# Numerical models of novel discrete-time chaotic systems

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### Abstract

The paper proposes a theoretical approach to designing novel dynamical systems based on known discrete maps. We show some algebraic transformations based on the derivative operators' discrete-time approximation. These operators' approximations are considered as defined by some nonlinear algebraic combinations of finite and infinite element numbers of maps. Such an approach allows us to use various differential operators, including fractional-order and complex fractional-order ones. As a result, matrix nonlinear algebraic equations are defined for the considered discrete map. The order of these equations depends on the number and order of the derivatives used to define the system dynamics. Such a formalized approach allows us to easily define the system's finite-difference equations and solve them using known numerical methods. The obtained solutions make it possible to determine the systems' motions, and we offer to add some external signals to increase the range where these motions are defined. One can consider such signals from a control theory viewpoint as control one and use known control approaches to define system equations in various state spaces. Using one of these approaches allows us to significantly increase the number of system outputs by using observability equations for defined system state variables. We show the use of our approach by designing and studying several chaotic systems based on a well-known logistic equation. Our studies prove the possibility of constructing novel systems that produce previously unknown chaotic signals significantly different from known ones. The given system equations not only allow the generation of new signals for use in various applications but also give us the possibility to improve system performance.

### Keywords

Chaotic systems, discrete-time dynamical systems, finite-difference equations, nonlinear models[1]

## 1. Introduction

In today's world, chaotic systems are widely used in various scientific and engineering applications due to their unique ability to produce unpredictable signals [1,2,3]. These systems are utilized to study and predict biological [4,5,6], meteorological [7,8,9], and financial [10,11,12] processes, and to design control systems for different technical systems, devices, and networks [13,14,15].

At the same time, the primary application of chaotic systems is in data encryption [16,17,18] and secure data transmission [19,20,21]. The increasing need for highly secure communication systems is driven by the necessity to transmit large amounts of data for critical infrastructure cyber-physical systems [22,23,24], while restricting access to unauthorized persons. This interest is further fueled by the emergence of Industry 4.0 and Industry 5.0.

[1]ITTAP'2025: 4th International Workshop on Information Technologies: Theoretical and Applied Problems, October 22–24, 2025, Ternopil, Ukraine, Opole, Poland

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Consequently, numerous chaotic systems have been developed [25,26,27], with many authors studying chaotic systems in the real domain and designing channels to provide desired system features.

However, these systems are often considered as single-channel dynamical systems, which limits their ability to perform multichannel data transmission.

The use of control theory approaches which are based on the defining dynamical system motions by state spaces which are appended with observability equations gives one the possibility to define as many system outputs as he desire [28,29,30]. This approach also eliminates the need for subjective design [31,32,33] of new chaotic systems as it involves transforming known systems. We demonstrate our method by considering a simple discrete-time chaotic system, but it can easily be extended to any system with chaotic dynamics.

Our paper is organized as follows: firstly, we consider the generalized first-order discrete-time map and show its usage to define the generalized discrete-time dynamical system. Then, we show the appending this system with control signal to have possibility to affect on system motions without changing its initial conditions. Thirdly, we consider the use of nonlinear matrix observablity equations to increase the number of system outputs. We illustrate the use of our approach by considering a well-known chaotic system based on the logistic map, and transforming it into controlled discrete-time dynamical system with two outputs.

## 2. Method

### 2.1. Transformation of the generalized discrete map into discrete-time dynamical system

Let us consider the generalized one-dimensional discrete map which is given as follows

$$x_i = f\left(x_{i-1}, \ldots x_{i-k}\right), \tag{1}$$

where $x_i$ and $x_{i-j}$ are current and $k$ previous map values and $f(.)$ is some nonlinear function. Here and further we assume that nonlinear function $f(.)$ is defined in some hyper-complex domain and map values are considered as hyper-complex numbers.

It is clear that (1) allows us to define the current map value $x_i$ by using information about previous map values $x_{i-j}$. Depending on the used nonlinear function such approach gives us the possibility to produce both regular and chaotic time series which can find their usage in various applications. At the same time, it is hard enough to use (1) for study map's general features by using known methods and approaches from the control theory. That is why we take into account that signals, which are produced by map (1), change their values in time and claim that one can consider (1) as some generalized discrete-time dynamical system. The map values $x_i$ and $x_{i-j}$ in this case can be studied as values which define system's current and previous states.

Since the motions of discrete-time dynamical systems, in common, are given by finite difference equations, we take into consideration some nonlinear operator which depends on current and previous values of system state variable and approximates $\alpha$-order differential operator as follows

$$\frac{d^\alpha x}{dt^\alpha} \approx D^\alpha\left(x_i, x_{i-1}, \ldots, x_{i-n}\right), \tag{2}$$

where $n$ is a number of previous values of system state variable which are used to define the desired $\alpha$-order discrete-time approximation of system coordinate.

One can consider $n$ as the array's size which is used to store previously defined values of system state variable. We do not use any limitation on the derivative order here and believe that $\alpha$ can be considered as some nonlinear time-depended function of system state variable in $m$-dimensional hyper complex plane. So the operator $D^{\alpha}(.)$ can be defined with a quite complex functions and dependencies and in the most general case use information about all system's previous states. For example, as it is performed while the Grunvald-Letnikov

$$D^{\alpha}(x_i, x_{i-1}, \ldots, x_{i-n}) = \lim_{h\to\infty} \frac{1}{h^{\alpha}} \sum_{0\le i<\infty} \frac{(-1)^i \Gamma(\alpha+1)}{\Gamma(i+1)\Gamma(\alpha-i+1)} f(x_i + (\alpha-i)h), \tag{3}$$

here $\Gamma(.)$ is a gamma-function

or Cauchy

$$D^{\alpha}(x_i, x_{i-1}, \ldots, x_{i-n}) = \frac{\Gamma(\alpha+1)}{2\pi I} \int \frac{f(w)}{(w-x)^{\alpha+1}} dw, \tag{4}$$

where $I$ is an imaginary unit

derivatives are being calculated.

Let us rewrite (1) into implicit form

$$0 = g(x_i, \ldots, x_{i-k}), \quad g(x_i, \ldots, x_{i-k}) = f(x_{i-1}, \ldots, x_{i-k}) - x_i \tag{5}$$

and add the operator $D_{\alpha}(.)$ into left and right-hand expressions of first equation in (5)

$$\begin{gathered} D^{\alpha}(x_i, x_{i-1}, \ldots, x_{i-n}) = h(x_i, x_{i-1}, \ldots, x_{i-k}, \ldots, x_{i-n}), \\ h(x_i, x_{i-1}, \ldots, x_{i-k} \ldots, x_{i-n}) = g(x_i, \ldots, x_{i-k}) + D^{\alpha}(x_i, x_{i-1}, \ldots, x_{i-n}), \end{gathered} \tag{6}$$

here we assume that $k<n$. Otherwise, variables in (6) should be counted for $i-k$ but not $i-n$.

Defined in such a way finite-difference equation allows us to construct $\alpha$ order discrete-time dynamical system which is based on some discrete map. Contrary to the initial map (1) which allows us to define only one time series from map values, the consideration of dynamical system (6) gives us the possibility to define several system outputs by taking system output and its derivatives as system state variables

$$y_j = \frac{d^{\alpha_j}}{dt^{\alpha_j}} x \approx D^{\alpha_i}(x_i, x_{i-1}, \ldots, x_{i-j}). \tag{7}$$

Thus, the proposed approach allows us to increase the order of considered system and generate more signals at the same time even for the system which is based on the one-dimensional map.

However, the used mathematical apparatus allows consider the generalized $q$-dimensional maps as well by replacing scalar variables and functions in (6) with matrices and vectors as follows

$$\begin{gathered} \boldsymbol{D}^{\alpha}(\boldsymbol{X}) = \boldsymbol{H}(\boldsymbol{X}), \\ \boldsymbol{X} = \begin{pmatrix} x_{1(i)} & x_{1(i-1)} & \cdots & x_{1(i-n)} \\ x_{2(i)} & x_{2(i-1)} & \cdots & x_{2(i-n)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{q(i)} & x_{q(i-1)} & \cdots & x_{q(i-n)} \end{pmatrix}; \boldsymbol{H}(\boldsymbol{X}) = \begin{pmatrix} h_1(\boldsymbol{X}) \\ h_2(\boldsymbol{X}) \\ \vdots \\ h_q(\boldsymbol{X}) \end{pmatrix}. \end{gathered} \tag{8}$$

where $x_{ji}$ means considering the $i$-th value of $j$-th variable and $h_j$ is a component of nonlinear vector-function $\boldsymbol{H}(\boldsymbol{X})$.

It is clear that (6) have the same form as it is used to define trajectories for any dynamical system which motion is given by matrix equation. This fact allows us to claim that it is found the clear interrelation between discrete map and difference equations of discrete-time dynamical system. One can use these interrelations to convert discrete map into finite-difference equations and vice verse to solve various problems of system study and design.

### 2.2. The generalized discrete-time dynamical system with external control signal

The main drawback of dynamical systems which motions are given by some discrete map is their considering without any external signals. From the control theory viewpoint one can claim that (1) and its transformation (8) define the system perturbed motions which are caused by non-zero initial conditions. It is clear that such motions can be useful to analyze system stability and serve as output of some special signals' generators.

At the same time the produced in such a way motions are uncontrollable and do not depend on any external signals. This fact decreases the set of possible system motion trajectories and makes them dependent only system parameters and its initial conditions.

We offer to avoid this problem and increase the domain, where system produces its output, by rewriting (8) in the most general way as follows

$$\boldsymbol{D}^{\alpha}(\boldsymbol{X})=\boldsymbol{H}_c(\boldsymbol{X},\boldsymbol{U}),\ \ \boldsymbol{U}=\begin{pmatrix}u_1 & u_2 & \dots & u_q\end{pmatrix}^T \quad (9)$$

and $\boldsymbol{H}_c(\boldsymbol{X},\boldsymbol{U})$ are q-sized vector-function of two arguments.

It is clear that nonlinear combinations of system state variables and external signals allows us to form wide range motions. At the same time, these nonlinear combinations can cause problems in controller design for the considered dynamical system. That is why we offer to simplify (9) by assuming that system motion is given by following equation

$$\boldsymbol{D}^{\alpha}(\boldsymbol{X})=\boldsymbol{H}_1(\boldsymbol{X})+\boldsymbol{H}_2(\boldsymbol{X})\boldsymbol{H}_3(\boldsymbol{U}), \quad (10)$$

here $\boldsymbol{H}_j(.)$ are some nonlinear vector-function.

Similar to the case of uncontrollable system, the first summand $\boldsymbol{H}_1(\boldsymbol{X})$in (10) defines system perturbed motion. The second one $\boldsymbol{H}_2(\boldsymbol{X})\boldsymbol{H}_3(\boldsymbol{U})$ allows us to extend the previously considered system for the case of controlled systems and by defining controlled motion which is governed by some vector control signal. The control channel can depend on system operation mode which define specific sets of state variables as well as nonlinearities in system feedforward channel.

The main feature of both systems (8) and (10) is their solution for some matrix operator $\boldsymbol{D}^{\alpha}(\boldsymbol{X})$ which can be considered as the generalized approximation for $\alpha$ order derivative operator. One can find that such form of difference equations is convenient to find system output. At the same time, according to (10) this output depends on both control signal and previous values of system state variables. It is clear that these values have been defined in previous steps of numerical solution of (10) in a similar way. Thus, one can consider (10) as some recursive dependency which initially depend on system initial conditions and control effect.

That is why, we offer to take into account following matrix operator

$$\boldsymbol{H}^{\alpha}(\boldsymbol{X})=\frac{\boldsymbol{D}^{\alpha}(\boldsymbol{X})-\boldsymbol{H}_1(\boldsymbol{X})}{\boldsymbol{H}_2(\boldsymbol{X})} \quad (11)$$

and use it to rewrite (10) in such a way

$$\boldsymbol{H}^{\alpha}(\boldsymbol{X})=\boldsymbol{H}_3(\boldsymbol{U}). \quad (12)$$

Equation (12) define system motion as function of control signal only and its solution allows us to define this motion as follows

$$\boldsymbol{X}=\boldsymbol{H}^{-\alpha}(\boldsymbol{H}_3(\boldsymbol{U})), \quad (13)$$

where $\boldsymbol{H}^{-\alpha}(.)$ is operator inverted to $\boldsymbol{H}^{\alpha}(.)$.

Analysis of (13) shows that the system output is defined as function of external control signal only. Possible non-zero initial conditions are taken into account while $\boldsymbol{H}^{-\alpha}(.)$ operator is being defined.

One more benefit to study discrete maps in form of dynamical system is the possibility to use known control approach which is based on transfer functions apparatus. Comparing (13) with transfer functions gives us the possibility to consider right-hand expression in (13) as the generalized discrete-time transfer function for dynamical system (10). Due to the nonlinearity of this system, definition of expression (13) can require to use numerical methods to solve nonlinear algebraic equations.

### 2.3. The generalized discrete-time controlled dynamical system with nonlinear observability equation

As it is known from the control theory, sometimes system state variables can have different nature and define various physical processes. That is why not all systems state variables are useful to transmit and/or utilize in practical applications as some signals. In this case, one can find it is very convenient to use observability equations' concept.

According to this concept some signals can be defined as the linear

$$\boldsymbol{Y}=\boldsymbol{A}\,\boldsymbol{X}+\boldsymbol{C}\,\boldsymbol{U}, \tag{14}$$

or nonlinear

$$\boldsymbol{Y}=\boldsymbol{G}(\boldsymbol{X},\boldsymbol{U}) \tag{15}$$

combinations of state variables and control signals.

Since the definition of these signals are based on the use corresponding matrices and vectors, it should be mentioned that matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ can have different sizes and these sizes are corresponded by sizes of scale matrices $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{G}(.)$ in (14) and (15). Size of these matrices depend on the desired size of system observed variables' matrix and can take any positive integer values from zero to infinity. Elements of these matrices depend on the interrelations between components of system state variables' matrix $\boldsymbol{X}$ and matrix of control inputs $\boldsymbol{U}$.

Thus, if one solves (15) for $\boldsymbol{X}$

$$\boldsymbol{X}=\boldsymbol{G}^{-1}(\boldsymbol{Y},\boldsymbol{U}) \tag{16}$$

and substitutes (16) and (10) following difference equation can be written down for the considered system in terms of output and control signals only

$$\boldsymbol{D}^{\alpha}(\boldsymbol{G}^{-1}(\boldsymbol{Y},\boldsymbol{U}))=\boldsymbol{H}_1(\boldsymbol{G}^{-1}(\boldsymbol{Y},\boldsymbol{U}))+\boldsymbol{H}_2(\boldsymbol{G}^{-1}(\boldsymbol{Y},\boldsymbol{U}))\,\boldsymbol{H}_3(\boldsymbol{U}). \tag{17}$$

Since (17) uses observed signals (16) we claim that (17) is the generalization of (11) for the case when observer variables $\boldsymbol{Y}$ differ from the system state variables $\boldsymbol{X}$. Otherwise, these equations are same.

Contrary to the solving both discrete-time state equation and observability equation which solution require to use some hardware resources for storing intermediate data, such an approach allows us to immediately define system output.

Analysis of (17) shows that the both right and left hand expressions in this equation depend on vector of output variables $\boldsymbol{Y}$. Since the left-hand expression can be considered as discrete-time implementation of differential equation we use this fact to generalize the concept of observability equations for the class of finite-difference equations

$$\boldsymbol{D}^{\beta}\boldsymbol{Y}=\boldsymbol{G}_{\beta}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{U}), \tag{18}$$

here $\boldsymbol{D}^{\beta}(.)$ is $\beta$-order finite-difference operator and $\boldsymbol{G}_{\beta}(.)$ is some nonlinear vector-function.

Contrary to (14) and (15) one can use (18) to define system output as combination of system state variables, its outputs, and control inputs. Solution of this equation for $\boldsymbol{X}$ can be given in form similar to (16)

$$\boldsymbol{X}=\boldsymbol{G}_{\beta}^{-1}(\boldsymbol{Y},\boldsymbol{U}), \tag{19}$$

which allows us to define system motion equation in output variables by replacing $\boldsymbol{G}^{-1}(.)$ with $\boldsymbol{G}_{\beta}^{-1}(.)$

$$\boldsymbol{D}^{\alpha}(\boldsymbol{G}_{\beta}^{-1}(\boldsymbol{Y},\boldsymbol{U}))=\boldsymbol{H}_1(\boldsymbol{G}_{\beta}^{-1}(\boldsymbol{Y},\boldsymbol{U}))+\boldsymbol{H}_2(\boldsymbol{G}_{\beta}^{-1}(\boldsymbol{Y},\boldsymbol{U}))\boldsymbol{H}_3(\boldsymbol{U}). \tag{20}$$

It is clear that operators $\boldsymbol{G}_{\beta}(.)$ and $\boldsymbol{G}_{\beta}^{-1}(.)$ in (18)-(20) are more complex than operators $\boldsymbol{G}(.)$ and $\boldsymbol{G}^{-1}(.)$ in(15)-(17) due to the using information about history of system outputs' changing. Thus, (20) can be considered as generalization for (17).

If one takes into account (11) and (12) equations similar to (13) can be written down for system (20)

$$\boldsymbol{Y}=\boldsymbol{H}_{\beta}^{-\alpha}(\boldsymbol{U}), \tag{21}$$

which allows us to define system outputs only as functions of its inputs and their previous values. Since (21) are based on (20) we claim that (21) is the generalization for (13).

## 3. Results and discussion

### 3.1. Study of exactly-defined dynamical system based on logistic map

We show an example of using the proposed approach by studying a logistic map

$$x_i=rx_{i-1}-rx_{i-1}^2, \tag{22}$$

where $x_i$ and $x_{i-1}$ are current and previous map's values and $r$ is a map factor

It is a well-known fact that such a map produce chaotic series for case

$$r\in\boldsymbol{r},\ \boldsymbol{r}=[3.6,4.0]. \tag{23}$$

Let us represent map (22) in form of the first-order discrete-time dynamical system which motions are given by finite-difference equation. We use the simplest approximation of derivative operator by backward difference here

$$\frac{d}{dt}x\approx\frac{x_i-x_{i-1}}{T}, \tag{24}$$

where $T$ is sample time.

If one divides (22) on sample time $T$ and subtract from it map's previous value $x_{i-1}$ following expression can be written down

$$\frac{x_i-x_{i-1}}{T}=\frac{(r-1)x_{i-1}-rx_{i-1}^2}{T}. \tag{25}$$

As one can see designed in such a way discrete-time dynamical system has weight factors which depend on sample time. Thus, these factors can take huge values for small sample time.

In the case of small sample time (25) can be rewritten in form of well-known logistic differential equation

$$\frac{d}{dt}x=\frac{r-1}{T}x-\frac{r}{T}x^2, \tag{26}$$

which proves the correctness of considered approach.

The Matlab-Simulink block diagram of discrete-time dynamical system (25) shown in Figure.1 and in Figure.2 block diagram of continuous-time dynamical system (20) is shown.

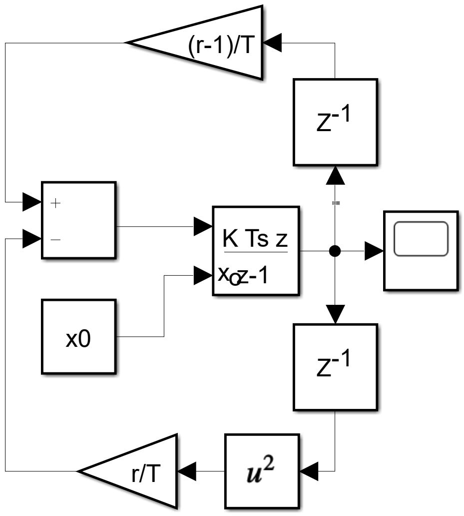



**Figure 1:** Block-diagram of designed discrete-time system.

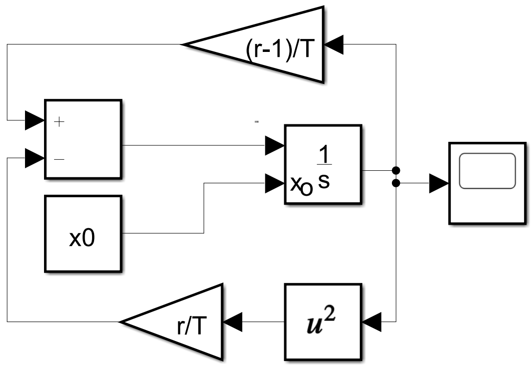



**Figure 2:** Block-diagram of designed continuous-time system.

As one can see from analysis of these block-diagrams they have the similar structure and depend in implementation of numerical integration blocks as well as necessity to form time delay in one sample time in discrete system according to (25).

We simulate both systems with the Euler numerical integration method and assume sample time $T$ and simulation step $h$ are equals. Due to the similarity of these time factors we get the same simulation results for both block-diagrams (**Figure 3.**). It is necessary to say that in the general case the using different sample time and simulation step makes no sense because logistic map and logistic equation produce different outputs.

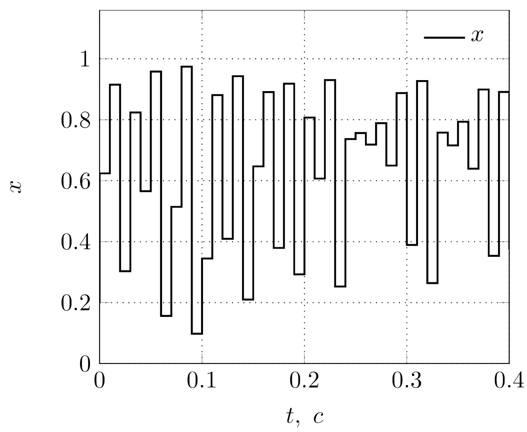



**Figure 3:** Numerical solution (25) and (26) for $r=3.9$, $T=0.01$, and $h=0.01$, $x_{-1}=0.2$.

Analysis of given in Figure.3 simulation result is similar to well-known output of logistic map and we consider this fact as the numerical proof of our approach's correctness. Moreover, if one analyzes (22) and (25), he find that applying Euler numerical integration method to (25) leads to implicit obtaining (22). This fact also proves the strong interrelation between the designed dynamical system and initial chaotic map and proves the possibility of occurring chaotic oscillations in discrete-time dynamical which are based on chaotic maps.

At the same time our approach allows not only to prove known fact but also it can be used to design some novel chaotic discrete-time systems.

### 3.2. Design 2nd order exactly-defined discrete-time chaotic system

Let us consider usage of our approach to construct the 2nd order chaotic system.

We offer to construct novel chaotic discrete time system by assuming existing of two state variables which are interrelated with following expressions

$$\frac{x_{1i}-x_{1(i-1)}}{T}=x_{2i}. \tag{27}$$

In other words we think that state variable $x_2$ approximates derivative of $x_1$ in discrete-time domain. Such a dependency is the simplest one and that is why it is taken for consideration. Although, it is not a strong requirement and one can use any linear or nonlinear dependencies in right-hand expression of (27) in general.

It is clear that if one equals (27) and (25), following equation can be written down

$$x_{2i}=\frac{(r-1)x_{1(i-1)}-rx_{1(i-1)}^2}{T} \tag{28}$$

and this equation is valid for each time moment which is defined by its number $i$ of sample time, including and previous time moment $(i-1)T$

$$x_{2(i-1)}=\frac{(r-1)x_{1(i-2)}-rx_{1(i-2)}^2}{T} \tag{29}$$

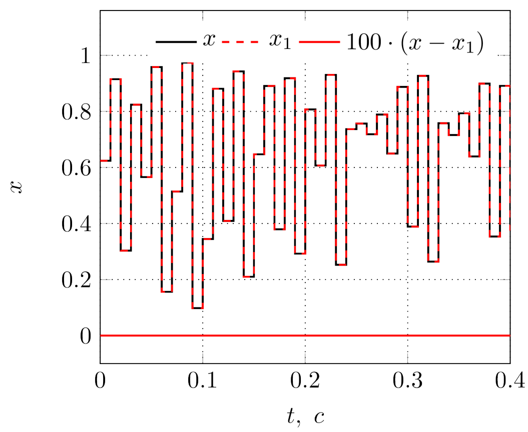
Expressions (28) and (29) give us the possibility to define following finite difference

$$\frac{x_{2i}-x_{2(i-1)}}{T}=\frac{(r-1)(x_{1(i-1)}-x_{1(i-2)})-r(x_{1(i-1)}^2-x_{1(i-2)}^2)}{T^2} \tag{30}$$

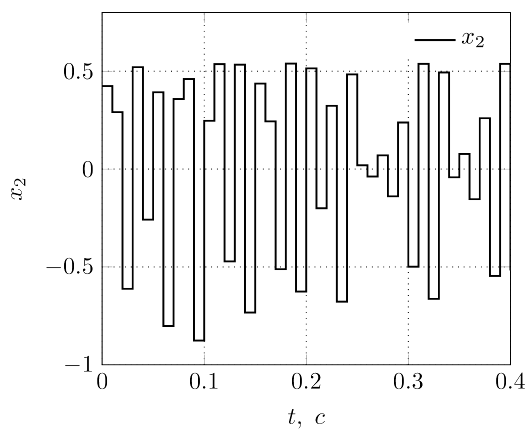
If one takes into account (27) he can rewrite the first summand in (30) and represent this equation as follows

$$\frac{x_{2i}-x_{2(i-1)}}{T}=\frac{(r-1)Tx_{2(i-1)}-r(x_{1(i-1)}^2-x_{1(i-2)}^2)}{T^2}. \tag{30}$$

Designed in such a way 2nd order discrete-time dynamical system is based on logistic map and defined with two state variables by equations (27) and (30). Contrary to (25) one can use this system as the source of two chaotic signals (Figure.4)




a) 1st and 2nd order systems' comparison




b) Additional output in 2nd order system

**Figure 4:** Numerical solution (27) and (30) for $r=3.9$, $T=0.01$, $x_{-1}=0.2$, and $x_{-2}=0$.

As one can see from Figure.4a the designed first and second order discrete-time dynamical systems produce the same output variables. Difference between these variables equals to zero which proves correctness transformations (27)-(30). Also the main benefit of system (27) and (30) is possibility to produce additional output variable (Figure.4b) which is not produced by classical logistic map. Thus, the designed 2nd order dynamical system can be considered as the source of two chaotic signals which are produced in the parallel way.

### 3.3. Design 2nd order exactly-defined discrete-time controlled chaotic system

Motions in the above-given discrete-time dynamical systems are caused by their only non-zero initial conditions and thus do not depend on external signals and conditions. One can find that this fact is a little drawback for such systems, because of the necessity to their reconfiguration to form other motions for the considered systems.

From the control theory such systems are called non-controlled and they find their applications in a few applications only. We avoid this drawback by change structure of the studied systems and consider them as controlled dynamical systems.

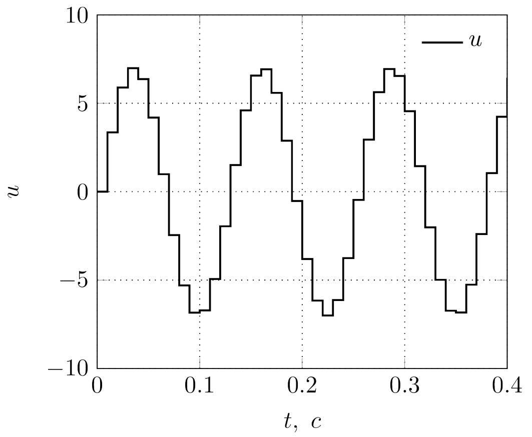
Since according to our approach we operate with discrete-time finite-difference equations, which are together with continuous-time differential equations are widely used in control theory, one can easy define controlled system by adding control signal into right-hand expression of selected equation.

If one adds control signal $u$ into (30), following equation can be written down

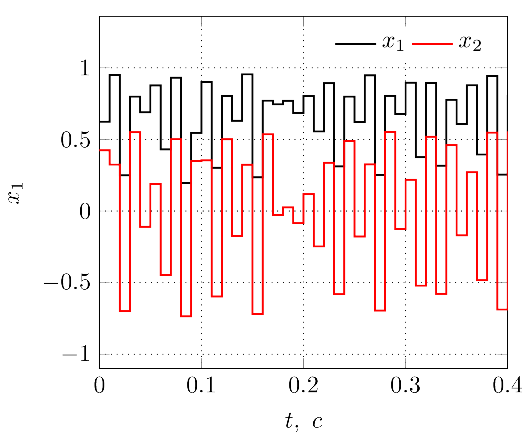
$$\frac{x_{2i}-x_{2(i-1)}}{T}=\frac{(r-1)Tx_{2(i-1)}-r\left(x_{1(i-1)}^2-x_{1(i-2)}^2\right)}{T^2}+u_i. \qquad (31)$$

Simulation results for system (31) are given in Figure.5a-Figure.5b. We assume that external control signal is defined as harmonic function

$$u_i=7\sin(50iT). \qquad (32)$$




a) External control signal




b) System output

**Figure 5:** Chaotic system under external signal.

Comparison of simulation results in Figure.5a, Figure.5b with Figure.4a, Figure.4b shows that even quite small external signal after a short time change system motion trajectory without any system reconfiguration. Moreover, defining these control signals as functions of system state variables and desired motions allows to solve systems' synchronization problems and makes different systems moves through the same trajectories. We leave solution of kind problems for our future research.

### 3.4. Design 2nd order exactly-defined discrete-time controlled chaotic system with observability equation

One can use all above-considered chaotic systems as the sources of unpredictable true random oscillations. It is a well-known fact that motions of such systems depend on its parameters, initial conditions and external signals. Changing of these factors allows us to generate oscillations in a quite wide range.

One can dramatically extend this range by performing some processing of the generated signals. We offer to perform such a processing by using so-called observability equations.

Classical control theory assumes operating the linear only observability equations but it is clear that the use of different nonlinear functions allows to increase the number of produced oscillations.

In this subsection we show results of applying linear

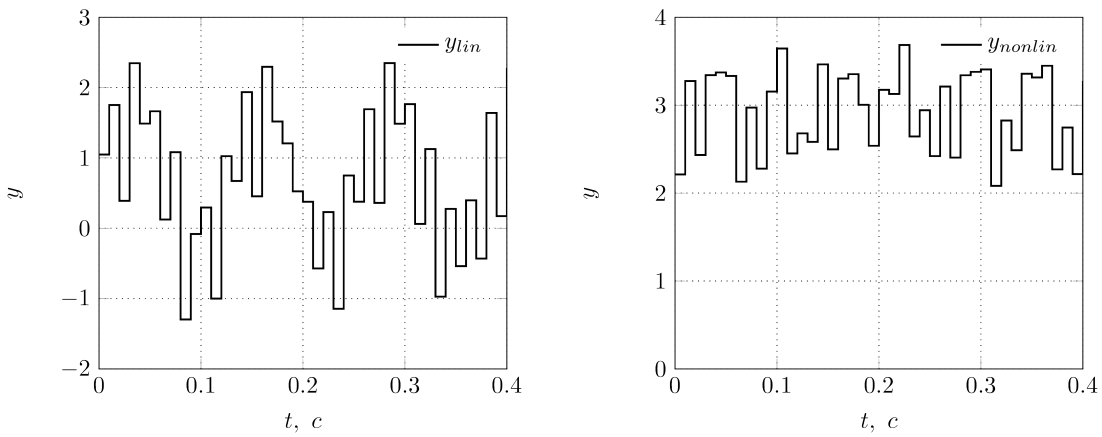
$$y_{i\,lin}=x_{1\,i}+x_{2\,i}+u_i/7. \qquad (33)$$

and nonlinear

$$y_{i\,nonlin}=2\cos(x_{2i})+x_{1i}^2+|u_i/7|. \qquad (34)$$

observability equations to controlled system, which motions are defined by (27) and (31). As one can see there no any restrictions on the using nonlinear functions in the observability equation. Moreover, despite we use analytical functions to rewrite state space equations in terms of output variables, one can use non-analytical ones as well without defining new motion equations.

Simulation results of discrete-time chaotic system based on (27) and (31), which outputs are defined according to (33) and (34), are given in Figure.6a and Figure.6b




a) System output according to (33) b) System output according to (34)

**Figure 6:** Chaotic signals which are produced after chaotic oscillations processing.

Analysis of results in Figure.6a and Figure.6b shows that applying various processing algorithms for generated signals allows us to change these signals in a very wide range, including forming dual-polar signals and signals with constant bias.

It should be mentioned that such a signal processing assumes that these system state variables are obtained in some ways and then the their values are substituted into observability equation. At the same time, as it have been already mentioned one can solve observability equation and rewrite system motion equations.

Let us show such a rewriting by considering linear observability equation (33). If one solve this equation for $x_{1i}$

$$x_{1i} = y_{i\,lin} - x_{2\,i} - u_i / 7. \tag{35}$$

Equations (27) and (31) can be rewritten in such a way

$$\frac{y_{ilin} - y_{(i-1)lin}}{T} = \frac{(T+1)x_{2i}}{T} - \frac{x_{2(i-1)}}{T} + \frac{1}{7T}(u_i - u_{i-1}). \tag{36}$$

$$\frac{x_{2i} - x_{2(i-1)}}{T} = \frac{(r-1)x_{2(i-1)}}{T} + u_i - \tag{37}$$

$$\frac{-r\left(\left(y_{(i-1)lin} - x_{2(i-1)} - u_{i-1}/7\right)^2 - \left(y_{(i-2)lin} - x_{2(i-2)} - u_{i-2}/7\right)^2\right)}{T^2}$$

As one can see above performed transformation gives us the possibility to define output variable in the explicit way. Furthermore, such an approach extend the class of used finite-difference equations by using external signal in each equation. Moreover, the current and previous values of this signal is used.

One can easy perform the similar transformations for (34) as well. As a result 3$^{rd}$ order system with two outputs is obtained. We find that this can reduce calculation time and improve calculation performance. In our study it allows us to decrease calculation time on 8.1%.

## 4. Conclusion

The proposed approach allows us to design discrete-time dynamical system based on the discrete maps. The main feature of designed in such a way dynamical systems is their high performance due to huge values of factors which are used to define system equations. Contrary to classical dynamical systems which operates with time constants more than tensor hundreds of simulation steps, the designed systems are defined with time constants which are equal or higher than simulation step. Another benefit of the proposed approach is the possibility to extend the order of dynamical system in comparing with initial discrete-time map. In joint with the use of observability equations' concept and using controlled form of equations this fact allows us to obtain some internal state variables to generate new signals for known systems. Generated in such a way signals can be calculated in parallel way by using various calculation nodes and thus improve system performance.

## Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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