Brief Study of the Relation between AGM Postulates $(\dot{-}7)$ and $(\dot{+}7)$ under Non-classical Logics

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Abstract. This paper studies the implication relations between the AGM postulates $(\dot{-}7)$ and $(\dot{+}7)$ under non-classical logics, namely intuitionistic logic, paraconsistent logic, G3 and G3'. In order to do so, some theorems on the implication relations between such postulates are drawn for logics with a special set of axioms. These are later used to deduce similar results under the four logics of interest. Then we discuss a possible solution for our lack of sufficient conditions for the implication relations to hold under some of the studied logics.

1 Introduction

On one hand, the well known *AGM theory of belief revision* breaks the problem of defining belief revision operations into two parts: a set of rationality postulates and constructions (i.e. a framework for defining effective methods) for operators that aim to fulfill such postulates. This theory also assumes some underlying logic that includes *classical propositional logic* and that is compact[1].[1]

On the other hand, besides classical logic, there exist the so called *non-classical logics*. The motivations for creating and studying them are varied. Examples of such logics are *intuitionistic logic*, *paraconsistent logic*, *G3* and *G3'*.

Here we present the first results of a research aimed to help finding a way of using the AGM theory of belief revision (if possible) under the non-classical logics mentioned. The research is focused only on studying the implication relations between the rationality postulates, with the primary objective of finding sufficient conditions for these implications to hold under those logics. The two results are (i) what are sufficient conditions for the AGM postulate for contraction $(\dot{-}7)$ to hold if the AGM postulate for revision $(\dot{+}7)$ holds, and (ii) what are sufficient conditions for the AGM postulate for revision $(\dot{+}7)$ to hold if the AGM postulate for contraction $(\dot{-}7)$ holds.

Since the rationality postulates are the foundation of any construction for a contraction or revision operation [1], the ultimate contribution of this research is knowledge that facilitates any attempts to create or adapt constructions under the logics considered.

[1] Compact as mentioned in remark 10.

The structure of the paper is as follows. Section 2 recalls some logic notions and basic notations. Section 3 briefly describes the motivation behind non-classical logics and introduces the logics studied. Section 4 summarizes the AGM theory and broadens the motivation for this research. Our contribution is given in section 5.

We build on the AGM theory of belief revision [1] and works on non-classical logics [2,3]. The reader may also refer to [4], which is an excellent reference on this theory of belief revision and played an important role in inspiring this work. Finally, we borrow most of our background and notations from [5,6,7].

2 Background

This section is intended to remark our definition of *logic* and some relevant properties on the notions of *logical consequence relation* and *logical consequence operator* implied by such definition. The properties will be particularly important in section 5.

2.1 Formal Propositional Language

In order to talk about a logic, one must first have a way of coding *propositions*. Thus we define a *formal propositional language* together with its *alphabet*:

Definition 1. *[3] The alphabet Σ is the countable set built from: a countable set of elements called* atoms*; the binary connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication); the unary connective $\neg$ (negation); and the* auxiliary symbols *for opening and closing parenthesis.*

Definition 2. *[7] The formal propositional language $\mathcal{P}$ is the set whose elements, called* formulas*, are strings over Σ built recursively using the following rules:*

1. *If α is an atom, then $\alpha \in \mathcal{P}$.*
2. *If $x, y \in \mathcal{P}$, then $(x \wedge y)$, $(x \vee y)$, $(x \rightarrow y) \in \mathcal{P}$.*
3. *If $x \in \mathcal{P}$, then $\neg x \in \mathcal{P}$.*

We will denote atoms with the Greek letters α, β and γ. Similarly, we will denote formulas with the letters a, x and y. The notation $x \leftrightarrow y$ will be used to abbreviate $(x \rightarrow y) \wedge (y \rightarrow x)$.

Auxiliary parenthesis will sometimes be omitted in the writing of formulas. In those cases the usual precedence for connectives applies: $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ should be processed in that same order.

When dealing with formulas, it will be useful to have a name for any set of formulas, thus:

Definition 3. *A* theory *is a subset of $\mathcal{P}$.*

We will use each of the symbols R, S and T to represent any theory. Similarly, T' will stand for any finite theory.

2.2 Logic

One important definition to keep in mind is that of *logic*. In this paper a *logic* is considered simply as a *formal theory*:

Definition 4. *[7] A* formal theory *or* logic $\mathcal{F}$ *is built from:*

1. *A countable set* $Ss_{\mathcal{F}}$ *of symbols called* the symbols of $\mathcal{F}$. *Each finite sequence of symbols will be called an* expression of $\mathcal{F}$.
2. *A subset* $Swff_{\mathcal{F}}$ *of the expressions of* $\mathcal{F}$ *called the* set of well-formed formulas of $\mathcal{F}$.
3. *A subset* $\Omega_{\mathcal{F}}$ *of* $Swff_{\mathcal{F}}$ *called the* set of axioms of $\mathcal{F}$.
4. *A finite set* $\{X_1, X_2, \ldots, X_n\}$ *of n-relations over* $Swff_{\mathcal{F}}$ [2], *called* rules of inference. *For each* $w \in Cfbf_{\mathcal{F}}$, *if there are* $f_1, \ldots, f_m \in Cfbf_{\mathcal{F}}$ *and* $j \in \{1, \ldots, n\}$ *such that* $\langle f_1, \ldots, f_m, w\rangle \in X_j$, *then* w *is a* direct consequence *of* $f_1, \ldots, f_m$ *by virtue of* X_j.

Moreover, from now on the following assumptions are made for all logics:

1. $Ss_{\mathcal{F}} = \Sigma$
2. $Swff_{\mathcal{F}} = \mathcal{P}$
3. The set $\Omega_{\mathcal{F}}$ is *closed under substitution*: if a formula x is in $\Omega_{\mathcal{F}}$, then any other formula obtained by replacing all occurrences of an atom α in x with another formula y is in $\Omega_{\mathcal{F}}$ too [2].
4. The only rule of inference is *modus ponens*: the n-relation over $\mathcal{P}$ defined as:

$$M.P. \stackrel{\text{def}}{=} \{\langle x, y, z\rangle \in \mathcal{P} \times \mathcal{P} \times \mathcal{P} \mid y = x \to z\} .$$

There are several other ideas related to the concept of logic. Amongst them, *proof* and *logical consequence* are of our immediate interest. The former is the base for the later, while the later, together with our definition of logic, imply certain relevant properties presented in the next subsection.

Definition 5. *[7] A* proof *or* deduction *in a formal theory* $\mathcal{F}$ *for* $w \in Swff_{\mathcal{F}}$ *from* $\Gamma \subseteq Cfbf_{\mathcal{F}}$ *is a finite sequence* $f_1, \ldots, f_n$, *where* $f_1, \ldots, f_n \in Cfbf_{\mathcal{F}}$, *that satisfies the following two conditions:*

1. $f_n = w$
2. *For each* $j \in \{1, \ldots, n\}$ *one of the following conditions is satisfied:* $f_j \in \Gamma$ *or* $f_j \in \Omega_{\mathcal{F}}$ *or* f_j *is a direct consequence of some of the previous well-formed formulas in the sequence by virtue of some rule of inference of* $\mathcal{F}$.

Definition 6. *[6] A* $w \in Cfbf_{\mathcal{F}}$ *is a* logical consequence *in the formal theory* $\mathcal{F}$ *from a* $\Gamma \subseteq Swff_{\mathcal{F}}$ *if there is a proof in* $\mathcal{F}$ *for* w *from* Γ.

[2] I.e. subsets of the Cartesian product of $Swff_{\mathcal{F}}$ with itself n times.

2.3 Consequence relations and operations

Another important concept that should be borne in mind is that of a *consequence relation*:

Definition 7. *[5] A* (logical) consequence relation $\vdash_{\mathcal{F}}$ *is a binary relation such that*

$$\vdash_{\mathcal{F}} \stackrel{def}{=} \{\langle T, x\rangle \in \wp(\mathcal{P}) \times \mathcal{P} \mid x \text{ is a logical consequence in } \mathcal{F} \text{ from } T\}$$

where $\mathcal{F}$ is a logic, $\wp(\mathcal{P})$ stands for the power set of $\mathcal{P}$ and $\wp(\mathcal{P}) \times \mathcal{P}$ denotes the Cartesian product of $\wp(\mathcal{P})$ and $\mathcal{P}$.

Subscripts (as in $\vdash_{\mathcal{F}}$) may be omitted from now on whenever there is no confusion about the underlying logic.

The notation $T \vdash x$ will be used to denote $\langle T, x\rangle \in\vdash$. Similarly, $T \nvdash x$ will be used to denote $\langle T, x\rangle \notin\vdash$. In the case that $T = \varnothing$, we will write $\vdash x$ to denote that $\varnothing \vdash x$, and we will use $\nvdash x$ to denote that $\varnothing \nvdash x$.

There are two kinds of consequence relations that are relevant to this work, namely:

Definition 8. *[5] A consequence relation is an* abstract consequence relation *if it has the following properties:*

$$\text{If } x \in T, \text{ then } T \vdash x\ .$$

$$\text{If } S \vdash x \text{ and } S \subseteq T, \text{ then } T \vdash x\ .$$

$$\text{If } T \vdash x \text{ and for every } y \in T,\ S \vdash y, \text{ then } S \vdash x\ .$$

Definition 9. *[5] A consequence relation is a* finitary consequence relation *if in addition to being abstract it satisfies:*

$$\text{If } T \vdash x, \text{ then there is a finite set } T' \subseteq T \text{ such that } T' \vdash x\ .$$

Remark 10. The condition introduced in definition 9 is usually called *compactness.*

Proposition 11. *[6] All consequence relations assumed in this paper are* finitary consequence relations.

Proof. The proof is straightforward given the definition of consequence relation and the definition of logic.

The last two definitions are used in proofs in section 5, and so is a certain kind of *consequence operator.* This is the motivation for the following two definitions:

Definition 12. *[5] A* (logical) consequence operator *is a function $Cn_{\vdash} : \wp(\mathcal{P}) \to \wp(\mathcal{P})$ such that*

$$Cn_{\vdash}(T) \stackrel{def}{=} \{x \mid T \vdash x\}\ .$$

The reader should note that whenever a consequence operator $Cn_{\vdash}$ is defined from a consequence relation $\vdash_{\mathcal{F}}$ dependent upon a logic $\mathcal{F}$, we will write $Cn_{\mathcal{F}}$ instead of $Cn_{\vdash_{\mathcal{F}}}$.

Remark 13. [5] By definition 12 and set theory, clearly

$$x \in Cn_{\vdash}(T) \text{ if and only if } T \vdash x$$

therefore both notations will be used interchangeably.

Definition 14. *[5] A consequence operator is an* abstract consequence operator *if it has the following properties:*

$$T \subseteq Cn(T) \ .$$

$$Cn(Cn(T)) = Cn(T) \ .$$

$$\text{If } S \subseteq T, \text{ then } Cn(S) \subseteq Cn(T) \ .$$

Proposition 15. *A consequence operator is abstract if and only if the consequence relation used to define it is abstract.*

Proof. Using set theory, it is a simple exercise to show this.

To end this section, we introduce one last definition that will prove useful later:

Definition 16. *A* cn-theory *is a theory closed under a consequence operator.*

The symbol A will be used to denote any cn-theory: i.e. $A = Cn(T)$, for some T, possibly A itself.

3 Studied Logics

Classical logic was created as a model for studying the truth of propositions [2]. Our main interest in this work is in *non-classical logics*. Such logics arise from certain modeling needs, for instance: the need of modeling *possibility* and *necessity*, the need for allowing *inconsistency* or the need for establishing truth in a *constructive* manner [7,2]. The second is the case of *paraconsistent logic* and *G3'*, while *intuitionistic logic* and *G3* fit into the third category.

In this section we present some logics of interest to this work and a way to compare them. Definitions of the logics are borrowed from [3].

Definition 17. Pos *(short for* positive logic*) is the logic whose set of axioms Ω_{Pos} has the following elements:*

$$\alpha \to (\beta \to \alpha) \ .$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \ .$$

$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)) \ .$$

$$\alpha \wedge \beta \to \alpha \ .$$

$$\alpha \wedge \beta \to \beta \ .$$

$$\alpha \to (\beta \to (\alpha \wedge \beta)) \ .$$

$$\alpha \to \alpha \vee \beta \ .$$

$$\beta \to \alpha \vee \beta \ .$$

Definition 18. Cw *is the logic whose set of axioms* Ω_{Cw} *has all the elements of* Ω_{Pos} *plus the following two axioms:*

$$\alpha \vee \neg\alpha \ .$$

$$\neg\neg\alpha \to \alpha \ .$$

Definition 19. Pac *(short for* paraconsistent logic*) is the logic whose set of axioms* Ω_{Pac} *has all the elements of* Ω_{Cw} *plus the following axioms:*

$$\alpha \to \neg\neg\alpha \ .$$

$$((\alpha \to \beta) \to \alpha) \to \alpha \ .$$

$$(\neg\alpha \vee \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta) \ .$$

$$(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg(\alpha \vee \beta) \ .$$

$$\neg(\alpha \to \beta) \leftrightarrow (\alpha \wedge \neg\beta) \ .$$

Definition 20. Int *(short for* intuitionistic logic*) is the logic whose set of axioms* Ω_{Int} *has all the elements of* Ω_{Pos} *plus the following axioms:*

$$\neg\alpha \to (\alpha \to \beta) \ .$$

$$(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha) \ .$$

Definition 21. G3 *(also known as* the logic of here and there*) is the logic whose set of axioms* Ω_{G3} *has all the elements of* Ω_{Int} *plus the following axiom:*

$$(\neg\beta \to \alpha) \to (((\alpha \to \beta) \to \alpha) \to \alpha) \ .$$

G3' is one last logic that we are interested in. A rigorous definition for G3' is omitted here, but it suffices to know than its set of axioms $\Omega_{G3'}$ has all the elements of Ω_{Cw} plus another four. Such additional axioms can be found in [3].

Only the last four logics mentioned are of importance in the following sections. The others just provide insight for comparing those logics, as it will be made clear shortly. The reader may refer to [2,3,7,8] for further reading.

3.1 Comparing logics

A usual way of comparing logics is by comparing their *set of theorems*:

Definition 22. $Cn_{\mathcal{F}}(\varnothing)$ *is the* set of theorems *of logic* $\mathcal{F}$*.*

Definition 23. *[2] A logic* $\mathcal{F}'$ *is* stronger than or equal to *a logic* $\mathcal{F}$ *if* $Cn_{\mathcal{F}}(\varnothing) \subseteq Cn_{\mathcal{F}'}(\varnothing)$*.*

In our case, the following lemma is useful:

Lemma 24. *Let* $\mathcal{F}$ *and* $\mathcal{F}'$ *be each any logic. If* $\Omega_{\mathcal{F}} \subseteq \Omega_{\mathcal{F}'}$*, then* $\mathcal{F}'$ *is stronger than or equal to* $\mathcal{F}$*.*

Proof. This follows from the definition of proof (i.e. definition 5). A detailed proof is omitted.

Example 25. Int is stronger than or equal to Pos, while G3 is stronger than or equal to Int.

4 Taking AGM into Non-classical Logics

By *AGM theory of belief revision* we mean the theory developed by Alchourron, Gärdenfors and Makinson in works like [4,9]. It is well known that their work has been dominant in the field of belief revision [10].

The AGM theory assumes a formal propositional language[3] as the means for coding propositions and some underlying logic that includes classical propositional logic and that is compact. Moreover, the AGM theory encompasses three different belief change operations over cn-theories: expansion, where a formula is introduced into a cn-theory together with all the formulas deducible from the new cn-theory; contraction, that ensures a formula cannot be deduced from a resulting cn-theory; and revision, which retracts everything in contradiction with a new formula and subsequently expands the contraction with the new formula. Defining the first of them is trivial, while the other two have to undergo a more elaborated process. [1]

The main focus of this theory of belief revision is defining revision and contraction. This process is split in two main parts: the AGM rationality postulates (named after the three authors) and constructions (i.e. a framework for defining effective methods) for operations that comply with those postulates. The former is concerned with stating what an *appropriate* revision or contraction operation is (i.e. *the what*); the later is intended to be the scheme of actual implementations of such operations (i.e. *the how*). Both parts are then equivalent by means of what are called *representation theorems*. [1]

We dare to say that the postulates part is the more important of the two. If we are to contribute to finding a way of using this theory under non-classical logics, the postulates can be used to find out if its rationale holds under such

[3] For which some details are left open. We assume this language to be $\mathcal{P}$.

logics in the first place. This is the reason why we have focused this research on the rationality postulates only. It must be noted, however, that even if we were successful, finding a way for the equivalence of the two main parts of this theory of belief revision to hold is yet another issue (which we conjecture would require less effort) in the way to reaching the overall goal.

Moreover, the postulates are further divided in postulates for contraction and postulates for revision, and for each case there is a *basic* set of postulates and a *supplementary* set [1]. The basic set for the contraction postulates is as follows [4]:

$$A \dot{-} x \text{ is a cn-theory whenever } A \text{ is a cn-theory.} \tag{$\dot{-}1$}$$

$$A \dot{-} x \subseteq A\ . \tag{$\dot{-}2$}$$

$$\text{If } x \notin Cn(A)\text{, then } A \dot{-} x = A\ . \tag{$\dot{-}3$}$$

$$\text{If } x \notin Cn(\varnothing)\text{, then } x \notin Cn(A \dot{-} x)\ . \tag{$\dot{-}4$}$$

$$\text{If } Cn(\{x\}) = Cn(\{y\})\text{, then } A \dot{-} x = A \dot{-} y\ . \tag{$\dot{-}5$}$$

$$A \subseteq Cn((A \dot{-} x) \cup \{x\}) \text{ whenever } A \text{ is a cn-theory.} \tag{$\dot{-}6$}$$

Since only two more postulates will be needed by the forthcoming results, the supplementary set for contraction and the complete set for revision are omitted here. Nevertheless, the reader can find both sets of postulates in [4].

The theory of belief revision also considers ways of bridging, so to speak, the postulates for contraction and the postulates of revision. This is done by means of two identities: the *Levi identity* and the *Harper identity* [4]. The former is of interest to this paper:

$$A \dot{+} x = Cn((A \dot{-} \neg x) \cup \{x\})\ . \tag{Levi}$$

In [1] such bridging between AGM postulates for revision and contraction means that if we were to create a construction for a revision or for a contraction operation, this would yield a construction for the other. In other words, one set of postulates is equivalent to the other. This equivalence, however, appears to be known only in the case of classical logic. Thus we will not assume that any parts of the sets of postulates for contraction and revision are equivalent in other logics. Indeed, discovering a way for the equivalence between the two sets of postulates to hold in other logics is the core of our research: their equivalence is what we consider indicates that the postulates hold.

The research we have carried so far has studied the implication relations between two AGM (supplementary) postulates and, therefore their possible equivalence. We have focused primarily on finding sufficient conditions for such relations to hold. These two postulates are:

$$A \dot{+} (x \wedge y) \subseteq Cn((A \dot{+} x) \cup \{y\}) \text{ for any cn-theory } A. \tag{$\dot{+}7$}$$

$$(A \dot{-} x) \cap (A \dot{-} y) \subseteq A \dot{-} (x \wedge y) \text{ for any cn-theory } A. \tag{$\dot{-}7$}$$

Their intention (aided by the rest of the supplementary postulates) is to specify the behavior of their corresponding operations when dealing with formulas in conjunctive form [1].

Before ending this section, we must not forget to mention a useful lemma:

Lemma 26. *Let the notation $A \dot{-} x$ stand for the set of formulas that is assigned to A and x by a contraction operation $\dot{-}$ that satisfies the AGM postulate ($\dot{-}$6) and is defined for a logic with an abstract consequence relation. Then it holds that: If $\{y\} \vdash x$, then $A \subseteq Cn\left((A \dot{-} x) \cup \{y\}\right)$.*

Proof. The proof is straightforward and thus is omitted.

5 On the Implication Relations between ($\dot{-}$7) and ($\dot{+}$7)

We studied several conditions for the implication relations between ($\dot{+}$7) and ($\dot{-}$7) to hold. Amongst the conditions for each case there is a logic, which implies certain properties.

In this section we present those two logics, describe briefly their relevant properties and enunciate the most important results on the implication relations.

5.1 Implication Relation from ($\dot{+}$7) to ($\dot{-}$7)

Definition 27. *Let* LT1 *be the logic whose set of axioms Ω_{LT1} has the following elements:*

$$\alpha \to (\beta \to \alpha) \ . \tag{1}$$

$$(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)) \ . \tag{2}$$

$$(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)) \ . \tag{3}$$

$$\alpha \wedge \beta \to \alpha \ . \tag{4}$$

$$\alpha \wedge \beta \to \beta \ . \tag{5}$$

$$\alpha \to (\beta \to (\alpha \wedge \beta)) \ . \tag{6}$$

$$\alpha \leftrightarrow \neg\neg\alpha \ . \tag{7}$$

$$\beta \to \alpha \vee \beta \ . \tag{8}$$

$$(\neg\alpha \vee \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta) \ . \tag{9}$$

$$(\neg\alpha \wedge \neg\beta) \leftrightarrow \neg(\alpha \vee \beta) \ . \tag{10}$$

$$\alpha \vee \neg\alpha \ . \tag{11}$$

Proposition 28. *All of the following hold under LT1:*

$$T \cup \{x\} \vdash y \text{ iff } T \vdash x \to y \ . \tag{12}$$

$$\text{If } \vdash x \leftrightarrow y, \text{ then } Cn(\{x\}) = Cn(\{y\}) \ . \tag{13}$$

$$\text{If } T \cup \{x\} \vdash a \text{ and } T \cup \{y\} \vdash a, \text{ then } T \cup \{x \vee y\} \vdash a \ . \tag{14}$$

$$\text{If } T \cup \{x\} \vdash a, \ T \cup \{y\} \vdash a \text{ and } (x \vee y) \in T, \text{ then } T \vdash a \ . \tag{15}$$

$$\vdash x \leftrightarrow \neg((\neg x \vee \neg y) \wedge \neg x) \ . \tag{16}$$

$$\vdash y \leftrightarrow \neg((\neg x \vee \neg y) \wedge \neg y) \ . \tag{17}$$

$$\vdash \neg(\neg x \vee \neg y) \leftrightarrow (x \wedge y) \ . \tag{18}$$

Proof. Detailed proofs are omitted.

(12): Any logic that satisfies axioms (1) and (2) also satisfies (12) [3].

(13): Holds given the definition of abstract consequence relation, (12) and axioms (4) and (5).

(14): Holds given the definition of abstract consequence relation, axiom (3), (12) and set theory.

(15): Holds given (14) and set theory.

(16), (17) and (18): It is a simple exercise to prove each under LT1.

Remark 29. (12) is usually called *deduction theorem* (as in [3]), while (14) is usually called *introduction of disjunctions in the premises* (as in [4]).

Theorem 30. *Let the notation $A \dot{-} x$ stand for the set of formulas that is assigned to A and x by a contraction operation $\dot{-}$ that satisfies AGM postulates ($\dot{-}$1), ($\dot{-}$2), ($\dot{-}$5) and ($\dot{-}$6), the Levi identity and is defined over the logic LT1. Such contraction operation satisfies ($\dot{-}$7) if it satisfies ($\dot{+}$7).*

Proof. Let us assume ($\dot{+}$7). We need to show that ($\dot{-}$7) holds. Let

$$a \in ((A \dot{-} x) \cap (A \dot{-} y)) \ . \tag{19}$$

By the previous and set theory, to prove ($\dot{-}$7) it suffices to show

$$a \in A \dot{-} (x \wedge y) \ . \tag{20}$$

Also by (19) and set theory, both of the following hold

$$a \in A \dot{-} x \ . \tag{21}$$

$$a \in A \dot{-} y \ . \tag{22}$$

With the help of ($\dot{-}$5), (13) and (16), (21) becomes

$$a \in A \dot{-} \neg\left((\neg x \vee \neg y) \wedge \neg x\right) \ . \tag{23}$$

Using only set theory, definition of abstract consequence operation, (Levi) and ($\dot{+}$7),

$$A \dot{-} \neg\left((\neg x \vee \neg y) \wedge \neg x\right) \subseteq Cn\left((A \dot{+} (\neg x \vee \neg y)) \cup \{\neg x\}\right) \ . \tag{24}$$

Then, by (23), (24) and set theory, it can be deduced that

$$a \in Cn\left((A \dot{+} (\neg x \vee \neg y)) \cup \{\neg x\}\right) \ . \tag{25}$$

Based on (22), (17) and a similar reasoning as the one developed from (23) to (25),

$$a \in Cn\left((A \dot{+} (\neg x \vee \neg y)) \cup \{\neg y\}\right) \ . \tag{26}$$

By set theory, definition of abstract consequence operation and (Levi),

$$(\neg x \vee \neg y) \in A \dot{+} (\neg x \vee \neg y) \ . \tag{27}$$

Recalling that $x \in Cn(T)$ *iff* $T \vdash x$ and based on (25), (26) and (27), (15) can be used to obtain

$$a \in Cn\left(A \dot{+} (\neg x \vee \neg y)\right) \ . \tag{28}$$

Given (28), by definition of abstract consequence operation, (Levi), (9), (18), (13) and ($\dot{-}$5), it can be turned into

$$a \in Cn\left((A \dot{-} (x \wedge y)) \cup \{\neg (x \wedge y)\}\right) \ . \tag{29}$$

By (21), ($\dot{-}$2), ($\dot{-}$6) and set theory,

$$a \in Cn\left((A \dot{-} (x \wedge y)) \cup \{(x \wedge y)\}\right) \ . \tag{30}$$

We know that, by (11), $\vdash (x \wedge y) \vee \neg (x \wedge y)$, so by ($\dot{-}$1) it holds that

$$((x \wedge y) \vee \neg (x \wedge y)) \in A \dot{-} (x \wedge y) \ . \tag{31}$$

Again, by $x \in Cn(T)$ *iff* $T \vdash x$, (29), (30) and (31), (15) can be used to get

$$a \in Cn\left(A \dot{-} (x \wedge y)\right) \ . \tag{32}$$

Which by $Cn(Cn(T)) = Cn(T)$ and ($\dot{-}$1) shows (20). □

Given the previous result, the definition of Pac and lemma 24, we have as a direct consequence the following corollary:

Corollary 31. *Let the notation $A \dot{-} x$ stand for the set of formulas that is assigned to A and x by a contraction operation $\dot{-}$ that satisfies AGM postulates ($\dot{-}$1), ($\dot{-}$2), ($\dot{-}$5) and ($\dot{-}$6), the Levi identity and is defined over the logic Pac. Such contraction operation satisfies ($\dot{-}$7) if it satisfies ($\dot{+}$7).*

Remark 32. A similar result could not be verified for the other logics mentioned earlier due to the steps taken in the proof of theorem 30. Specifically (16), (17) and (18) do not hold for Int, G3 or G3'[4]: only the $\rightarrow$ side of (16), (17), and the $\leftarrow$ side of (18) hold for Int and G3; also, so to speak *symmetrically*, only the $\leftarrow$ side of (16), (17), and the $\rightarrow$ side of (18) hold for G3'. Moreover, it is well known that (11) is not a theorem of Int or G3 (see [3]), which is needed by the proof of theorem 30. In the light of this situation, the validity of a similar result as corollary 31 under Int, G3 or G3' remains unknown.

5.2 Implication Relation from ($\dot{-}$7) to ($\dot{+}$7)

When studying the implication relation from ($\dot{-}$7) to ($\dot{+}$7), we found out that the assumption of Int as the underlying logic provides sufficient properties for this implication to hold.

Remark 33. The same as in LT1, the deduction theorem, the theorem of introduction of disjunctions in the premises and propositions (13) and (15) hold in Int.

Proposition 34. *In addition to the properties mentioned by the previous remark, the following hold in Int:*

$$\vdash \neg x \leftrightarrow \neg (x \wedge y) \wedge (\neg x \vee y) \ . \tag{33}$$

$$Cn (R \cup T) \cap Cn (S \cup T) \subseteq Cn ((Cn (R) \cap Cn (S)) \cup T) \ . \tag{34}$$

$$Cn (A \cup \{x \wedge y\}) \subseteq Cn ((A \dot{-} (\neg x \vee y)) \cup \{x \wedge y\}) \ . \tag{35}$$

Proof. Detailed proofs are omitted.

(33): It is a simple exercise to prove under Int.

(34): Holds given the definition of finitary consequence relation, the theorem of introduction of disjunctions in the premises and the axioms of Int dealing with conjunctions and disjunctions.

(35): Clearly $\{x \wedge y\} \vdash \neg x \vee y$ holds under Int, so by definition of abstract consequence relation, lemma 26 and set theory, (35) is easy to prove.

Theorem 35. *Let the notation $A \dot{-} x$ stand for the set of formulas that is assigned to A and x by a contraction operation $\dot{-}$ that satisfies AGM postulates ($\dot{-}$1), ($\dot{-}$2), ($\dot{-}$5) and ($\dot{-}$6), the Levi identity and is defined over the logic Int. Such contraction operation satisfies ($\dot{+}$7) if it satisfies ($\dot{-}$7).*

Proof. Let us assume ($\dot{-}$7). We need to show that ($\dot{+}$7) holds. Let

$$a \in A \dot{+} (x \wedge y) \ . \tag{36}$$

[4] This is assured by the use of truth tables (see [3]).

By the previous and set theory, to prove $(\dotplus 7)$ it suffices to show

$$a \in Cn\left((A \dotplus x) \cup \{y\}\right) \ . \tag{37}$$

Based on (Levi), the definition of abstract consequence relation and the axioms of Int dealing with conjunctions, the previous proposition is equivalent to

$$a \in Cn\left((A \dot{-} \neg x) \cup \{x \wedge y\}\right) \ . \tag{38}$$

Thus, to prove $(\dotplus 7)$ it suffices to show proposition (38). But by (33), (13), and $(\dot{-}5)$ the following proposition is equivalent to (38), so it suffices to show it in order to show $(\dotplus 7)$:

$$a \in Cn\left((A \dot{-} (\neg (x \wedge y) \wedge (\neg x \vee y))) \cup \{x \wedge y\}\right) \ . \tag{39}$$

Taking into account $(\dot{-}7)$, set theory and the definition of abstract consequence operation, it can be shown that

$$\begin{aligned} &Cn\left(((A \dot{-} \neg (x \wedge y)) \cap (A \dot{-} (\neg x \vee y))) \cup \{x \wedge y\}\right) \\ &\quad \subseteq Cn\left((A \dot{-} (\neg (x \wedge y) \wedge (\neg x \vee y))) \cup \{x \wedge y\}\right) \ . \end{aligned} \tag{40}$$

With the definition of abstract consequence operation, set theory, $(\dot{-}1)$, (34) and the previous step, it can be shown that

$$\begin{aligned} &Cn\left((A \dot{-} \neg (x \wedge y)) \cup \{x \wedge y\}\right) \cap Cn\left((A \dot{-} (\neg x \vee y)) \cup \{x \wedge y\}\right) \\ &\quad \subseteq Cn\left((A \dot{-} (\neg (x \wedge y) \wedge (\neg x \vee y))) \cup \{x \wedge y\}\right) \ . \end{aligned} \tag{41}$$

By (41) and set theory, to prove (39) it suffices to show both of the following:

$$a \in Cn\left((A \dot{-} \neg (x \wedge y)) \cup \{x \wedge y\}\right) \ . \tag{42}$$

$$a \in Cn\left((A \dot{-} (\neg x \vee y)) \cup \{x \wedge y\}\right) \ . \tag{43}$$

It is easy to realize that (42) follows from (Levi) and (36). Then, by $(\dot{-}2)$, set theory and definition of abstract consequence operation,

$$Cn\left((A \dot{-} \neg (x \wedge y)) \cup \{x \wedge y\}\right) \subseteq Cn\left(A \cup \{x \wedge y\}\right) \ . \tag{44}$$

A direct consequence of (42), set theory and the previous is

$$a \in Cn\left(A \cup \{x \wedge y\}\right) \ . \tag{45}$$

Finally, by the previous proposition, (35) and set theory it is easy to see that (43) holds. □

Given the previous result, the definition of G3 and lemma 26, we have:

Corollary 36. *Let the notation $A \dot{-} x$ stand for the set of formulas that is assigned to A and x by a contraction operation $\dot{-}$ that satisfies AGM postulates $(\dot{-}1)$, $(\dot{-}2)$, $(\dot{-}5)$ and $(\dot{-}6)$, the Levi identity and is defined over the logic G3. Such contraction operation satisfies $(\dotplus 7)$ if it satisfies $(\dot{-}7)$.*

Remark 37. A similar result could not be verified for the other logics mentioned earlier due to the steps taken in the proof of theorem 35. Specifically, only the ← side of (33) does not hold for Pac or G3'.

5.3 Avoiding the Problems Found Going from ($\dot{-}7$) to ($\dot{+}7$)

Given the primary objective of this research (i.e. finding sufficient conditions for the implication relations to hold), the author is focused on discovering further useful assumptions. In this subsection we discuss an option that seems promising and is left for future work.

When analyzing the proofs of theorems 30 and 35, one can eventually pose the question of how would the proofs be affected by assuming some containment relation between contractions of the same cn-theory with respect to formulas related in some way. Amongst various candidates for solving this question, the following two examples are particularly interesting at first glance:

$$\text{If } \{x\} \vdash y, \text{ then } A \dot{-} y \subseteq A \dot{-} x\,. \qquad (\dot{-}5')$$

$$\text{If } \{x\} \vdash y, \text{ then } A \dot{-} y \supseteq A \dot{-} x\,. \qquad (\dot{-}5'')$$

With ($\dot{-}5'$) and using a slightly different logic than Pos, it would be possible to deduce similar results as corollary 36 for Pac and G3'. Unfortunately ($\dot{-}5'$) turns out to be in contradiction with the postulates for contraction: one fundamental assumption of it is that $x \notin A \dot{-} y$ whenever $\{x\} \vdash y$ and $\nvdash x$, but at the same time it can be shown, using ($\dot{-}1$), ($\dot{-}2$) and ($\dot{-}6$), that if $\vdash y$, then $A \dot{-} y = A$.

In the case of ($\dot{-}5''$), we conjecture that it will not have similar problems as ($\dot{-}5'$). With the help of ($\dot{-}5''$) it would be possible to bypass some of the problems encountered in finding similar results as corollary 31 for Int, G3 and G3'.

6 Conclusions and Future Work

We have introduced some non-classical logics and summarized the AGM theory of belief revision. We have shown several implication relations between the AGM postulates ($\dot{+}7$) and ($\dot{-}7$). Finally, we have given an example of an open issue regarding the quest for additional conditions for the implication relations to hold.

Future work will focus on currently open issues and on studying the possible equivalence of other AGM postulates, such as ($\dot{+}8$) and ($\dot{-}8$), under the logics already studied.

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