Description Logics with Epsilon Individuals

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Abstract

We investigate the extension of description logics (DLs) with definite descriptions—that is, references to individuals based on descriptions of their properties. Specifically, we introduce the syntax and semantics for ε-individuals, modelled on the ε-terms that Hilbert introduced for first-order logic. We present sound and complete reasoning algorithms for the logics that result from adding ε-individuals to several well-known DLs. In particular, for the extension of the basic DL $\mathcal{ALC}$ with ε-individuals, we provide a tableau calculus and show that the language without TBoxes is as expressive as the language with TBoxes; both also share EXPTIME-completeness of reasoning. In the case of the extension $\mathcal{ALCO}$ of the language with nominals, we give a reduction to the language $\mathcal{ALCO}_u$ with the universal role and show that reasoning remains EXPTIME-complete. Finally, for the lightweight DL $\mathcal{ELO}$, we show that the usual saturation calculus can be extended for ε-individuals, while maintaining the PTIME complexity.

Keywords

Description Logics, Definite Descriptions, Epsilon Individuals

1. Introduction

In semantic modelling, it is often desirable to have a way to refer to individuals by describing their properties. For example, one might want a way of referring to 'the king of France' based on a formalisation of the 'king of' relation and the country of France. In engineering, a notation referring to 'the temperature sensor on the exhaust pipe of the generator…' would be similar to tagging systems commonly used in the engineering of industrial plants. Such references to individuals based on their properties are known as *definite descriptions*, and they have been extensively studied in both logic and philosophy [1, 2, 3, 4]. The key questions that have to be answered when attempting a formalisation of definite descriptions within the model semantics framework include the following: (a) What does a definite description 'the thing with property C' refer to when there is no such thing? (b) What if there are several things with that property? (c) If there are several syntactically identical references, do they refer to the same thing?

The best known formalisation of definite descriptions is probably the one for first-order logic as introduced by Hilbert [5]. In this calculus, so-called iota terms $\iota x.\phi$ are added to the syntax for this purpose, where x is a bound variable and ϕ is a formula describing the required property. Such terms may only be used in contexts where both existence and uniqueness of such values have first been established, which addresses questions (a)–(c). However, when transforming proofs, it can happen that an ι-term is moved outside the context in which existence and uniqueness are guaranteed. To address this issue, Hilbert also introduced epsilon terms $\varepsilon x.\phi$; such a term may always be used, and it (i) denotes an arbitrary domain element if no element satisfies ϕ and (ii) denotes one of the domain elements that satisfy ϕ if there are several. Moreover, it is generally agreed that (iii) syntactically identical occurrences of ε-terms should denote the same value even when this value is not known. What is less clear from Hilbert's work is whether $\varepsilon x.\phi = \varepsilon x.\psi$ when ϕ and ψ are equivalent but not syntactically identical. This property was not needed for Hilbert's purposes, but subsequent work has explored the consequences of this choice, as well as possible relaxations of (iii), for different logics [6, 7, 8, 9, 10].

In the context of description logics (DLs), Artale et al. [11] have recently investigated the addition of individuals ιC, corresponding to Hilbert's ι-terms, to common DLs such as $\mathcal{ALC}$ and $\mathcal{EL}$. Rather than

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adopting Hilbert's approach to questions (a) and (b) above, however, they have based their formalisation on *free logics* [12, 13], where iota terms are permitted not to denote *anything*. In particular, in their semantics a nominal $\{\iota.C\}$ is interpreted as a domain element d if the concept C is interpreted as the singleton set $\{d\}$, and as empty otherwise. They have shown that extending $\mathcal{ALC}$ and $\mathcal{EL}$ with nominals, the universal role, and such ι-individuals does not increase the complexity of reasoning compared to the original languages.

While Artale et al. have demonstrated the viability of incorporating definite descriptions into DLs, we believe that adopting the free logic approach to semantics is a significant departure from the conventional DL framework. Hähnle [14] has shown that it is in many ways more natural to let undefined terms denote an unknown domain element than to deal with partiality in the semantics. In this paper, we therefore investigate the consequences of adding ε-individuals $\varepsilon.C$ to DLs instead, addressing questions (a)–(c) in spirit of Hilbert's ε-terms in first-order logic. Specifically, we present the syntax and semantics of this addition, and discuss reasoning methods and complexity for $\mathcal{ALC}$ (where ε-individuals can only appear in the ABox), $\mathcal{ALCO}$, and $\mathcal{ELO}$. In our formalisation, we adopt the so-called *intentional semantics* of ε-individuals, where only syntactically identical occurrences of such individuals are required to be interpreted identically to satisfy property (iii). This contrasts with the *extensional semantics*, in which the ε-individuals for all equivalent concepts must denote the same domain element—an alternative we leave for future work.

Our results can be summarised as follows. After having formulated ε-individuals in the context of $\mathcal{ALC}$ (Section 2), we provde a tableau calculus for this extension and show that the result of adding ε-individuals to the language without TBoxes renders it as expressive as the language with TBoxes; EXPTIME-completeness of reasoning is also preserved (Section 3). For the extension $\mathcal{ALCO}$ of $\mathcal{ALC}$ with nominals, we present a reduction of the language with ε-individuals to one with the universal role, $\mathcal{ALCO}_u$, thereby showing that reasoning remains in EXPTIME (Section 4). Finally, for the lightweight DL $\mathcal{ELO}$, we show that the usual saturation calculus can be extended to support ε-individuals, while preserving PTIME completeness of concept subsumption (Section 5).

Full proofs of all claims in this paper can be found in the technical report [15].

2. $\mathcal{ALC}$ with ε-Individuals

In this section, we introduce the syntax and semantics of ε-individuals in the context of $\mathcal{ALC}$. The definitions for $\mathcal{ALCO}$ and $\mathcal{ELO}$, given in Sections 4 and 5, follow the same pattern.

We begin with the syntax of $\mathcal{ALC}$ extended with ε-individuals, which we call $\mathcal{ALC}^\varepsilon$. We first do this for concepts and individuals, where the latter includes ε-individuals $\varepsilon.C$ as a new syntactic category, which can be added not only to $\mathcal{ALC}$ but also to any DL. These individuals may be used wherever an individual name is allowed, which, in the case of $\mathcal{ALC}$, means only in ABox assertions. However, we will later consider logics $\mathcal{ALCO}$ and $\mathcal{ELO}$ with nominals, where ε-individuals may also appear in concept descriptions.

Definition 1 ($\mathcal{ALC}^\varepsilon$ Syntax)**.** *Let N_C, N_R, and N_O be sets of* concept names, role names*, and* individual names*, respectively. Then, $\mathcal{ALC}^\varepsilon$ concepts and individual descriptions, C and τ, are defined by the following grammar:*

$$C ::= A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C,$$
$$\tau ::= a \mid \varepsilon.C,$$

where A, r, and a range over N_C, N_R, and N_O, respectively. An assertional axiom *is an expression of the form $C(\tau)$ or $r(\tau_1, \tau_2)$ where C is a concept, r a role name, and τ, τ_1, τ_2 individual descriptions. An $\mathcal{ALC}^\varepsilon$* concept inclusion axiom *is of the form $C \sqsubseteq D$ for concepts C and D. Then, $\mathcal{ALC}^\varepsilon$* ABox *and* TBox *are sets of $\mathcal{ALC}^\varepsilon$ assertional and concept inclusion axioms, respectively. An $\mathcal{ALC}^\varepsilon$* knowledge base *(*KB*) is a pair $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ consisting of an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$.*

Note that all $\mathcal{ALC}^\varepsilon$ concepts are also in plain $\mathcal{ALC}$, and the same holds for TBoxes.

We now move on to the semantics, beginning with the definition of interpretations for concepts and individual descriptions. For later discussion, it is convenient to first define interpretations that impose no restrictions on the interpretation of an individual $\varepsilon.C$, apart from the requirement that all syntactically identical occurrences be interpreted the same, as stated in property (iii) in the introduction. We then constrain interpretations to satisfy the central intended property of ε-individuals—property (ii) in the introduction. It is common in the literature on ε-terms (e.g. [7, 9]) to refer to the resulting semantics as *intensional.*

Definition 2 (Interpretations and Intentional Interpretations). *An* interpretation *is a pair* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, *where* domain $\Delta^\mathcal{I}$ *is a non-empty set, and* $\cdot^\mathcal{I}$ *is a function mapping each concept name* A *to* $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, *each role name* r *to* $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, *and each individual description* τ *to* $\tau^\mathcal{I} \in \Delta^\mathcal{I}$. *Then, for each* $\mathcal{ALC}^\varepsilon$ *concepts* C, D *and role name* r, *let*

$$(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}, \qquad (C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}, \qquad (\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I},$$
$$(\exists r.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \textit{there exists } \langle x, y\rangle \in r^\mathcal{I} \textit{ such that } y \in C^\mathcal{I}\},$$
$$(\forall r.C)^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \textit{for all } \langle x, y\rangle \in r^\mathcal{I}, y \in C^\mathcal{I}\}.$$

An interpretation $\mathcal{I}$ *is* intensional *if* $(\varepsilon.C)^\mathcal{I} \in C^\mathcal{I}$ *for each concept* C *such that* $C^\mathcal{I} \neq \emptyset$,.

We emphasise that ε-individuals based on semantically equivalent concepts may not be interpreted the same; for instance, there is no guarantee that $(\varepsilon.(C \sqcap D))^\mathcal{I} = (\varepsilon.(D \sqcap C))^\mathcal{I}$. An alternative approach would be to define the semantics of $\varepsilon.C$ as a function of the extension $C^\mathcal{I}$. The properties of such semantics, usually referred to as *extensional*, are left as future work.

We also note that the definition imposes no restriction on $\varepsilon.C^\mathcal{I}$ when $C^\mathcal{I} = \emptyset$. In this case, the value may be any domain element, potentially different ones for syntactically different concept descriptions, but still committed to be the same one for all syntactically identical C.

We move on to the semantics of $\mathcal{ALC}^\varepsilon$ KBs.

Definition 3 (Semantics of Axioms and Knowledge Bases). *Given an* $\mathcal{ALC}^\varepsilon$ *axiom* ϕ *and interpretation* $\mathcal{I}$, *the* satisfaction *of* ϕ, *written* $\mathcal{I} \models \phi$, *is defined, for different forms of* ϕ, *as follows:*

$$\mathcal{I} \models C(\tau) \;\textit{ if }\; \tau^\mathcal{I} \in C^\mathcal{I}, \qquad \mathcal{I} \models r(\tau_1, \tau_2) \;\textit{ if }\; \langle \tau_1^\mathcal{I}, \tau_2^\mathcal{I}\rangle \in r^\mathcal{I}, \qquad \mathcal{I} \models C \sqsubseteq D \;\textit{ if }\; C^\mathcal{I} \subseteq D^\mathcal{I}.$$

Then, $\mathcal{I}$ satisfies *a KB* $\mathcal{K} = (\mathcal{A}, \mathcal{T})$, *written* $\mathcal{I} \models \mathcal{K}$, *if it satisfies each axiom in* $\mathcal{A}$ *and in* $\mathcal{T}$. *A KB* $\mathcal{K}$ *is* satisfiable *if there exists an interpretation* $\mathcal{I}$ *that satisfies* $\mathcal{K}$. *It is* intensionally satisfiable *if there exists such an intensional interpretation.*

The extension of the language by intensionally interpreted ε-individuals is conservative, in the sense that every 'standard' interpretation can be extended to an intensional one.

Theorem 4 (Embedding Theorem). *Let* $\mathcal{K}$ *be an* $\mathcal{ALC}^\varepsilon$ *KB that mentions no individual descriptions of the form* $\varepsilon.C$. *Then,* $\mathcal{K}$ *is satisfiable if and only if* $\mathcal{K}$ *is intensionally satisfiable.*

Proof. The backward direction follows from the definition. For the forward direction, assume that $\mathcal{K}$ is satisfiable and let $\mathcal{I} \models \mathcal{K}$. Let $f : \mathcal{P}(\Delta^\mathcal{I}) \to \Delta^\mathcal{I}$ be a *choice function*—that is, a function that selects an element in the argument subset of $\Delta^\mathcal{I}$ if this subset is non-empty and an arbitrary element of $\Delta^\mathcal{I}$ otherwise. Let an intentional interpretation $\mathcal{I}_I$ have domain $\Delta^\mathcal{I}$, interpret all concept, role, and individual names as $\mathcal{I}$, and have $\varepsilon.C^{\mathcal{I}_I} = f(C^\mathcal{I})$ for each C. □

3. The Calculus for $\mathcal{ALC}^\varepsilon$ and its Complexity

An initial observation regarding reasoning in $\mathcal{ALC}^\varepsilon$ is that ε-individuals in ABox axioms of the form $C(\varepsilon.D)$ can be used to express arbitrary TBox axioms: for each concept C, we have $\mathcal{I} \models (\neg C)(\varepsilon.C)$ if and only if $C^\mathcal{I} = \emptyset$, and thus, for every two concept descriptions C and D,

$$\mathcal{I} \models C \sqsubseteq D \iff (C \sqcap \neg D)^\mathcal{I} = \emptyset \iff \mathcal{I} \models (\neg C \sqcup D)(\varepsilon(C \sqcap \neg D)). \tag{1}$$

This equivalence immediately yields a lower bound on expressive power (and as a result, efficiency of reasoning): $\mathcal{ALC}^{\varepsilon}$ under intensional semantics without TBoxes is at least as expressive as $\mathcal{ALC}$ with general TBoxes. Thus, $\mathcal{ALC}^{\varepsilon}$ ABox satisfiability is EXPTIME-hard [16, 17, 18, 19], exceeding the PSPACE-completeness of this problem for standard $\mathcal{ALC}$ (note here that concept satisfiability in $\mathcal{ALC}^{\varepsilon}$ coincides with that in $\mathcal{ALC}$). In the next section, we will see that EXPTIME is also an upper bound, both with and without TBox.

Before this, however, we introduce a tableau calculus for the satisfiability problem in $\mathcal{ALC}^{\varepsilon}$, focusing on ABox; in light of equivalence (1), this calculus applies also to $\mathcal{ALC}^{\varepsilon}$ KBs. Our calculus operates concepts in negation normal form—that is, with negation applied only to concept names. Reasoning about ε-individuals $\varepsilon.C$ requires distinguishing two cases: one where $C^{\mathcal{I}}$ is non-empty, in which case $\mathcal{I} \models C(\varepsilon.C)$, and one where $C^{\mathcal{I}}$ is empty. The latter is expressible as $(\neg C)(\varepsilon.C)$, but this introduces an additional negation that disrupts the negation normal form and thus requires re-normalisation. This, in turn, complicates the reasoning process when saturating open branches. So, we instead introduce a new auxiliary form of ABox axioms, $\overline{C}$, which expresses the emptiness of a concept C. However, we emphasise that such axioms are introduced purely as a reasoning aid, as $\overline{C}$ is semantically equivalent to $(\neg C)(\varepsilon.C)$.

Definition 5. *A* Ξ-ABox *is an $\mathcal{ALC}^{\varepsilon}$ ABox that additionally allows for* Ξ-axioms *of the form $\overline{C}$, where C is an $\mathcal{ALC}^{\varepsilon}$ concept. For an interpretation $\mathcal{I}$, we let $\mathcal{I} \models \overline{C}$ if $C^{\mathcal{I}} = \emptyset$.*

We are now ready to define the negation normal form.

Definition 6 (Negation Normal Form)**.** *An $\mathcal{ALC}^{\varepsilon}$ concept C is in* negation normal form *(*NNF*) if negation appears only in front of concept names. A Ξ-ABox $\mathcal{A}$ is in* NNF *if, for every axiom of the form $C(\tau)$ in $\mathcal{A}$, the concept C is in NNF.*

As usual, every concept C and, thus every Ξ-ABox $\mathcal{A}$ can be transformed in polynomial time into an equivalent concept and ABox, respectively, in NNF, which are denoted $\sim C$ and $\sim \mathcal{A}$.

Note that we do not require the concepts within ε-individuals in an ABox in NNF to be in NNF. In fact, such transformations must be avoided, as they may alter the syntactic identity of individual descriptions. For instance, if C_1 and C_2 are syntactically different, then $\varepsilon.C_1$ and $\varepsilon.C_2$ may be interpreted differently, but if their respective NNFs $\sim C_1$ and $\sim C_2$ are syntactically identical, then $\varepsilon.(\sim C_1)$ and $\varepsilon.(\sim C_2)$ must be interpreted by the same domain element. Moreover, the concept C in a Ξ-axiom $\overline{C}$ need not be in NNF, even if the containing ABox is.

The tableau calculus for $\mathcal{ALC}^{\varepsilon}$, which is defined next, is based on the one by Bucheit et al. [17]. The calculus operates on a set $\mathcal{S}$ of Ξ-ABoxes in NNF and consists of six rules, each attempting to replace an ABox in $\mathcal{S}$ by one or two new ABoxes. The goal is to demonstrate unsatisfiability by deriving a *clash*—that is, the presence of both $A(\tau)$ and $\neg A(\tau)$ in an ABox for some $A \in N_C$ and individual description τ. If every ABox in the set contains a clash, then the calculus concludes that each ABox in the input $\mathcal{S}$ is unsatisfiable. Conversely, if an ABox contains no clash and no rule is applicable to it, then the input $\mathcal{S}$ contains a satisfiable ABox.

Definition 7 (Tableau Calculus for $\mathcal{ALC}^{\varepsilon}$)**.** *Given a finite set $\mathcal{S}$ of Ξ-ABoxes, where, in each $\mathcal{A} \in \mathcal{S}$, certain individual descriptions are designated to be* ancestors *of others, a* rule application *replaces some ABox $\mathcal{A}$ in $\mathcal{S}$ by one or two ABoxes $\mathcal{A}'$ and (potentially) $\mathcal{A}''$ (where the ancestry relation for common individual descriptions is inherited from $\mathcal{A}$) by the following rules:*

1. *if $\mathcal{A}$ includes $(C_1 \sqcap C_2)(\tau)$, but not both $C_1(\tau)$ and $C_2(\tau)$, then $\mathcal{A}' := \mathcal{A} \cup \{C_1(\tau), C_2(\tau)\}$;*
2. *if $\mathcal{A}$ includes $(C_1 \sqcup C_2)(\tau)$, but neither $C_1(\tau)$ nor $C_2(\tau)$, then $\mathcal{A}' := \mathcal{A} \cup \{C_1(\tau)\}$ and $\mathcal{A}'' := \mathcal{A} \cup \{C_2(\tau)\}$;*
3. *if $\mathcal{A}$ includes $(\forall r.C)(\tau)$ and $r(\tau, \tau')$, but not $C(\tau')$ then $\mathcal{A}' := \mathcal{A} \cup \{C(\tau')\}$;*

4. *if $\mathcal{A}$ includes $(\exists r.C)(\tau)$ such that there is no τ' with $C(\tau')$ and $r(\tau,\tau')$ in $\mathcal{A}$, and there is no ancestor τ'' of τ that is* blocked*—that is, has an ancestor τ''' such that $C'(\tau'') \in \mathcal{A}$ implies $C'(\tau''') \in \mathcal{A}$ for every concept C'—then $\mathcal{A}' := \mathcal{A} \cup \{C(b), r(\tau, b)\}$, where b is a fresh individual name with ancestors τ and all ancestors of τ (assuming that N_O is sufficiently large);*
5. *if $\mathcal{A}$ includes $C_1(\varepsilon.C_2)$, but neither $(\sim C_2)(\varepsilon.C_2)$ nor $\overline{C_2}$, then $\mathcal{A}' := \mathcal{A} \cup \{(\sim C_2)(\varepsilon.C_2)\}$ and $\mathcal{A}'' := \mathcal{A} \cup \{\overline{C_2}\}$;*
6. *if $\mathcal{A}$ includes $\overline{C}$ and an individual description τ is mentioned in an axiom in $\mathcal{A}$, but $(\sim(\neg C))(\tau)$ is not in $\mathcal{A}$, then $\mathcal{A}' := \mathcal{A} \cup \{(\sim(\neg C))(\tau)\}$.*

Among the rules in Definition 7, the first four are inherited from the standard tableau calculus for $\mathcal{ALC}$, while the last two address reasoning with ε-individuals. In particular, Rule 5 splits the branch into two: if an ABox includes some $\varepsilon.C$, then either $\varepsilon.C$ satisfies C or C is empty. Then, Rule 6 is essentially a reframing of the standard TBox-rule in terms of Ξ-axioms.

We argue the correctness of our calculus by means of soundness and completeness theorems.

Theorem 8 (Soundness)**.** *Let $\mathcal{S}'$ be a set of Ξ-ABoxes obtained from a set $\mathcal{S}$ of Ξ-ABoxes in NNF by an application of a rule in Definition 7. Then, each ABox in $\mathcal{S}'$ is in NNF. Moreover, if each ABox in $\mathcal{S}'$ is intensionally unsatisfiable, then each ABox in $\mathcal{S}$ is intensionally unsatisfiable.*

Proof sketch. Let $\mathcal{A}'$ and (potentially) $\mathcal{A}''$ be the Ξ-ABoxes obtained by a rule application to a Ξ-ABox $\mathcal{A}$. We show that if $\mathcal{A}$ is intensionally satisfiable, then either $\mathcal{A}'$ or $\mathcal{A}''$ is intensionally satisfiable. We concentrate on Rules 5 and 6 here, since the other rules are standard.

Assume first that Rule 5 is applied to $C_1(\varepsilon.C_2)$ and $\mathcal{I}$ is an intensional interpretation such that $\mathcal{I} \models C_1(\varepsilon.C_2)$. If $C_2^{\mathcal{I}} \neq \emptyset$ then $(\varepsilon.C_2)^{\mathcal{I}} \in C_2^{\mathcal{I}}$ since $\mathcal{I}$ is intensional, and so $\mathcal{I} \models C_2(\varepsilon.C_2)$, implying $\mathcal{I} \models (\sim C_2)(\varepsilon.C_2)$. Otherwise, $\mathcal{I} \models \overline{C_2}$ by definition.

Assume now that Rule 6 is applied to $\overline{C}$. By definition, $C^{\mathcal{I}} = \emptyset$, and so $\tau \notin C^{\mathcal{I}}$ implies $\tau \in (\neg C)^{\mathcal{I}} = (\sim(\neg C))^{\mathcal{I}}$ for every individual description τ. □

Theorem 9 (Completeness)**.** *Assume that no rules in Definition 7 is applicable to a set $\mathcal{S}$ of Ξ-ABoxes in NNF. If $\mathcal{A} \in \mathcal{S}$ contains no clash then $\mathcal{A}$ is intensionally satisfiable.*

Proof sketch. Given an ABox $\mathcal{A}$ as described, we construct an intentional interpretation $\mathcal{I}_{\mathcal{A}}$ satisfying $\mathcal{A}$. The domain $\Delta^{\mathcal{I}_{\mathcal{A}}}$ of $\mathcal{I}_{\mathcal{A}}$ consist of all individual descriptions mentioned in $\mathcal{A}$. Then, for each individual description τ appearing in $\mathcal{A}$, we set $\tau^{\mathcal{I}_{\mathcal{A}}} = \tau$; in particular, for every concept C, if $\varepsilon.C$ appears in $\mathcal{A}$ then $(\varepsilon.C)^{\mathcal{I}_{\mathcal{A}}} = \varepsilon.C$. Moreover, for $\varepsilon.C$ not mentioned in $\mathcal{A}$, we set $(\varepsilon.C)^{\mathcal{I}_{\mathcal{A}}} = f(C^{\mathcal{I}_{\mathcal{A}}})$ for a choice function f for $\Delta^{\mathcal{I}_{\mathcal{A}}}$. Finally, for each concept name A, we set $A^{\mathcal{I}_{\mathcal{A}}} = \{\tau \mid A(\tau) \in \mathcal{A}\}$, and for each role name r, we define $r^{\mathcal{I}_{\mathcal{A}}}$ as follows: if $r(\tau_1, \tau_2) \in \mathcal{A}$, then $\langle \tau_1, \tau_2 \rangle \in r^{\mathcal{I}_{\mathcal{A}}}$, and if τ_1 is blocked by τ_1' and $r(\tau_1', \tau_2) \in \mathcal{A}$, then $\langle \tau_1, \tau_2 \rangle \in r^{\mathcal{I}_{\mathcal{A}}}$. It is a routine to show that $\mathcal{I}_{\mathcal{A}}$ is intentional and satisfies $\mathcal{A}$. □

Theorems 8 and 9 give us the following result; note that termination follows from the same arguments as for plain $\mathcal{ALC}$ (with TBox) [17].

Theorem 10 (Calculus Correctness)**.** *Let $\mathcal{A}$ be an $\mathcal{ALC}^{\varepsilon}$ ABox. The tableau calculus in Definition 7 terminates on $\{\sim\mathcal{A}\}$ after a finite number of rule applications. Moreover, $\mathcal{A}$ is satisfiable if and only if the set of ABoxes after the termination contains an ABox without a clash.*

We finish the section with the complexity of $\mathcal{ALC}^{\varepsilon}$ reasoning.

Theorem 11 (Complexity)**.** *The problem of $\mathcal{ALC}^{\varepsilon}$ KB satisfiability is EXPTIME-complete.*

As mentioned above, the lower bound follows from the facts that we can encode TBoxes in $\mathcal{ALC}^{\varepsilon}$ ABoxes and that $\mathcal{ALC}$ concept satisfiability with TBox is EXPTIME-complete [18, 19]. For the upper bound, note that, as for plain $\mathcal{ALC}$, the calculus in Definition 7 may run in time higher than exponential, and so it cannot be used as a justification of the upper bound. So, we defer the argument to Theorem 16, where we prove the hardness for a larger logic, $\mathcal{ALCO}^{\varepsilon}$. Note that, although the addition of nominals does not increase the complexity, we treat $\mathcal{ALC}^{\varepsilon}$ in a separate section, both to show how ε individuals can be handled in a calculus, and to start with a simpler formalism without nesting.

4. Reduction of $\mathcal{ALCO}^{\varepsilon}$ to $\mathcal{ALCO}_u$

The treatment of ε-individuals in $\mathcal{ALC}^{\varepsilon}$ was simplified by the fact they do not occur in concepts, and thus cannot be nested. The situation changes when the language is extended to allow individual descriptions to appear inside concepts, as is the case in DLs with *nominals*, such as $\mathcal{ALCO}$. In this section, we study the extension $\mathcal{ALCO}^{\varepsilon}$ of this logic with ε-individuals. Rather than extending our tableau calculus, we propose a reduction of $\mathcal{ALCO}^{\varepsilon}$ to $\mathcal{ALCO}_u$, the extension of $\mathcal{ALC}$ with nominals and the universal role. This reduction-based approach enables us to rely on existing results for $\mathcal{ALCO}_u$, and offers the practical advantage of supporting the reuse of existing theorem prover implementations for reasoning with ε-individuals.

Definition 12 (Syntax of $\mathcal{ALCO}^{\varepsilon}$)**.** *Concepts and individual descriptions in $\mathcal{ALCO}^{\varepsilon}$ are defined by extending Definition 1 with concepts of the form $\{\tau\}$, called* nominals*, with τ an individual description. Then, $\mathcal{ALCO}^{\varepsilon}$ axioms and KBs are defined the same as for $\mathcal{ALC}^{\varepsilon}$ in Definition 1.*

For example, $\varepsilon.(\{\varepsilon.C\} \sqcup \{\varepsilon.D\})$ is an $\mathcal{ALCO}^{\varepsilon}$ individual description with nested ε-individuals.

Definition 13 (Semantics of $\mathcal{ALCO}^{\varepsilon}$)**.** Interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ *for $\mathcal{ALCO}^{\varepsilon}$ extend Definition 2 by interpreting $\{\tau\}^{\mathcal{I}} = \{\tau^{\mathcal{I}}\}$ for every individual description τ.* Intensionality, satisfaction*, and (*intentional*)* satisfiability *then applies to $\mathcal{ALCO}^{\varepsilon}$ as in Definitions 2 and 3.*

Although we do not adopt the UNA, we can express $\tau_1 \neq \tau_2$ in $\mathcal{ALCO}^{\varepsilon}$ as $\{\tau_1\} \sqcap \{\tau_2\} \sqsubseteq \bot$.

As said above, we next reduce $\mathcal{ALCO}^{\varepsilon}$ reasoning to reasoning in the standard DL $\mathcal{ALCO}_u$. Formally, it is the same as $\mathcal{ALCO}^{\varepsilon}$ except that it does not allow for ε-individuals, but uses N_R extended with a special role u that is interpreted as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ in every interpretation $\mathcal{I}$.

Definition 14 (Reduction of $\mathcal{ALCO}^{\varepsilon}$)**.** *Let a_C be a fresh unique individual name for each concept C. Then, the $\mathcal{ALCO}_u$* reduction *of an $\mathcal{ALCO}^{\varepsilon}$ KB $\mathcal{K}$ is the $\mathcal{ALCO}_u$ KB obtained from $\mathcal{K}$ by first adding, for every ε-individual $\varepsilon.C$ mentioned in $\mathcal{K}$, the ABox axiom $(\neg\exists u.C \sqcup C)(\varepsilon.C)$, and then replacing, in each axiom (including the ones added at the first step), every occurrence of an ε-individual $\varepsilon.C$ that is not part of another ε-individual with a_C.*

We now show that this reduction indeed preserves the satisfiability of KBs. The key part of the proof is the iterative construction of an $\mathcal{ALCO}^{\varepsilon}$ intentional interpretation from an $\mathcal{ALCO}_u$ interpretation. This construction is not trivial as it must guarantee the intensionality not only for the ε-individuals occurring in the KB, but for all such individuals.

Theorem 15. *An $\mathcal{ALCO}^{\varepsilon}$ KB is intensionally satisfiable if and only if its $\mathcal{ALCO}_u$ reduction is satisfiable.*

Proof sketch. Let $\mathcal{K}$ be a $\mathcal{ALCO}^{\varepsilon}$ KB and $\mathcal{K}^*$ be its $\mathcal{ALCO}_u$ reduction.

For the forward direction, assume that $\mathcal{I} \models \mathcal{K}$ for an intentional interpretation $\mathcal{I}$. Define the $\mathcal{ALCO}_u$-interpretation $\mathcal{I}^*$ with $\Delta^{\mathcal{I}^*} = \Delta^{\mathcal{I}}$ as follows. For each concept, role, or individual name x in the language of $\mathcal{K}$, set $x^{\mathcal{I}^*} = x^{\mathcal{I}}$, and for each a_C obtained from the corresponding $\varepsilon.C$, set $a_C^{\mathcal{I}^*} = \varepsilon.C^{\mathcal{I}}$. It follows immediately from the definitions that $C^{\mathcal{I}} = (C^*)^{\mathcal{I}^*}$ for every concept C appearing in $\mathcal{K}$ and its corresponding replacement C^* as in Definition 14, and hence all axioms in $\mathcal{K}^*$ that are reductions of the axioms in $\mathcal{K}$ are satisfied by $\mathcal{I}^*$. Moreover, the reductions of the added axioms $(\neg\exists u.C \sqcup C)(\varepsilon.C)$ are also satisfied, which can be shown by simple analysis of two cases: whether $(C^*)^{\mathcal{I}^*}$ is empty or not.

For the backward direction, let $\mathcal{I}^* \models \mathcal{K}^*$ for an interpretation $\mathcal{I}^*$. We construct an $\mathcal{ALCO}^{\varepsilon}$-interpretation $\mathcal{I}$ as the limit of interpretations, each handling subsequent 'layers' of ε-nesting. To this end, we first let X_0 be the union of N_C, N_O, N_R, and the set of all $\varepsilon.C$ such that C contains no individual descriptions, and then, for each $n \geq 0$, let X_{n+1} be the set of all $\varepsilon.C$ such that C mentions at least one individual description from X_n and all individual descriptions mentioned in it are from $X_1, \ldots, X_n$. Using these sets, we next construct a sequence of interpretations $\mathcal{I}_n$, $n \geq 0$. First, we let $\mathcal{I}_0$ be defined as follows: $\Delta^{\mathcal{I}_0} = \Delta^{\mathcal{I}^*}$; then for each concept, role, and individual name x, we let $x^{\mathcal{I}_0} = x^{\mathcal{I}^*}$; then, for each $\varepsilon.C \in X_0$, if a_C appears in $\mathcal{K}^*$, then we set $\varepsilon.C^{\mathcal{I}_0} = a_C^{\mathcal{I}^*}$, and otherwise we set $\varepsilon.C^{\mathcal{I}_0} = f(C^{\mathcal{I}^*})$, where f is a choice function; finally, for every other $\varepsilon.C$, we set $\varepsilon.C^{\mathcal{I}_0} = d$ for an arbitrary $d \in \Delta^{\mathcal{I}_0}$ (note that, for $\mathcal{I}$, the latter will be all redefined later). Then, for each $n \geq 0$, we define $\mathcal{I}_{n+1}$ as follows: $\Delta^{\mathcal{I}_{n+1}} = \Delta^{\mathcal{I}_n}$; for each $\varepsilon.C \in X_{n+1}$, if a_C appears in $\mathcal{K}^*$, then we set $\varepsilon.C^{\mathcal{I}_{n+1}} = a_C^{\mathcal{I}^*}$, and otherwise, we set $\varepsilon.C^{\mathcal{I}_{n+1}} = f(C^{\mathcal{I}_n})$, where f is again a choice function; finally, for every other concept, role name, or individual description x, we set $x^{\mathcal{I}_n+1} = x^{\mathcal{I}_n}$. We now define $\mathcal{I} = \bigcup_{i \geq 0} \mathcal{I}'_n$, where, for each $n \geq 0$, $\mathcal{I}'_n$ is the restriction of $\mathcal{I}_n$ to $\bigcup_{i=0}^{n} X_n$. We can then show that $\mathcal{I}$ is indeed an intentional interpretation of $\mathcal{K}$. □

To conclude this section, we argue the complexity of reasoning in $\mathcal{ALCO}^\varepsilon$.

Theorem 16. *The problem of $\mathcal{ALCO}^\varepsilon$ KB satisfiability is EXPTIME-complete.*

Proof. The lower bound is argued in Theorem 11. For the upper bound, we can first observe that the reduction in Definition 14 can be realised in polynomial time, then apply Theorem 15 to reduce our problem to satisfiability of $\mathcal{ALCO}_u$ KBs, and finally apply the observation of Artale et al. [11] that the result of Passay and Tinchev [20] about the EXPTIME membership of satisfiability of formulas in Propositional Dynamic Logic extended with nominals and the universal modality can be easily adapted to satisfiability of $\mathcal{ALCO}_u$ KBs. □

5. Concept Subsumption in $\mathcal{ELO}^\varepsilon$

In this section, we study the impact of adding ε-individuals to the lightweight DL $\mathcal{ELO}$. As with plain $\mathcal{ELO}$, satisfiability in the extended logic $\mathcal{ELO}^\varepsilon$ is trivial, since all KBs are satisfiable. Thus, in line with standard practice for $\mathcal{EL}$-based DLs, our reasoning problem is *concept subsumption*—that is, the problem of checking whether a KB $\mathcal{K}$ entails, over intentional interpretations, an inclusion axiom $C_1 \sqsubseteq C_2$. We will write this entailment as $\mathcal{K} \models_{int} C_1 \sqsubseteq C_2$. We show that concept subsumption $\mathcal{ELO}^\varepsilon$ can be reduced to the case where $\mathcal{K}$ has empty ABox and all inclusion axioms (including $C_1 \sqsubseteq C_2$) are in a simple normal form. We then extend the standard $\mathcal{ELO}$ concept subsumption algorithm to handle $\mathcal{ELO}^\varepsilon$ in this form. Before proceeding, we note that adding ε-individuals to a more standard DL $\mathcal{EL}$ is not very interesting, as concept subsumption in the resulting logic coincides with concept subsumption in plain $\mathcal{EL}$.

Definition 17 ($\mathcal{ELO}^\varepsilon$)**.** *Let $\mathcal{ELO}^\varepsilon$ be the sublanguage of $\mathcal{ALCO}^\varepsilon$ with the concepts grammar*

$$C ::= A \mid C \sqcap C \mid \exists r.C \mid \{\tau\} \mid \top.$$

The semantics of $\mathcal{ELO}^\varepsilon$ is inherited from $\mathcal{ALCO}^\varepsilon$, with $\top^\mathcal{I} = \Delta^\mathcal{I}$ for every interpretation $\mathcal{I}$.

We next define the normal form for $\mathcal{ELO}^\varepsilon$ KBs, which has no ABox and no complex concepts.

Definition 18 (Normal Form)**.** *An $\mathcal{ELO}^\varepsilon$ KB $\mathcal{K}$ is in* normal form *if its ABox is empty and inclusion axioms are of the following forms, where each C, C_1, C_2, D is $\top$, a concept name, or a nominal $\{\tau\}$, for τ an individual name or an ε-individual $\varepsilon.A$ with a concept name A:*

$$C \sqsubseteq D, \qquad C_1 \sqcap C_2 \sqsubseteq D, \qquad C \sqsubseteq \exists r.D, \qquad \exists r.C \sqsubseteq D.$$

We next show that each $\mathcal{ELO}^\varepsilon$ KB can be easily normalised.

Proposition 19. *For each $\mathcal{ELO}^\varepsilon$ KB $\mathcal{K}$, we can construct, in polynomial time, an $\mathcal{ELO}^\varepsilon$ KB $\mathcal{K}'$ in normal form such that, for every axiom ϕ in the signature of $\mathcal{K}$, $\mathcal{K} \models_{int} \phi$ if and only if $\mathcal{K}' \models_{int} \phi$.*

Proof sketch. Given an $\mathcal{ELO}^\varepsilon$ KB $\mathcal{K}$ we can first eliminate complex concepts in ε-individuals by replacing each $\varepsilon.C$ in $\mathcal{K}$ that is not nested within another ε-individual with $\varepsilon.A_C$, for A_C a fresh concept name, and adding the corresponding axiom $C \equiv A_C$ (as usual, expressible using concept inclusions). Then, we can apply usual normalisation as for $\mathcal{ELO}$. □

The following procedure takes as input an $\mathcal{ELO}^\varepsilon$ KB $\mathcal{K}$ in normal form and concept names A, B. This procedure determines whether $\mathcal{K} \models_{int} A \sqsubseteq B$. Note that the procedure can be generalised to checking whether $\mathcal{K} \models_{int} C \sqsubseteq D$ for arbitrary concepts C, D by applying the algorithm to the normalisation of $\mathcal{K} \cup \{A \equiv C, B \equiv D\}$, where A, B are fresh concept names.

Definition 20 (Entailment Procedure for $\mathcal{ELO}^\varepsilon$). *Let $\mathcal{K}$ be an $\mathcal{ELO}^\varepsilon$ KB in normal form and A, B be concept names. For each concept C and role name r mentioned in $\mathcal{K}$, initialise the following sets of concepts and pairs of concepts, respectively: $S(C) = \{C, \top\}$ and $R(r) = \emptyset$. Update these sets according to the following rules until no rules can be applied, where the relation $\rightsquigarrow_R$ on concept pairs is defined so that $C \rightsquigarrow_R D$ if and only if there exist concepts $C_1, \ldots, C_k$ such that $C_1 = C$ or $C_1 = \{\tau\}$ for some individual description τ, $(C_j, C_{j+1}) \in R(r_j)$ for some role name r_j for each $j = 1, \ldots, k-1$, and $C_k = D$:*

(CR1) if $C' \in S(C)$, $C' \sqsubseteq D \in \mathcal{K}$, and $D \notin S(C)$, then set $S(C) := S(C) \cup \{D\}$;

(CR2) if $C_1, C_2 \in S(C)$, $C_1 \sqcap C_2 \sqsubseteq D \in \mathcal{K}$, and $D \notin S(C)$, then set $S(C) := S(C) \cup \{D\}$;

(CR3) if $C' \in S(C)$, $C' \sqsubseteq \exists r.D \in \mathcal{K}$, and $(C, D) \notin R(r)$, then set $R(r) := R(r) \cup \{(C, D)\}$;

(CR4) if $(C, D) \in R(r)$, $D' \in S(D)$, $\exists r.D' \sqsubseteq E \in \mathcal{K}$, and $E \notin S(C)$, then set $S(C) := S(C) \cup \{E\}$;

(CR5) if $\{a\} \in S(C) \cap S(D)$, $A \rightsquigarrow_R D$ and $S(D) \not\subseteq S(C)$, then set $S(C) := S(C) \cup S(D)$;

(CR6) if $A \rightsquigarrow_R C$ and $C \notin S(\{\varepsilon.C\})$, then set $S(\{\varepsilon.C\}) := S(\{\varepsilon.C\}) \cup \{C\}$.

Output yes if and only if $B \in S(A)$.

The correctness of the algorithm follows from the soundness and completeness theorems.

Theorem 21 (Soundness). *Let $\mathcal{K}$ be an $\mathcal{ELO}^\varepsilon$ KB in normal form and A, B be concept names. Let $\mathcal{I}$ be an intensional interpretation such that $\mathcal{I} \models \mathcal{K}$ and $A^\mathcal{I} \neq \emptyset$. The following conditions are satisfied by the initial sets $S(C)$ and $R(r)$ for each two concepts C, C', and role r in the procedure in Definition 20 applied to $\mathcal{K}$, A, and B:*

(I1) $C' \in S(C) \implies \mathcal{I} \models C \sqsubseteq C'$,

(I2) $(C, C') \in R(r) \implies \mathcal{I} \models C \sqsubseteq \exists r.C'$.

Moreover, if these conditions are satisfied before the application of a rule, then they are satisfied after this application.

Proof sketch. For the initial sets, both conditions hold by construction.

Let now S and R be the sets before the application of a rule. In this sketch, we concentrate only on the rules relevant to the ε-individuals: *(CR5)* and *(CR6)*. Since the sets only expand, we only need to show that the conditions are satisfied for the added elements (which means, in particular, that in this sketch we concern only about *(I1)*).

Let the rule be *(CR5)*. Since $A \rightsquigarrow_R D$, there are $C_1, \ldots, C_k$ such that $C_1 = A$ or $C_1 = \{\tau\}$, $C_k = D$, and $(C_j, C_{j+1}) \in R(r_j)$ for some r_j for each j. Since $A^\mathcal{I} \neq \emptyset$ and, by definition, $\{\tau\}^\mathcal{I} \neq \emptyset$, we have $C_1^\mathcal{I} \neq \emptyset$. Thus, by definition of $\rightsquigarrow_R$ and condition *(I2)*, $D^\mathcal{I} \neq \emptyset$. Since, $D^\mathcal{I} \subseteq \{a^\mathcal{I}\}$ and $C^\mathcal{I} \subseteq \{a^\mathcal{I}\}$ by *(I1)*, we conclude that $C^\mathcal{I} \subseteq \{a\}^\mathcal{I} = D^\mathcal{I}$. As such, if some C' is in $S(D)$ (and thus in $S(C)$ after the rule application), we have $C^\mathcal{I} \subseteq D^\mathcal{I} \subseteq (C')^\mathcal{I}$ as needed.

Let the rule be *(CR6)*. As for *(CR5)*, $C^\mathcal{I} \neq \emptyset$. Since $\mathcal{I}$ is intensional, we have $\varepsilon.C^\mathcal{I} \in C^\mathcal{I}$. □

Theorem 22 (Completeness). *Let $\mathcal{K}$ be an $\mathcal{ELO}^\varepsilon$ KB in normal form, A, B be concept names, and S, R be sets of concepts and pairs of concepts, obtained by the procedure in Definition 20 applied to $\mathcal{K}$, A, and B so that no further rules can be applied. Then, $\mathcal{K} \models_{int} A \sqsubseteq B$ implies $B \in S(A)$.*

Proof. Let $\mathcal{K} \models_{int} A \sqsubseteq B$, but assume that $B \notin S(A)$. We will construct an intensional interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$, but $A^{\mathcal{I}} \not\subseteq B^{\mathcal{I}}$.

Let $C \sim C'$ for concepts C and C' if and only if $C = C'$ or there is some $\{a\} \in S(C) \cap S(C')$. The inapplicability of rule *(CR6)* implies that $\sim$ is an equivalence relation. Moreover, for each C, C', D such that $C \sim C'$ and for each $r \in N_R$, we have $S(C) = S(C')$ and $(C, D) \in R(r)$ implying $(C', D) \in R(r)$. Indeed, the former follows from the inapplicability of rule *(CR5)*. For the latter, note that there must be some iteration of the procedure where (C, D) is added to $R(r)$ by rule *(CR3)*. Thus, there is some $C_1 \in S(C)$ with $C_1 \sqsubseteq \exists r.D \in \mathcal{K}$. By the former, $S(C) = S(C')$, and so $C_1 \in S(C')$. Thus, $(C', D) \in R(r)$ since rule *(CR3)* is not applicable.

We are now ready to construct $\mathcal{I}$. First we set $\Delta^{\mathcal{I}}$ to be the set of all equivalence classes $[C]$ over $\sim$ for concepts C such that $A \rightsquigarrow_R C$. Then, we set $(A')^{\mathcal{I}} = \{[C] \mid A' \in S(C)\}$ for each concept name A' such that $A \rightsquigarrow_R A'$ and $(A')^{\mathcal{I}} = \emptyset$ for each other concept name A'. Moreover, we set $r^{\mathcal{I}} = \{\langle [C], [D] \rangle \mid (C, D) \in R(r)\}$ for each role name r. Finally, we set $\tau^{\mathcal{I}} = [\{\tau\}]$ for each individual description τ with $A \rightsquigarrow_R \{\tau\}$, and, for other individual descriptions τ, $\tau^{\mathcal{I}} = [A]$ if τ is an individual name and $\tau^{\mathcal{I}} = f((A')^{\mathcal{I}})$ for a choice function f on $\Delta^{\mathcal{I}}$ if $\tau = \varepsilon.A'$ (recall that $f(\emptyset)$ is an arbitrary element by definition of a choice function).

Interpretation $\mathcal{I}$ is intensional: for each concept name A' with $(A')^{\mathcal{I}} \neq \emptyset$, if $A \rightsquigarrow_R \{\varepsilon.(A')\}$, $(\varepsilon.(A'))^{\mathcal{I}} \in (A')^{\mathcal{I}}$ since *(CR6)* is inapplicable, and otherwise $\varepsilon.C^{\mathcal{I}} = f(C^{\mathcal{I}}) \in C^{\mathcal{I}}$.

The rest of the proof is mostly a result of the following claim.

Claim 23. *For each $[C] \in \Delta^{\mathcal{I}}$ and D as in Definition 18, $[C] \in D^{\mathcal{I}}$ if and only if $D \in S(C)$.*

Proof. Consider each possible form of D. If $D = \top$ then the claim follows since $\top \in S(C)$. If D is a concept name, then it follows by construction of $D^{\mathcal{I}}$. Finally, let $D = \{\tau\}$. On the one hand, $[C] \in \{\tau\}^{\mathcal{I}}$ implies $\tau^{\mathcal{I}} = [C]$, $[C] = [\{\tau\}]$ by definition of $\tau^{\mathcal{I}}$, and, since $\{\tau\}$ is in $S(\{\tau\})$ initially and $[\{\tau\}] \sim C$, we have $\{\tau\} \in S(C)$. On the other hand, if $\{\tau\} \in S(C)$, we have $[C] = [\{\tau\}]$ by definition of $\sim$, and so $\tau^{\mathcal{I}} = [C]$, implying $[C] \in \{\tau\}^{\mathcal{I}}$. ☐

In order to show that $A^{\mathcal{I}} \not\subseteq B^{\mathcal{I}}$, we observe that $[A] \in A^{\mathcal{I}}$ by construction, but, by Claim 23, $[A] \notin B^{\mathcal{I}}$, since $B \notin S(A)$.

We are left to show that $\mathcal{I} \models \mathcal{K}$. To this end, consider each form of an axiom in $\mathcal{K}$.

Let $C \sqsubseteq D$. For every C' with $[C'] \in C^{\mathcal{I}}$, we have $C \in S(C')$ by Claim 23. Since rule *(CR1)* is inapplicable, $D \in S(C')$, and so $[C'] \in D^{\mathcal{I}}$ also by Claim 23.

Let $C \sqsubseteq C_1 \sqcap C_2$. This case is similar to the previous one, except that we use rule *(CR2)*.

Let $C \sqsubseteq \exists r.D$. For each C' with $[C'] \in C^{\mathcal{I}}$, $C \in S(C')$ by Claim 23. Since rule *(CR3)* is inapplicable, $(C', D) \in R(r)$. Then, $\langle [C'], D \rangle \in r^{\mathcal{I}}$ implies $[D] \in D^{\mathcal{I}}$ and so $[C'] \in (\exists r.D)^{\mathcal{I}}$.

Let $\exists r.C \sqsubseteq D$. For each D' with $[D'] \in (\exists r.C)^{\mathcal{I}}$, there exists some $[C'] \in \Delta^{\mathcal{I}}$ such that $\langle [D'], [C'] \rangle \in r^{\mathcal{I}}$ and $[C'] \in C^{\mathcal{I}}$. Thus, there is some $C'' \in [C']$ such that $(D', C'') \in R(r)$. Since $[C''] = [C'] \in C^{\mathcal{I}}$, we have $C \in S(C'')$ by Claim 23, and so $D \in S(D')$ since rule *(CR4)* is inapplicable. Finally, Claim 23 gives us $[D'] \in D^{\mathcal{I}}$, as needed. ☐

Theorem 24 ($\mathcal{ELO}^{\varepsilon}$ Correctness)**.** *Given an $\mathcal{ELO}^{\varepsilon}$ KB $\mathcal{K}$ in normal form and concept names A, B, the procedure of Definition 20 terminates. Moreover, $\mathcal{K} \models_{int} C \sqsubseteq D$ if and only it answers yes.*

Proof. First, observe that the procedure is always terminating, because both S and R are subsets of finite sets and every rule increases one such set; nothing is ever removed. The soundness and completeness follow from Theorems 21 and 22. ☐

Finally, we show that the complexity of concept subsumption in $\mathcal{ELO}^{\varepsilon}$ is the same as in $\mathcal{EL}$.

Theorem 25. *Concept subsumption in $\mathcal{ELO}^{\varepsilon}$ is PTIME-complete.*

Proof. The lower bound is inherited from $\mathcal{EL}$, for which it follows from the PTIME-completeness of satisfiability of propositional Horn clauses [21]. The upper bound is by Proposition 19 and since the procedure in Definition 20 is polynomial, because no rules remove any elements from the sets and there only polynomial number of possible elements in total. ☐

6. Conclusion

In this paper, we studied the effect of adding Hilbert-style ε-individuals with intensional semantics to the description logics $\mathcal{ALC}$, $\mathcal{ALCO}$, and $\mathcal{ELO}$. We demonstrated that existing reasoning algorithms, including tableaux for $\mathcal{ALC}$ satisfiability and TBox saturation for $\mathcal{ELO}$ concept subsumption, can be extended to ε-individuals; however, the proofs of correctness of these algorithms—especially completeness—are far from trivial. In most cases (but not always), the complexity of reasoning is also preserved. For future work, we suggest studying the extensional semantics of ε-individuals in the context of DLs. The inclusion of reasoning about ε-individuals in other calculi for DL, such as hyper-tableaux [22], is also interesting.

Declaration on Generative AI

During the preparation of this work, the authors used GPT-4 in order to grammar and spelling check.

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