, \Delta_m)$ such that all conditionals in Δ_i are tolerated by $\bigcup_{j \geq i} \Delta_j$. If the sets Δ_i are chosen inclusion-maximally, starting from Δ_0, then the partition is called the *Z-partition* of Δ and denoted as $Z(\Delta) = (\Delta_0, \dots, \Delta_m)$. For all $\delta \in \Delta$, we define $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ in the Z-partition. The *System Z ranking model* of Δ is then defined as follows:

$$\kappa^Z_\Delta(\omega) = \begin{cases} 0 & \text{iff } \text{fal}_\Delta(\omega) = \emptyset, \\ 1 + \max_{\delta \in \text{fal}_\Delta(\omega)} Z_\Delta(\delta) & \text{otherwise.} \end{cases}$$

We denote the inductive inference operator constructed from the mapping $\Delta \mapsto \kappa^Z_\Delta$ and $\mathbf{I}^{\text{ocf}}$ as $\mathbf{C}^Z$, and the corresponding inference relation as $\mathbf{C}^Z(\Delta) = \mid\!\sim^Z_\Delta$. This operator satisfies the (IRK) principle.

Proposition 11. $\mathbf{C}^Z$ *satisfies (IRK).*

Lexicographic Inference Lexicographic inference [13] is based on an order relation $\preceq^{\text{lex}}$ over integer vectors, which is defined via $(x_1, \ldots, x_m) \preceq^{\text{lex}} (y_1, \ldots, y_m)$ iff there exists $1 \leq j \leq m$ such that $x_j \leq y_j$ and for all $k > j$, $x_k = y_k$.

For a belief base Δ with $Z(\Delta) = (\Delta_0, \ldots, \Delta_m)$, let $\text{lex}_\Delta(\omega) = (|\text{ver}_{\Delta_1}(\omega)|, \ldots, |\text{ver}_{\Delta_m}(\omega)|)$ for every possible world ω. Now a total preorder $\preceq^{\text{lex}}_\Delta$ over Ω can be constructed such that $\omega \preceq^{\text{lex}}_\Delta \omega'$ iff $\text{lex}_\Delta(\omega) \preceq^{\text{lex}} \text{lex}_\Delta(\omega')$.

We denote the inductive inference operator defined via this total preorder as $\mathbf{C}^{\text{lex}} = \mathbf{I}^{\text{tpo}}(\preceq^{\text{lex}}_\Delta)$.

Proposition 12. $\mathbf{C}^{\text{lex}}$ *satisfies (IRK).*

p-Entailment A formula A *p-entails* another formula B given a conditional belief base Δ, $A \mathrel{|\!\!\sim}^{\text{p}}_\Delta B$, iff $A \mathrel{|\!\!\sim}_\kappa B$ holds for all ranking functions κ with $\kappa \models \Delta$. Equivalently, $A \mathrel{|\!\!\sim}^{\text{p}}_\Delta B$ holds iff $\Delta \cup \{(\overline{B}|A)\}$ is inconsistent [12]. We denote the respective inductive inference operator implementing p-entailment as $\mathbf{C}^{\text{p}}$, i.e. $\mathbf{C}^{\text{p}}(\Delta) = \mathrel{|\!\!\sim}^{\text{p}}_\Delta$.

Proposition 13. $\mathbf{C}^{\text{p}}$ *satisfies (IRK).*

Skeptical c-Inference Skeptical c-inference [14] with respect to Δ is defined by taking all c-representations of Δ into account, i.e. $A \mathrel{|\!\!\sim}^{\text{c-sk}}_\Delta B$ holds iff $A \mathrel{|\!\!\sim}_{\kappa^c} B$ holds for all c-representations κ^c of Δ. Let $\mathbf{C}^{\text{c-sk}}$ denote the respective inductive inference operator with $\mathbf{C}^{\text{c-sk}}(\Delta) = \mathrel{|\!\!\sim}^{\text{c-sk}}_\Delta$.

Proposition 14. $\mathbf{C}^{\text{c-sk}}$ *satisfies (IRK).*

Strategic c-Inference A c-revision is called *strategic* [15] if it uses a so-called *selection strategy* $\sigma : (\kappa, \Delta) \mapsto \vec{\eta}$ to choose a solution for the constraint satisfaction problem described in Definition 6. Selection strategies were first described for c-representations in [7].

It was shown in [3] that strategic c-revisions with selection strategies that satisfy a postulate called (IP-ESP$^\sigma$), *impact preservation with respect to equivalent subproblems*, satisfy (GRK). Therefore, this class of OCF-revision operators induces inference relations that satisfy (IRK).

Proposition 15. *Let $*_\sigma$ be a revision operator that satisfies (IP-ESP$^\sigma$), and let $\mathbf{E}^*_\sigma$ be the inductive epistemic operator induced by $*_\sigma$. Then the inductive inference operator $\mathbf{C}^*_\sigma = (\mathbf{I}^{\text{ocf}} \circ \mathbf{E}^*_\sigma)$ satisfies (IRK).*

Elementary Inductive Inference Chandler and Booth proposed a method to define conditional TPO-revision operators $\circledast$ from propositional TPO-revision operators $*$ [16]. If $*$ is one of the so-called *elementary revision operators* [17], then it was shown in [4] that $\circledast$ satisfies (QK). Therefore, $\circledast$ induces inference relations satisfying (IRK).

Proposition 16. *Let $\circledast$ be a conditional TPO-revision operator defined from an elementary revision operator as described above. Then the inductive inference operator $\mathbf{C}^{\circledast} = (\mathbf{I}^{\text{tpo}} \circ \mathbf{E}^{\circledast})$ satisfies (IRK).*

9. Kinematics and Syntax Splitting

In this section, we are going to briefly discuss the relationship between kinematics and *syntax splitting*, another principle for efficient reasoning in local contexts.

9.1. Syntax Splitting

Syntax splitting was first introduced as a property for propositional belief revision operators by [18] and later adapted for inductive inference operators in [7].

The core idea behind syntax splitting is quite similar to the idea behind the kinematics principles: unrelated information should not influence the reasoning process. This is implemented by partitioning the belief base syntactically such that one can focus on the locally relevant parts while ignoring the rest.

The difference lies in the splittings that are utilized: While case splittings split a belief base according to exclusive premises (e.g. information about penguins and non-penguins), syntax splittings split according to the syntax with which the information is expressed (e.g. information about penguins and politics).

Let $\Delta \in (\mathcal{L}|\mathcal{L})$ be a belief base including conditionals over the language $(\mathcal{L}|\mathcal{L})$ defined over the alphabet Σ. Let $\Sigma_1, \Sigma_2 \subseteq \Sigma$ such that $\Sigma_1 \cap \Sigma_2 = \emptyset$, and let $\mathcal{L}_1, \mathcal{L}_2$ be the languages defined over Σ_1, Σ_2, respectively. Syntactically splitting Δ over Σ_1, Σ_2 means partitioning $\Delta = \Delta_1 \cup \Delta_2$ such that $\Delta_1 \subseteq (\mathcal{L}_1|\mathcal{L}_1)$ and $\Delta_2 \subseteq (\mathcal{L}_2|\mathcal{L}_2)$.

According to [7], the postulate of syntax splitting (SynSplit) consists of two parts: *(syntactic) relevance* and *(syntactic) independence*. Let Δ be a conditional belief base that syntactically splits into $\Delta = \Delta_1 \cup \Delta_2$, and let $\mathbf{C}$ be an inductive inference operator with $\mathbf{C}(\Delta) = \mid\!\sim_\Delta$.

(SynRel) For $A, B \in \mathcal{L}_i$ and $i \in \{1, 2\}$: $\quad A \mid\!\sim_\Delta B \quad$ iff $\quad A \mid\!\sim_{\Delta_i} B$.

(SynInd) For $A, B \in \mathcal{L}_i, C \in \mathcal{L}_j, i, j \in \{1, 2\}$, and $i \neq j$: $\quad A \mid\!\sim_\Delta B \quad$ iff $\quad AC \mid\!\sim_\Delta B$.

(SynSplit) $\mathbf{C}$ satisfies both (SynRel) and (SynInd).

9.2. Syntactic Relevance and Case Relevance

The postulate (SynRel) is quite similar to (CaseRel$^{\text{IR}}$). This becomes more obvious when we observe the following consequence of (CaseRel$^{\text{IR}}$): For all A_i and $B, C \in \mathrm{Sc}(A_i)$, it holds that $B \mid\!\sim_\Delta C$ iff $B \mid\!\sim_{\Delta_i} C$. Both postulates ensure that only one of the subsets from the respective splitting is relevant for the local reasoning process. The difference between (SynRel) and (CaseRel$^{\text{IR}}$) lies in their applicability. While (SynRel) helps for queries that split over sub-alphabets, (CaseRel$^{\text{IR}}$) helps with queries that concern only certain cases. Consider the following example from [7] for a scenario in which (SynRel) is helpful while (CaseRel$^{\text{IR}}$) is not.

Example 3 ("Penguins"). Let $\Sigma^{\text{pen}} = \{p, b, f, d, v\}$. The belief base $\Delta^{\text{pen}} = \{(f|b), (b|p), (\overline{f}|p), (\overline{v}|d)\}$ encodes that birds (b) can fly (f), penguins (p) are birds but cannot fly ($\overline{f}$), and dark objekts (d) are not visible at night ($\overline{v}$). Syntactically splitting Δ^{pen} over $\Sigma_1^{\text{pen}} = \{p, b, f\}$ and $\Sigma_2^{\text{pen}} = \{d, v\}$ yields: $\Delta_1^{\text{pen}} = \{(f|b), (b|p), (\overline{f}|p)\}$, $\Delta_2^{\text{pen}} = \{(\overline{v}|d)\}$. To answer the query "$pb \mid\!\sim_{\Delta^{\text{pen}}} f$?": (SynRel) implies that only Δ_1^{pen} needs to be considered, whereas (CaseRel$^{\text{IR}}$) does not help since $p \vee b$ and d are not exclusive, and no helpful case splitting can be found.

For a contrary scenario, consider the following example from [3].

Example 4 ("Furniture"). Let $\Sigma^{\text{fur}} = \{k, l, c\}$, encoding information about furniture, with the atoms referring to kitchen items (k), items from the new collection (c), and items requiring heavy lifting (l). Consider the belief base $\Delta^{\text{fur}} = \{(l|kc), (\bar{l}|k\overline{c}), (\overline{c}|\overline{k})\}$ .. Case splitting with $A_1 = k$ and $A_2 = \overline{k}$ yields: $\Delta_1^{\text{fur}} = \{(l|kc), (\bar{l}|k\overline{c})\}$, $\Delta_2^{\text{fur}} = \{(\overline{c}|\overline{k})\}$. To answer the query "$k \mid\!\sim_{\Delta^{\text{fur}}} l$?": (SynRel) does not help since there is no syntax splitting, but (CaseRel$^{\text{IR}}$) implies that only Δ_1^{fur} needs to be considered.

9.3. Syntactic Independence and Case Irrelevance

After observing the similarity between (CaseRel$^{\text{IR}}$) and (SynRel), one could assume that similar parallels exist between (CaseIrr$^{\text{IR}}$) and (SynInd). However, the latter two postulates are quite different in nature.

The property (CaseIrr$^{\text{IR}}$) concerns itself with the plausibility of the cases themselves, stating that this information should not matter at all when focusing on one of the cases via conditionalization. In that way, the kinematics principle for inductive reasoning ensures that plausible propositional beliefs are treated independently from conditional beliefs. Syntax splitting, on the other hand, does not apply special treatment to plausible propositional beliefs.

The postulate (SynInd) essentially states that when locally reasoning about one (syntactically limited) area, extending the premise by (syntactically) unrelated information should not influence the local reasoning results at all. This is very different from the scenarios with which the kinematics principle concerns itself, since (IRK) assumes exclusive cases and only restricts reasoning for premises which belong to one of these cases.

9.4. Combination of (SynSplit) and (IRK)

Syntax splitting and kinematics can be combined to yield high-quality inferences in an efficient way since having both properties enables us to split belief bases in multiple ways.

Example 5. Consider a combination of Examples 3 and 4, i.e. let our belief base Δ contain information about penguins (Σ^{pen}) as well as information about furniture (Σ^{fur}). Note that Σ^{pen} and Σ^{fur} are clearly disjoint. Moreover, we hold the plausible propositional belief $\neg k$, since we believe the item in front of us to not be a kitchen item (but that does not stop us from reasoning about hypothetical alternatives). In summary, our belief base looks as follows: $\Delta = (\Delta^{\text{pen}} \cup \Delta^{\text{fur}} \cup \{\neg k\})$.

Now assume that we obtain the inference relation $\mid\!\sim_\Delta$ via an inductive reasoning operator that satisfies both (SynSplit) and (IRK). Again, we wish to answer the query "$k \mid\!\sim_\Delta l$?" from Example 4. We first apply (SynSplit) and find that $k \mid\!\sim_\Delta l$ iff $k \mid\!\sim_{\Delta^{\text{fur}} \cup \{\neg k\}} l$ since $\Delta = \Delta^{\text{pen}} \cup (\Delta^{\text{fur}} \cup \{\neg k\})$ syntactically splits over Σ^{pen} and Σ^{fur}. Afterwards, we can apply (IRK) to obtain $k \mid\!\sim_{\Delta^{\text{fur}} \cup \{\neg k\}} l$ iff $k \mid\!\sim_{\Delta^{\text{fur}}_1} l$ just like in Example 4, ignoring $\{\neg k\}$ as well because of (CaseIrr$^{\text{IR}}$). Therefore, although our belief base Δ is much more complex now, the combination of syntax splitting and kinematics allowed us to reduce the conditionals that need to be considered to the locally relevant minimum.

The combination of syntax splitting and kinematics is very promising, not just because of high-quality inferences, but in particular for increasing the computational efficiency of model-based inductive inference. If we needed to consider all possible worlds in Example 5, we would have to deal with $2^{|\Sigma_1|+|\Sigma_2|} = 256$ worlds. Syntax splitting allows us to reduce this to $2^{|\Sigma_2|} = 8$ worlds, and finally kinematics halves the amount of worlds again since only models of k need to be considered. Therefore, utilizing these principles could be essential for making large-scale conditional belief bases manageable.

10. Conclusions and Future Work

In this paper, we have presented a kinematics principle for inductive reasoning (IRK) based on the principle of Generalized Ranking Kinematics (GRK) for belief revision from [3]. Moreover, we have investigated the relationship between inductive inference operators and revision operators for OCFs and TPOs, building on recent work by [6], to motivate our version of the kinematics principle. We have evaluated inference relations from the literature with respect to (IRK) and have been able to show that several well-known inference relations—System Z, lexicographic inference, p-entailment, and skeptical c-inference—satisfy this kinematics principle. Finally, we compared (IRK) to the postulate (SynSplit) for inductive inference relations from [7].

Future work includes the evaluation of more approaches from the literature, for example System W [19] and its extension for a weaker notion of consistency [20]. Furthermore, comparing our approach to Weydert's version of subset independence for inference [5] would be interesting. Moreover, the relationship and interplay with other techniques for efficiency or notions of relevance has not yet been deeply investigated. Especially the relationships between case splittings, syntax splittings, or its extensions like conditional syntax splittings [21] or semantic splittings [22], and other forms of modularity with respect to conditional belief bases and OCF/TPO models deserve attention, both for non-monotonic reasoning and belief revision.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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