# Method of coordinate free for identification of the plane curves and its applications to motion planning*

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**Abstract**

In this paper we present a method for representing a continuous curve in the plane by a function defined on a unit interval, which we call the angular characteristic function of the curve. This representation is invariant under translations, rotations and universal scaling and is well suited for curve shape recognition. We have used this function for feature extraction in an experimental system for optical recognition of handwritten characters and the approach, briefly described in this paper, has shown good results. For graphic primitives - circular arcs and line segments - the angular characteristic function representing them has a particularly simple form. This circumstance allows using the angular characteristic function in motion planning problems and in robotics. As an example of such applications, we present an algorithm that can be used in a system for automatic program generation for a robotic manipulator used for a cutting process. This is an algorithm for approximating a smooth curve in the plane by a sequence of such graphic primitives forming a continuous smooth curve (in which each segment is conjugate to its neighbors).

**Keywords**

flat curve representation, image recognition, feature extraction, trajectory control

## 1. Introduction

Over the past few years, as machine learning and computer vision have experienced rapid advancements, image classification and recognition as well as increasing efficiency of the robot's sensor and control information processing have gained widespread application across diverse domains [1-3]. For instance, it is employed in security for facial recognition technology, in transportation for vehicle recognition, and in aerospace for aviation remote sensing, among other areas. The process of identifying graphical patterns stands out from other methods of pattern recognition due to its distinct approach. It involves several preliminary stages of preprocessing the input information. In the initial phases, the image undergoes transformations to make it suitable for subsequent analysis. Following this, a critical step involves selecting the type and quantity of informational attributes that can facilitate pattern recognition, whether with or without a reference sample [4]. Among these methods, the most crucial stage is the selection of informative attributes. These attributes play a pivotal role in distinguishing a pattern from a group of surrounding objects.

A method for representing a continuous plane curve is proposed in this paper and applications of this method to image recognition problems and robotics are discussed. The essence of the method is that a curve on a plane is assigned a function, defined on a unit interval, which we call the function of the angular characteristic of the curve. This representation is invariant under translations, rotations, and uniform scaling, which makes it very convenient for image recognition problems. The

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method was designed primarily for optical handwritten symbol recognition [5]. There are known studies on image recognition, in which algorithms for recognizing curves in an image based on curve segmentation were proposed [6-8]. An algorithm for automatic segmentation of a curve on a plane (such segmentation in which each segment is well approximated by either a straight-line segment or a circular arc) can be used to control a robot manipulator [9, 10], and is also promising for trajectory control of mobile robots [11, 12].

Strokes on paper that a person perceives as 1D curved lines correspond to curvilinear structures in a digital image. Those curvilinear structures possess a width and hence are not one-dimensional manifold. But their so called medial-axis, or simply center line is one dimensional manifold. Some detectors of curvilinear structures search for exactly these central lines. As an example, the algorithm described by Steger in the paper [13] can be considered. Usually, the detected central line is represented by an ordered sequence of vertices. Of course, some central lines may be closed, and then such a sequence of vertices defines a polygon. It is possible that the central lines may intersect, and in general, the output of such a detector can be considered as a planar graph. More precisely, an algorithm has been developed that transforms a black-and-white digital image into a planar graph in such a way that the points in the image corresponding to the graph vertices lie on imaginary central lines.

To avoid misunderstandings, two remarks should be made here. First, the term "extract a planar graph from an image" usually has a different meaning, namely, the extraction procedure is usually associated with image segmentation, and its result does not apply to curvilinear structures [14]. Second, to the best of the authors' knowledge, there are no known algorithms for detecting curvilinear structures that would directly transform an image into a planar graph. On the other hand, detecting curvilinear structures is a fairly broad area in which completely different problems are often set. The current state of affairs in this area can be found in the thesis [15].

The purpose of transforming an image into a planar graph of the type described above was to extract features for recognition from the planar graph. To analyze the obtained graph, procedures were carried out to identify connected subgraphs and construct spanning trees for connected subgraphs. This type of planar graph analysis can be called topological analysis. The chains of a planar graph correspond to broken lines with two ends. It is for recognizing their shape that we use the angular characteristic function.

Thus, we used the angular characteristic function for representation of continuous but not smooth plane curves. The angular characteristic function of a non-smooth curve (in particular, a broken line) has discontinuities. It is easy to see that if the graph of the angular characteristic function has the form of a broken line, that is, if the angular characteristic function is a piecewise linear function, then such an angular characteristic function represents a continuous smooth curve of a special type, which we call a polysegment line. A polysegment line is a sequence of conjugate arcs of circles and line segments. As shown in Section 2, using the angular characteristic function, one can construct an efficient algorithm for approximating a plane curve with a polysegment line.

# 2. Methods

## 2.1. Function of the angular characteristic of a smooth regular curve

A smooth plane curve given parametrically as $\gamma(t) \in \mathbb{R}^2$ can be considered as a spatial trajectory of a point moving along the plane, with the velocity vector of the point being defined as $\dot{\gamma}$. and the acceleration vector as $\ddot{\gamma}$.It is easy to prove that for the angular velocity $\omega$o f rotation of the vector $\dot{\gamma}$, the formula (1) is valid.

$$\omega = \frac{\dot{\gamma} \wedge \ddot{\gamma}}{|\dot{\gamma}|^2}, \tag{1}$$

where $a \wedge b$ denotes the external product of vectors $a, b \in \mathbb{R}^2$. If we denote as $S(a, b)$ the oriented area of the parallelogram, spanned by vectors $a, b$, then in any Cartesian basis the equalities (2) holds true.

$$\forall a, b \in \mathbb{R}^2 \; a \wedge b = S(a, b) = a_x b_y - a_y b_x. \tag{2}$$

Let us now proceed to the parameterization of the $\gamma$ by a natural parameter $s$, imagining that an imaginary point moves along the plane so that at any moment $|\dot{\gamma}| = 1$. So, if we denote the length of $\gamma$ by $L$, then equality $\forall s \in [0. L] \; |\dot{\gamma}(s)| = 1$ holds true, and will denote $\dot{\gamma}(s)$ as $\vec{e}(s)$ to emphasize this fact. For an arbitrary point $M(s_0) \in \gamma$, we define $k(s_0)$ as (3):

$$k(s_0) = \lim_{\Delta s \to 0} \frac{\angle(\vec{e}(s_0+\Delta s), \vec{e}(s_0))}{\Delta s}. \tag{3}$$

It should be noted that $|k(s_0)| = \frac{1}{R}$, where $R$ is the radius of the tangent circle, is the curvature of the curve $\gamma$at point $M(s_0)$. Note that as $\forall s \in [0. L] \; |\dot{\gamma}(s)| = 1$ $\lim_{\Delta s \to 0} \frac{\angle(\vec{e}(s_0+\Delta s), \vec{e}(s_0))}{\Delta s} = \lim_{\Delta t \to 0} \frac{\angle(\vec{e}(s_0+|\dot{\gamma}(s)|\Delta t), \vec{e}(s_0))}{|\dot{\gamma}(s)|\Delta t} = \omega$, therefore, $k$ is analytically determined by the formula (4)

$$\forall s \in [0. L] \; k(s) = \dot{\gamma}(s) \wedge \ddot{\gamma}(s). \tag{4}$$

We define the function $\alpha(s)$ by the formula (5):

$$\alpha(s) = \int_0^s k(\tau)\, d\tau \tag{5}$$

Assuming some uncertainty, we can give the following clear geometric definition of the function $\alpha(s)$: Let $\vec{v}_0 = \vec{v}(0)$ be a tangent vector to $\gamma$ at endpoint $A$. Then $\alpha(s)$ is the angle between the vectors $\vec{v}_0$ and $\vec{v}(s)$ (see Fig.1).

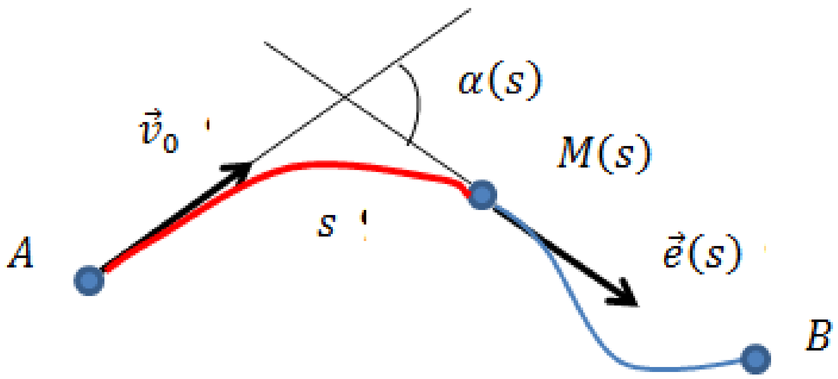


**Figure 1:** Geometrical interpretation of function $\alpha(s)$.

The function $\alpha(s)$ is defined on the interval $[0, L]$, where $L$ is the length of the curve $\gamma$. We define on the interval $I = [0,1]$ the function $\theta(x)$ by expression (6)

$$\theta(x) = \alpha(x \cdot L). \tag{6}$$

We will call $\theta(x)$ the function of the angular characteristic of the curve $\gamma$ (or simply the angular characteristic of $\gamma$). As will be shown below, this function has properties that allow it to be effectively used in symbol recognition tasks.

## 2.2. Properties of an angular characteristic function

1) Invariance to similarity transformations of the plane.

The most important property of the function of angular characteristics is its invariance to similarity transformations of the plane. Below we provide the necessary explanations: A similarity is a transformation of Euclidean plane which maps lines to lines and preserves the sizes of angels. The set of all similarities is the similarity group $S(2)$. Each similarity transformation can be viewed as a composition of translation, rotation, and uniform scaling. Let $\gamma$ be a regular curve with endpoints $A$ and $B$. Applying an arbitrary transformation $f \in S(2)$ to the plane transforms $\gamma$ into a regular curve $\gamma'$ with endpoints $f(A)$ and $f(B)$. Thus, if $\theta_1$ is the function of the angular characteristic of $\gamma$, then it corresponds to the function of the angular characteristic $\theta_2$ of the curve $\gamma'$. The statement about the invariance of the function of the angular characteristic to similarity transformations means that equality (7) holds.

$$\forall x \in [0,1]\ \theta_1(x) = \theta_2(x). \tag{7}$$

2) If the curve γ is a line segment, then its function of the angular characteristic $\theta$ of the γ is defined by (8)

$$\forall x \in [0,1]\ \ \theta(x) = 0. \tag{8}$$

3) If the curve is an arc of a circle cut off by an angle $\varphi$, then regardless of the radius of the circle, the function of the angular characteristics of this curve will be a linear function (9): A key novelty of our approach is the incorporation of a stability objective to encourage consistent outputs. We consider two types of stability:

$$\theta(x) = \ \varphi \cdot x. \tag{9}$$

It should be noted that the curves forming the symbol in the image can be approximated by curves made up of segments and arcs of a circle. The last two properties of the angular characteristics function make it easy to imagine the angular characteristics of these curves. In addition, the idea of approximating curves by joints of segments and arcs suggests an idea of using segmentation methods to recognize curves.

### 2.3. Function of angular characteristic of broken line

We denote a broken line by a sequence of vertices $\{A_0, A_1, \dots, A_n\}$. Let $(x_i, y_i)$ denote Cartesian coordinates of the $i$-th vertex. Then the broken line corresponds to a sequence of $n$ vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ given by equalities (10).

$$\vec{v}_i = \ (x_i - x_{i-1}, y_i - y_{i-1}) \tag{10}$$

We define the sequences $\{l_1, l_2, \dots, l_n\}$ and $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. by equalities (11) - (12).

$$l_i = |\vec{v}_i| \tag{11}$$

$$\varphi_i = \begin{cases} 0, if\ i = 1 \\ Ang(\vec{v}_i, \vec{v}_{i-1}), if\ i \neq 0 \end{cases} \tag{12}$$

Let $x_0, x_1, \dots, x_n$ be a sequence of points on the $X$ axis given by the equations (13)

$$x_i = \begin{cases} 0, if\ i = 0 \\ \displaystyle\sum_{k=1}^{i} l_k\,, if\ i \neq 0 \end{cases} \tag{13}$$

So, the interval $[0, L]$, where $L = \sum_{i=1}^{n} l_i$ is the length of broken line $\{A_0, A_1, \dots, A_n\}$, contains all points $x_0, x_1, \dots, x_n$. Now we define on the interval $[0, L]$ the step function $\alpha$ by (14)

$$\alpha(x) = \sum_{i=1}^{n} \varphi_i \cdot \chi_{D_i}(x), \tag{14}$$

where $\chi_{D_i}(x)$ is the indicator function of interval $D_i$ (15), and sequence of intervals $\{D_1, D_2, \dots, D_n\}$ is defined by equations (16).

$$\chi_{D_i}(x) = \begin{cases} 1 \ if \ x \in D_i \\ 0 \ if \ x \notin D_i \end{cases} \tag{15}$$

$$D_i = \begin{cases} [x_{i-1}, x_i) \ if \ 1 \leq i < n \\ (x_{i-1}, x_i] if \ i = n \end{cases} \tag{16}$$

The function of the angular characteristic $\theta$ of a broken line will be determined (as for a regular smooth curve, see (10)) by equation $\theta(x) = \alpha(x \cdot L)$. An example of a broken line (strictly speaking, a planar graph representing a tree with two leaves) and the corresponding function of the angular characteristic is shown below (see Fig. 2)

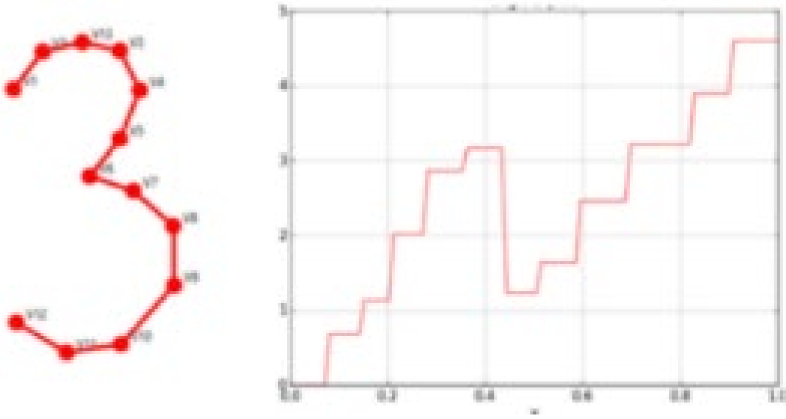

**Figure 2:** Broken line (planar graph of real image) and plot of its angular characteristic function

## 2.4. An optimal approximation of the smooth plane curve by polysegment line and its application in motion planning and robotics

Some problems in motion planning and robotics require that the plane curve representing a spatial trajectory be approximated by a special type of line, which we will call polysegment lines, defined as follows:

***Definition 1***. We define a polysegment line as a pair $(A, B)$, where $A = \{v_1, v_2, \dots, v_n\}$ is a sequence of $n$ vertices $\forall j \in \{1, .., n\}$ $v_j \in \mathbb{R}^2$ defining the polyline $p$, and $B = \{s_1, s_2, \dots, s_{n-1}\}$ is a sequence of $n - 1$ segments. Each segment $s_k$ where $k \in \{1, .., n - 1\}$ is a plane curve with two ends at vertices $v_k$ and $v_{k-1}$ which is either a line segment or an arc of a circle, connecting vertices $v_k$ and $v_{k-1}$. A polysegment line in which any two adjacent segments $s_j$ and $s_{j+1}$are either conjugate circular arcs or a straight-line segment, conjugate with an arc of a circle, we will call a conjugate polysegment line.

As an example of the problems that arise in the field of robotic automation of manufacturing, we will consider the problem of automatic generation of a program in the AS language for a robot manipulator cutting from a flat sheet of metal. In simple terms, the robot manipulator program in the AS language is a linear sequence of commands with the mnemonics "move" and "circle", determining the movement of the manipulator's working tool, respectively, along a straight line or along an arc of a circle. Accordingly, the spatial trajectory of the working tool (the cutter in this case) and the trace of the trajectory on the surface being processed are described by polysegment lines. An attempt to solve the problem by using one of the algorithms for approximating a smooth line with a broken line leads to the following issue: The operands of the move and circle commands determine the speed of movement of the working tool. If the manipulator program specifies the movement of the working tool along a broken line, then regardless of how exactly the speeds of movement on the trajectory segments were determined, the manipulator physically cannot execute such a program exactly as it is. From the point of view of physics, the reason is completely clear: the movement defined by such a program causes infinite radial acceleration of the working tool at the moments of passing the vertices of the broken line. In practice, if you load such a program into the manipulator controller, then during execution, the manipulator will stop moving every time the working tool passes over one of the vertices of an imaginary broken line on the surface of the metal sheet (the so-called trajectory trace on the workpiece surface). For some types of robotic cutting (e.g., plasma cutting), the accuracy of temporal parameters of the trajectory is critical. If the speed of the working tool (cutting tool) differs significantly from the prescribed one, this will lead to failure or even damage to the cutting tool. This problem is eliminated if we use the approximation of a smooth curve by a conjugate polysegment line instead of the approximation by a polyline. It is clear that it is desirable to have as few conjugation points as possible on a polysegment line approximating a given smooth plane curve. It is also clear that by reducing the number of segments, we reduce the achievable approximation accuracy. Taking into account the last two remarks, we can formulate the problem of optimal approximation of a smooth curve on a plane as follows: For a given maximum permissible approximation error sigma, find a polysegment line with a minimum number of segments that approximates the given plane curve with an approximation error less than or equal to sigma. As will be shown below, using the angular characteristic function allows us to construct a simple and effective algorithm for solving this problem.

## 2.5. Function of angular characteristic of broken line

The idea underlying the proposed algorithm arises from a simple observation: the graph of the angular characteristic function of an arbitrary conjugate polysegment line is a broken line. The corresponding statement is given below in expanded form as ***Lemma 1***. The angular characteristic function of a smooth curve on a plane will, of course, be a continuous smooth function. However, a continuous smooth function can be approximated by a broken line with any given approximation accuracy. This leads to the assumption that a broken line approximating the angular characteristic function of an arbitrarily smooth gamma curve on a plane will be the angular characteristic function of a polysegment line approximating the gamma curve. Of course, any angular characteristic function corresponds to a set of lines on a plane, and we must somehow choose an approximating polysegment line from this set (which will obviously be a set of polysegment lines). For an arbitrary angular characteristic function phi, there is an analytical expression that defines a set of curves on a plane for which phi will be a function of the angular characteristic. This set is parameterizable, and the parameterization has a simple geometric meaning. We give the precise formulation of these assertions as ***Lemma 2***. Based on the assertions of ***Lemma 1*** and ***Lemma 2***, it is easy to describe the set of polysegment lines defined by a piecewise linear continuous function of the angular characteristic. Finally, having at our disposal the two lemmas mentioned above, we formulate an algorithm for the optimal approximation of a smooth curve on a plane by a polysegment line.

***Lemma 1***: The angular characteristic function of an arbitrary polysegment line $(\{v_1,,..,v_n\},\{s_1,...,s_{n-1}\})$ with the number of vertices $n > 2$ will be a continuous piecewise linear function, analytically defined by formulas (19).

First, we formulate an easily proven statement. Let $\gamma$ be an arbitrary smooth curve on a plane with two ends at points $A$ and $B$. Denote by $\vartheta$ the function of the angular characteristic of this curve. Let the length of the $\gamma$ be equal to $L$. Let us choose an arbitrary point on the curve $M$. Let us denote by $l_1$ the length of the arc $AM$ of the $\gamma$ (in this case, of course, the length of the arc $MB$ will be equal to $L - l_1$). Let us denote by $\theta_1$ the function of the angular characteristic of the curve $AM$ and by $\theta_2$ the function of the angular characteristic of the curve $MB$. Then the function phi can be expressed through the functions $\theta_1$ and $\theta_2$, and analytically the relationship between the functions $\vartheta$, functions $\theta_1$ and $\theta_2$ is expressed by formula (17).

$$\forall x \in [0,1]\ \theta(x) = \begin{cases} \theta_1\left(\frac{Lx}{l_1}\right), & if\ x \in [0, \frac{l_1}{L}) \\ \theta_1(1) + \theta_2\left(\frac{\left(x - \frac{l_1}{L}\right)L}{L - l_1}\right), & if\ x \in (\frac{l_1}{L}, 1] \end{cases} \tag{17}$$

Let us denote as $l_k$ the length of the $k$-th segment $s_k$ of the polysegment line $(\{v_1, \ldots, v_n\}, \{s_1, \ldots, s_{n-1}\})$ and as $\theta_k$ the function of the angular characteristic of the $k$ -th segment, considered as a separate plane curve. Obviously, $\theta_k$ is determined by the formula $\forall x \in [0,1]\ \theta_k(x) = \alpha_k x$ , where $\alpha_k = 0$ if $s_k$ is straight line segment, otherwise, if $s_k$ is circular arc, then $\alpha_k = \frac{l_k}{R_k}$, were $R_k$ is the radius of the circle, corresponded to $s_k$. Let us denote as $\{x_k\}$ the sequence of $n$ values given by the formula $\forall k \in \{1, \ldots, n-1\}\ x_k = \frac{1}{L}\sum_{i=1}^{k} l_k$, where $L = \sum_{i=1}^{n-1} l_k$ is a total length of the polysegment line (). According to this definition $\forall k \in \{1, \ldots, n-1\}\ x_k \in [0,1]$ and $x_{n-1} = 1$. Using formula (17) and taking into account previous notion, we obtain the formula (18) for the values $\theta(x_k)$ of the angular characteristic function $\theta$ on the set of points $\{x_k\}$.

$$\theta(x_k) = \sum_{i=1}^{k} \alpha_k \tag{18}$$

The points $\{x_k\}$ divide the unit interval into $n-1$ intervals $\{I_k\}\ I_1 = [0, x_1)\ \forall k \in \{2, \ldots, n-1\}\ I_k = (x_{k-1}. x_k]$. On each of these intervals the function $\theta$ will be a linear function, thus $\theta$ is defined on $[0,1]$ by formula (19)

$$\forall x \in I_k\ \theta(x) = \theta_{k-1} + \left(\frac{\theta_k - \theta_{k-1}}{l_k}\right)x. \tag{19}$$

***Lemma 2***: For a given triple $(v_0, e_0, l)$ where vector $v_0 = \left[v_{0x}, v_{0y}\right]^T$ defines a position of the point on the plane, $e_0 \in \mathbb{R}^2$ is a unit vector, and $l > 0$ is a scalar parameter, an arbitrary angular characteristic function $\varphi$ uniquely determines a plane curve $\gamma$, given by equation (20)

$$\gamma(s) = v_0 + \int_0^l \left[R\left(\varphi\left(\frac{x}{l}\right)\right) e_0\right] dx. \tag{20}$$

where $R(\alpha)$ is the orthogonal matrix of the two dimension rotation operator, defined by expression (21):

$$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \tag{21}$$

Let us denote by $\theta$ an arbitrary continuous function defined on the interval $[0,1]$ and satisfying the condition $\theta(0) = 0$. For this function $\theta$ and an arbitrary triple $(v_0, e_0, l)$, the formula (4) uniquely determines a curve on the plane for which $\theta$ will be a function of the angular characteristic. If we denote by $\mathcal{X}$ the set of all triples $(v_0, e_0, l)$ and denote by $\mathcal{Y}^\theta$ the set of all plane curves for which $\theta$ will be an angular characteristic function, then the formula (20) defines the mapping $\beta: \mathcal{X} \to \mathcal{Y}^\theta$. Obviously, two curves defined by the formula (20) for $\theta$ and two different triples $t_1, t_2 \in \mathcal{X}$ cannot be identical, so the mapping $\beta$ is injective. It is also obvious that any plane curve $\gamma \in \mathcal{Y}^\theta$can be represented by the formula (20) for the function $\theta$ and some triple $t \in \mathcal{X}$, thus the image of the set $\mathcal{X}$ under the mapping $\beta$ coincides with the set $\mathcal{Y}^\theta$. This means that the mapping $\beta$ is bijective, that is, the sets $\mathcal{X}$ and $\mathcal{Y}^\theta$ are equivalent and we can use the notation $\mathcal{Y}_t^\theta$, where $t \in \mathcal{X}$ for the elements of $\mathcal{Y}^\theta$.

Now we can formulate an algorithm for constructing a polysegment line that approximates a smooth function plane curve $\gamma$ with two ends $A$ and $B$.

1. Step 1: Calculate the triple of parameters $(v_0, e_0, l)$ for the given curve, i.e., calculate the length $l$of the curve and calculate the tangent vector $e_0 = \dot{\gamma}(0)$(we assume that the position of point $A$, that determines $v_0$, is known
2. Step 2: Calculate the angular characteristic function $\theta$ of the plane curve $\gamma$.

3. Step 3. Build an optimal approximation of the angular characteristic function $\theta$ by a piecewise linear function $\theta^*$ with a given accuracy $\varepsilon > 0$.

Step 4. Build an approximating smooth function $\gamma$ polysegment line $\mathcal{Y}^{\theta^*}_{(v_0,e_0,l)}$.

## 2.6. Spanning tree of the connected component, cycle space and cycle's generation procedure

The spanning tree $T$ of a connected graph $G(V,E)$is the maximal acyclic connected subgraph (tree) of a given graph. All vertices of a graph are also vertices of its spanning tree (See Fig.3). Let $E_T$ be the set of edges of the spanning tree $T$, and $E_c = E - E_T$. If $E_C = \emptyset$, then $G$ is a tree. Otherwise, the cycle generation procedure described below is applied to the connected cyclic graph. Let $(a,b) \in E_C$ be the arbitrary edge of $G$, that is not an edge of spanning tree $T$. Then there is only one path $\gamma$ in $T$, connecting the vertices $a$ and $b$. If we add an edge $(a,b)$ to the set of edges $E(\gamma)$, then we get a cycle of the graph $G$ corresponding to the edge $(a,b)$. So, mapping $f: E_C \to C(G)$ , where $C(G)$ is a set of cycles of the graph $G$,is defined. It is easy to show that a mapping $f$ is a one-to-one mapping. (Suppose the opposite: Let $e_1, e_2 \in E_C$ and $f(e_1) = f(e_2)$. From this it follows that there is a path in the spanning tree $T$ containing one of the edges $e_1, e_2$, which is impossible). Let denote by $B \supset C(G)$ image of set $E_C$ under mapping $f$. In the sequel, we will denote by $B(T)$ the set of cycles $B$ defined in this way for an arbitrary spanning tree $T$ of a connected graph $G$. In graph theory, it is proved that the set of cycles $B$ is a basis of the linear space $V$ of the cycles over the field $\mathbb{Z}_2$.

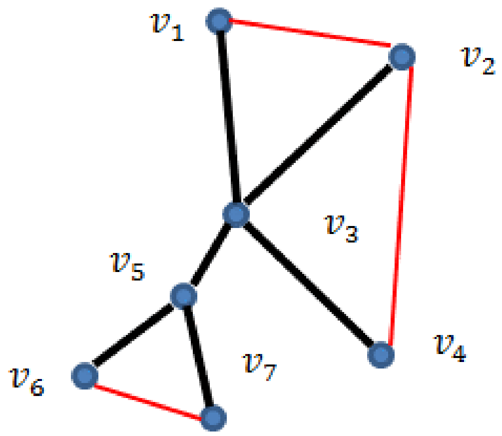


**Figure 3:** Spanning tree of the connected graph. The edges of the spanning tree are highlighted in black. Edges belonging to the set $E_C$ are highlighted in red.

In view of the importance of the concept of a linear space of the cycles for understanding graph analysis procedures, we give a brief description of this mathematical construction below. The elements (vectors) of the linear space $V$ are some sets of edges of $G$. Each element of $V$ can be represented as linear combinations of cycles of basis $B$, considered as sets of edges. For arbitrary $c \in V$ scalar multiplication is defined over $\{0,1\}$ as follows:

1. $0 * c = \{\}$, which shows that the zero-vector (the null set) is in Cycle Space
2. $1 * c = c$, which is identity

Let $a, b \in V$ then the addition operation $c = a + b$ be defined as a symmetric difference of the sets of edges: $a, b \in 2^{E(G)}$, $c = a \Delta b$. It is easy to show that $2^{E(G)}$ is an Abel group, relative to the addition operation defined in this way. It can be proved that for an arbitrary spanning tree $T$ of a connected graph $G$, the corresponding set of cycles $B(G)$, defined above, will be the basis of the linear space of cycles $V$ over the field $\mathbb{Z}_2$, with scalar product $*: \mathbb{Z}_2 \times V \to V$ and addition operations $+: V \times V \to V$ defined above. The procedure of cycles generation for a given base $B$ consists in generating the set of cycles $Q = \{c_1 + c_2 | c_1, c_2 \in B \; E(c_1) \cap E(c_2) \neq \emptyset\}$ and building the union $C = B \cup Q$. The constructed set of cycles $C$ does not contain all the cycles of the graph, but it is enough for solving the problems of removing the artifacts of the planar graph and generating the chains and cycles for proper symbol recognition. The need to carry out the cycle generation operation can be explained with the example shown in Fig.4.

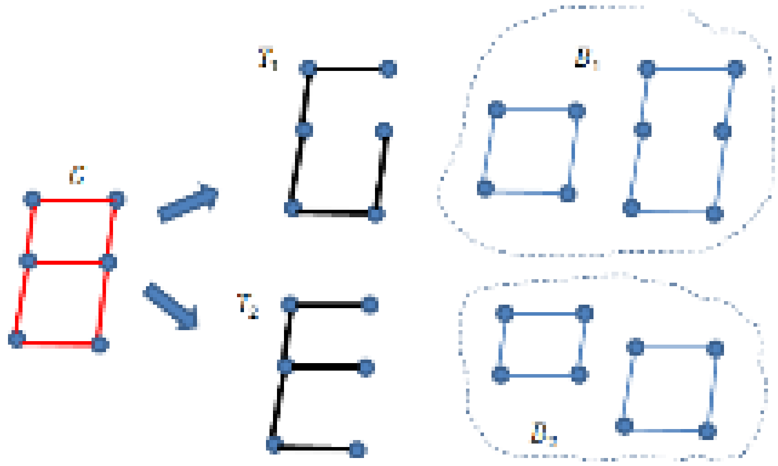

**Figure 4:** Two spanning trees $T_1$ and $T_2$ of the graph $G$ and the corresponding bases of the cycle space $B_1$ and $B_2$.

The fact is that it is impossible to predict in advance which of the spanning trees will be generated by the program. In this example, if the program generates the $T_1$ spanning tree, then the corresponding cycle base $B_1$ will not contain a "small" cycle (the top loop of the "8" symbol).

## 3. Results

### 3.1. The flat curve identification

The function of the angular characteristics of the curve completely determines the geometric shape of this curve. More precisely, this function can be viewed as a mapping of the set of regular curves to the functional space $D \subset L^2$ of functions defined on the interval $I = [0,1]$. As shown in the "Properties of angular characteristics function" section, this mapping is invariant with to plane similarity transformations. The concept of the function of the angular characteristics is easily transferred to broken lines. The functions of the angular characteristics of a broken line can be put in accordance with the feature vector (the procedure for extracting features is described in detail in the "feature extraction" section), which allows it to be classified. As mentioned above, the paths in the planar graph correspond to broken lines on the plane, which explains the key role of the concept of the function of the angular characteristics in the proposed method of recognizing handwritten characters.

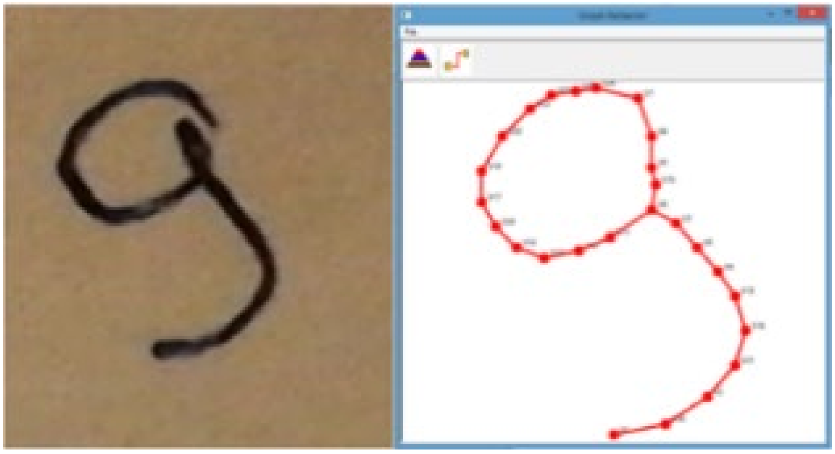

**Figure 5:** A fragment of a digital photo (left image) and a screenshot of the "GraphRedactor" window (right image) with the image of the planar graph of the digit symbol "9" obtained when converting an image into a graph.

According to the definition of a planar graph, there is a one-to-one mapping of the set of vertices of the graph to the set of points on the plane. Accordingly, each graph edge corresponds to a line segment in the plane, and the segments corresponding to different edges can intersect only at the end points. Each path in the graph corresponds to a simple polyline on the image plane. Suppose we have a digital photo of handwritten symbols drawn with thin lines (for example, written in pen on paper). After applying the conversion procedure to this image, each isolated symbol will correspond to a connected component of the resulting planar graph. The curves forming the images of the symbol will be approximated by polylines (simple or closed) corresponding to the paths and cycles in the graph (see Fig.5).

Thus, two algorithms for converting a raster image into a planar graph and software implementing these algorithms were developed and used for experiments with recognizing decimal digits in real photographs. As a result, it was demonstrated that the input data for character recognition procedures, including the feature extraction procedure, are connected components of a planar graph. Thus, the planar graph obtained as a result of applying the conversion procedure contains all the information necessary for recognition, and after converting an image into a planar graph, the original image is not used in recognition procedures.

The key idea of the proposed flat curve identification method is the use of an angular characteristic function to identify the shape of a simple smooth curve without self-intersections.

### 3.2. Feature extraction

An arbitrary function $f$ defined on an interval $[a, b]$ can be considered as a periodic function $\forall x \in \mathbb{R}\; f(x+T) = f(x)$ with a period $T = b - a$ (See Fig. 6) By definition, the function of the angular characteristics $\theta$ of arbitrary broken line is defined on the unit interval $I = [0,1]$ and $\theta(0) = 0$, but in the general case $\theta(1) \neq \theta(0)$. Consequently, a periodic function $\theta$ in the general case will have jump discontinuities. Due to Gibbs phenomenon, the partial sum of the Fourier series will be poorly approximated to the original function $\theta$. To solve this problem, we use the following method: for a given function of angular characteristic $\theta$, we define an even function $\Psi$ defined on the interval $[-1,1]$ by equation (22)

$$\Psi(x) = \theta(1 - |x|) \tag{22}$$

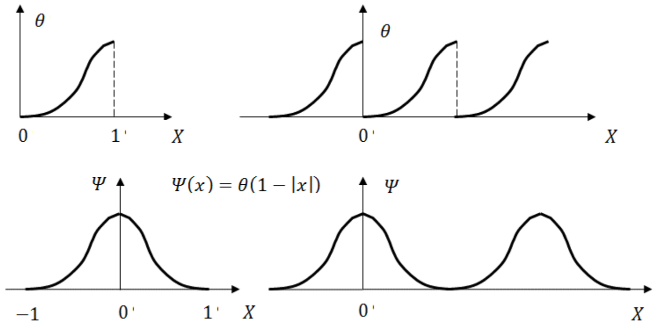


**Figure 6:** Angular characteristic function $\theta$ and corresponding function $\Psi$.

As $\Psi$ is a piecewise continuous bounded function it is possible to define its absolute value $\|\Psi\|$ by (23)

$$\|\Psi\| = \sqrt{\int_{-1}^{1} \Psi^2(x)\,dx}\,. \tag{23}$$

Let $\Omega = L^2([-1,1])$ be a $L^2$ space of functions, defined on interval $[-1,1]$. Then, $\Psi \in \Omega$, due to the fact that the norm $\|\Psi\| < \infty$ is defined by (2). As $\Omega$ is a $L^2$ space, $\Omega$ is a Hilbert space, where inner product of $f_1, f_2 \in \Omega$ is defined by (24):

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1(x) \cdot f_2(x) dx. \tag{24}$$

Each even function $g \in \Omega$ can be represented as a Fourier series in form (25)

$$g(x) = \sum_{i=1}^{\infty} c_i \cdot \varphi_i(x). \tag{25}$$

where functions $\varphi_i$ is defined by equations (26)

$$\varphi_i(x) = \begin{cases} \frac{1}{\sqrt{2}}\ if\ i = 1 \\ \cos(\pi \cdot i \cdot x)\ if\ 1 < i \end{cases} \tag{26}$$

The system of functions (25) is orthonormal, which means that (27) holds true.

$$\forall i, j\ \langle \varphi_i, \varphi_j \rangle = \delta_{ij}. \tag{27}$$

The coefficients $c_1, c_2 \ldots$ of Fourier series expansion (24) are defined as the projections (27) of the vector $g \in \Omega$ onto the vectors of the orthonormal basis $\varphi_1, \varphi_2 \ldots$:

$$c_i = \langle g, \varphi_i \rangle = \int_{-1}^{1} g(x) \cdot \varphi_i(x) dx. \tag{28}$$

For a given $N$, the $N$-th partial sum of the Fourier series (29) will approximate the original function.

$$g(x) \approx \sum_{i=1}^{N} c_i \cdot \varphi_i(x) \tag{29}$$

We will consider the real numbers $c_1, c_2 \ldots c_N$, as components of the vector $\vec{g} \in \mathbb{R}^N$, so in some orthonormal basis of $\mathbb{R}^N$ $\vec{g} = (c_1, c_2 \ldots c_N)$.Let designate by $g \rightarrow \vec{g}$ correspondence between even function $g \in \Omega$ and vector $\vec{g} \in \mathbb{R}^N$ of Fourier series coefficients. Then, for arbitrary even functions $f_1, f_2 \in \Omega$ approximations (30)-(31) hold true.

$$\langle f_1, f_2 \rangle \approx \vec{f_1} * \vec{f_2}. \tag{30}$$

$$\|f - g\| \approx |\vec{f} - \vec{g}|. \tag{31}$$

Both Hilbert space and Euclidean space are metric spaces. A metric in space (a set of points $D$) may be defined by the distance function $d$, defined for all pairs of points $A, B \in D$, which satisfies four conditions:

$$\forall A, B \in D \ \ d(A,B) = d(B,A)$$
$$\forall A, B \in D \ \ d(A,B) \geq 0$$
$$\forall A \in D \ \ d(A,A) = 0$$
$$\forall A, B, C \in D \ \ \rho(AB) \leq \rho(AC) + \rho(CB)$$

As is known, in the Euclidean space $\mathbb{R}^N$, distance $d$ between $\vec{A}, \vec{B} \in \mathbb{R}^N$is defined as $d(\vec{A}, \vec{B}) = |\vec{A} - \vec{B}|$. Similarly, for a Hilbert space $L^2$ metric $d^*$ is defined by equation (32), where $f_1, f_2 \in L^2$

$$d^*(f_1, f_2) = \|f_1 - f_2\|. \tag{32}$$

It is proved that conditions 1) −4) are satisfied for the metric $d^*$. Suppose that for a given function of the angular characteristic $\theta$, the function $\Psi$ is determined by the ratio (21) and $\Psi \to \vec{\Psi}$. If we consider the vector $\vec{\Psi}$ as a feature vector of the Euclidean feature space, then based on the above considerations, it can be argued that such a method of feature extraction will be good suitable for curve classification tasks. But in practice we need to classify the step characteristic functions of broken lines, containing discontinuous jumps. It is highly desirable to smooth out such functions. To do this, we apply the operation described below to the vector $\vec{\Psi}$ of the Fourier expansion coefficients of the function $\Psi$. Let denote by $g(x)$ Gaussian function, defined as (33)

$$g(x) = \frac{1}{\sqrt{2 \cdot \pi \cdot \sigma}} \cdot e^{-\frac{x^2}{2 \cdot \sigma^2}} \tag{33}$$

where $\int_{-\infty}^{\infty} g(x)\, dx = 1$, i.e., $g(x)$ is normalized. As $g$ is an even function, equations (34)-(35), for arbitrary $\omega \in \mathbb{R}$ holds true.

$$\int_{-\infty}^{\infty} g(x) \cdot \cos(\omega \cdot x)\, dx = \int_{-\infty}^{\infty} g(x) \cdot e^{-i\omega x} dx. \tag{34}$$

$$\int_{-\infty}^{\infty} g(x) \cdot cos(\omega \cdot x)\, dx = e^{-\frac{\sigma^2 \cdot \omega^2}{2}}. \tag{35}$$

For the values of the parameter $\sigma$, satisfying the non-equality $\sigma < \frac{2}{3}$, we can assume that $g(x) \approx x \ if \ x \notin [-1,1]$. Then the values $g_k^*$ defined by (36)

$$g_k^* = \begin{cases} 1 & if\ k = 0 \\ e^{-\frac{\sigma^2 \cdot (k-1)^2}{2}} & if\ k > 1 \end{cases} \tag{36}$$

can be considered as the Fourier expansion coefficients of the function $g^*$, approximating the Gaussian function $g(x)$. Let $\vec{\Psi} = (\varphi_1, \varphi_2, \dots, \varphi_N)$ be the vector of the Fourier coefficients of the function $\Psi$, and $\vec{g}^* = (g_1^*, g_2^*, \dots, g_N^*)$ the vector of the Fourier coefficients of the function $g^*$. We define the vector $\vec{\Psi}_s = (\varphi_{s1}, \varphi_{s2}, \dots, \varphi_{sN})$ by equalities (37).

$$\varphi_{sk} = \varphi_k \cdot g_k^*. \tag{37}$$

According to the convolution theorem for Fourier series, we can consider the components of the vector $\vec{\Psi}_s$ as the Fourier expansion coefficients of the function $\Psi_s(x)$, which is a result of application of a filter with Gaussian kernel $g^*$ to the function $\Psi$ (38)

$$\Psi_s(x) = \int_{x-1}^{x+1} \Psi(t) \cdot g^*(t)\, dt. \tag{38}$$

Of course, the choice of the number n and the values of the parameter t strongly affect the quality of recognition.

## 4. Discussion

The method for identifying plane curves described in the article was practically tested. We developed software that implements this method and used it for experiments with character recognition of decimal digits on real photos. At the first stages of the recognition procedure, the raster image was transformed into a planar graph. As mentioned above, the planar graph obtained as a result of applying the conversion procedure contains all the necessary information for recognition, and after converting the image into a planar graph, the original image is not used in the recognition procedures. We have developed a simple text format for storing information about a planar graph in the form of s - expressions. To visualize the resulting planar graphs, a special stand-alone application was developed (we called it "Graph Redactor"), which also allows the user to create artificial planar graphs using GUI tools (we used such artificially created planar graphs in experiments with character recognition software). All software was written in Python 2 language (Anaconda package). Since we used a small database to classify planar graphs corresponding to handwritten characters, the experimental results can only be considered as a proof of concept. However, we still consider it necessary to note that during the experiments, we observed high reliability of the implemented recognition method.

## 5. Conclusion

The suggested approach for recognizing plane curves was created with the intention of being utilized in tasks involving the recognition of handwritten characters. The core concept behind this method for identifying flat curves involves employing a unique function to discern the shape of a simple, continuous curve devoid of any self-intersections. Our primary focus was on character recognition techniques that involve the transformation of character images into a planar graph. This planar graph, generated through the conversion process, contains all the essential information required for recognition. As an outcome of our efforts, we designed software to put these algorithms into practice and applied it to conduct experiments involving the recognition of decimal digits in actual photographs.

Due to the radical difference between the proposed method and the existing ones, the software was created as proof of concept solely to confirm the operability of the method. For the experiments, images of the symbols of the digits 1, 2, 3, 5 were used (numbers for which the planar graphs corresponding to their symbols do not contain cycles were used) generated in a relatively small number of instances (from 10 to 20 for each symbol) by the participant of the experiment. Based on this sample, a small database was created that allowed character recognition. Then, an experiment was conducted to recognize characters generated by the same participant in the experiment. In the course of several similar experiments, 100% recognition accuracy was observed (no recognition errors). We emphasize that such experiments on small samples cannot be used for comparison with other methods and are only suitable as proof of concept.

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## Declaration on Generative AI

The authors have not employed any Generative AI tools

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