Clipping method of unpromising variants of a solution for the problem of integer linear programming with Boolean variables

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Abstract

This paper introduces an innovative clipping method (CM) for solving integer linear programming problems with Boolean variables, addressing the computational complexity inherent in these NP-complete problems. The proposed method is based on a rank-based approach that transforms the unitary n-dimensional cube Bn into a graph structure, where the vertices are organized by ranks according to the number of ones in their binary representations. The methodology employs six distinct clipping strategies (L1 - L6) that systematically eliminate non-promising solution paths during the search process. The fundamental strategy involves introducing a "double corridor" concept that establishes upper and lower bounds for feasible solutions, significantly reducing the search space. The method's key innovation lies in its ability to predict and exclude non-optimal paths through careful analysis of weighted graph traversals, where weights correspond to objective function coefficients and constraint parameters. Experimental results demonstrate that the proposed CM achieves algorithmic complexity, which represents a substantial improvement over exponential-time algorithms. Comparative analysis against well-known methods, including Balas' algorithm, the P-method, and exhaustive search approaches, reveals superior performance in terms of elementary operations count, solution time, and decision-making probability within time constraints. The method proves particularly effective for higher-dimensional problems when reasonable time limits are established, making it applicable to various practical optimization scenarios in cryptanalysis, network topology analysis, and resource allocation problems.

Keywords

Integer linear programming, Boolean variables, clipping, solution method, clipping strategies, algorithmic complexity,

1. Introduction

Across numerous research domains spanning computer science, engineering, economics, and operations research, practitioners frequently encounter optimization problems that can be formulated as integer linear programming (ILP) tasks with Boolean variables. These problems arise naturally in various application areas, including resource allocation, scheduling, network design, cryptanalysis, facility location, portfolio optimization, and combinatorial auction design [1]. The prevalence of such formulations stems from their remarkable expressive power and ability to capture discrete decision-making processes inherent in real-world scenarios.

Nevertheless, practical experience has consistently demonstrated that solving these optimization problems presents significant computational challenges. The primary difficulties include prohibitively

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complex algorithms and exponential growth in solution time relative to problem size. Memory requirements also scale unfavorably with dimensionality and the fundamental computational intractability characteristic of NP-complete problems [2]. These limitations become particularly acute when dealing with large-scale instances encountered in industrial applications, where problem dimensions may extend to thousands or millions of variables and constraints.

The integer programming framework, for which Boolean variable problems constitute a special case, has been recognized as extraordinarily versatile. Indeed, a vast array of combinatorial optimization problems (including the traveling salesman problem, the knapsack problem, the set covering problem, the maximum clique problem, and the graph coloring problem) can be naturally formulated as integer programming instances [**?**]. This universality, while theoretically elegant, has significant computational implications. The ability to encode virtually any NP-complete problem within the integer programming paradigm simultaneously confirms both the framework's power and its inherent computational difficulty. As established by complexity theory, unless $P = NP$, there is no polynomial-time algorithm to solve general integer programming problems [3].

In standard terminology, the phrase "integer programming" typically refers specifically to integer linear programming, wherein both the objective function and constraints are linear expressions of the decision variables. However, the broader landscape encompasses mixed-integer programming (MIP), where some variables are restricted to integer values. In contrast, others remain continuous, as well as mixed-integer nonlinear programming (MINLP), which incorporates nonlinear objective functions or constraints [3]. These nonlinear variants introduce additional layers of computational complexity beyond their linear counterparts. However, ILP techniques have demonstrated effectiveness across this spectrum of problem types, which is valuable for pure-integer formulations, pure-binary formulations where variables are restricted to 0,1, and hybrid formulations involving arbitrary combinations of decision variables with real value, integer value, and binary value [4].

The research community has developed numerous methodologies and algorithmic approaches to address integer programming problems. Contemporary commercial optimization software packages, including CPLEX, Gurobi, and XPRESS, routinely accept integer and binary variable restrictions as standard input specifications [5]. These sophisticated solvers employ branch-and-bound frameworks, automatically constructing search trees that systematically partition the solution space while computing linear programming relaxations at each node to establish bounds on optimal solutions. Advanced implementations incorporate cutting plane techniques, heuristic methods, and pre-solving procedures to enhance computational performance. Despite these algorithmic refinements, integer linear programming problems generally require substantially greater computational resources than their continuous linear programming counterparts. Although linear programming can be solved in polynomial time using interior-point or simplex methods [6], integer programming instances of comparable size may demand orders of magnitude more computation time, with solution times often showing exponential growth as the dimensions of the problem increase.

This computational gap between continuous and discrete optimization motivates ongoing research into specialized algorithms that exploit problem structure, develop tighter formulations, and employ novel search strategies to mitigate the inherent complexity of integer programming. The present work contributes to this research direction by introducing a clipping method specifically designed for integer linear programming problems with Boolean variables, aiming to reduce both algorithmic complexity and practical solution times through strategic elimination of unpromising solution candidates.

2. Literature Review

The application of integer linear programming with Boolean variables has been extensively investigated across multiple research domains, demonstrating the versatility and computational challenges inherent in this optimization paradigm. This section examines representative studies that illustrate both the breadth of applications and the evolution of solution methodologies.

Mihailescu [7] introduced a groundbreaking approach to cryptanalysis through linear approximation

methods, demonstrating that the Data Encryption Standard (DES) cipher could be compromised using integer programming formulations. His methodology successfully attacked 8-round DES using 2^{21} known plain texts and 16-round DES with 2^{47} known plain texts. Notably, when plain texts consisted of natural English sentences encoded in ASCII format, the cipher text-only attack required merely 2^{29} samples. This work established a fundamental connection between cryptographic security analysis and discrete optimization.

Building upon this foundation, Borghoff [8] presented an advanced framework applying mixed-integer linear programming to cryptanalysis of modern stream and block ciphers. Her investigation focused on Trivium, recommended by the eSTREAM project, and the lightweight cipher Ktantan, demonstrating how nonlinear multivariate Boolean equation systems inherent in cryptographic primitives can be reformulated as MILP problems. The methodology encompassed critical design decisions, including conversion techniques for transforming Boolean equations into real-valued constraints, strategic variable selection, and exploitation of the CPLEX commercial solver capabilities. More recently, Kölbl et al. [9] extended these techniques to analyze the SIMON block cipher family, achieving improved bounds on the number of active S-boxes through refined MILP models that incorporate differential and linear cryptanalysis constraints simultaneously.

Buchheim and Rinaldi [10] developed an innovative polyhedral approach to nonlinear Boolean optimization that significantly reduces model size compared to conventional formulations. Their methodology avoids transformation to a normal form, thereby eliminating the introduction of auxiliary variables that typically inflate the issue dimensions. By efficiently reducing the problem to degree-two polynomials through minimal dimension extension, they established connections to the maximum cut problem, demonstrating that the corresponding polytope constitutes a face of an appropriate cut polytope. Experimental validation using challenging instances from the Max-SAT Evaluation 2007 revealed that their general-purpose implementation outperformed specialized Max-SAT solvers in numerous cases.

Subsequently, Rostami et al. [11] extended this framework by introducing strengthened linearization techniques for pseudo-Boolean optimization, incorporating valid inequalities derived from the structure of the objective function. Their computational experiments on benchmark instances demonstrated solution time reductions of up to 40% compared to standard linearization approaches. Furthermore, Dlask et al. [12] investigated preprocessing techniques for Boolean optimization problems, showing that problem-specific reduction rules combined with constraint propagation can eliminate up to 60% of variables in certain problem classes before invoking MILP solvers.

Götz et al. [13] addressed the challenge of integrating optimization techniques into self-adaptive software systems through model-driven development paradigms. Their work compared Integer Linear Programming and Pseudo-Boolean Optimization approaches for automatically generating optimization problems from architectural models at runtime. The study provided comprehensive scalability analysis for pipe-and-filter application architectures, demonstrating practical feasibility despite the inherent computational complexity. Their methodology automated the translation from high-level system specifications to concrete optimization formulations, significantly reducing development effort and error potential.

Recent advances in this domain include the work of Jamshidi et al. [14], who developed transfer learning techniques for configuration optimization in software product lines. Their approach leverages MILP formulations to identify near-optimal configurations across related software systems, achieving speedups of two orders of magnitude compared to traditional search-based methods. Additionally, Siegmund et al. [15] investigated performance-influence models for configurable systems, utilizing Boolean satisfiability constraints encoded as integer programming problems to predict system performance across vast configuration spaces.

Savage et al. [16] introduced a sophisticated methodology for reconciling experimental phosphoproteomic data with prior knowledge of cellular signaling networks. Unlike prevalent approaches based on Bayesian networks, Boolean models, or ordinary differential equations, their technique employs integer linear programming on interaction graphs to encode qualitative behavioral constraints. The framework provides four fundamental operations: (i) identifying topology-consistent explanations for

experimental observations or determining nearest feasible explanations when inconsistencies exist; (ii) computing minimal node correction sets to restore consistency; (iii) extracting optimal subgraphs that best reflect experimental scenarios; and (iv) identifying candidate edges whose inclusion would maximally improve model-data concordance. Validation using EGFR/ErbB signaling data from primary hepatocytes demonstrated the method's ability to identify context-specific inactive interactions and propose biologically plausible network refinements. The freely available SigNetTrainer toolbox facilitates widespread application of this methodology.

Extending these biological applications, Razzaq et al. [17] developed MILP-based methods for metabolic network reconstruction that simultaneously optimize network topology and flux distributions. Their approach integrates transcriptomic and metabolomic data to infer context-specific metabolic models, achieving prediction accuracies exceeding 85% for growth phenotypes across diverse environmental conditions.

Terci et al. [18] formulated optimal bitwise register allocation as an integer linear programming problem, addressing a critical challenge in compiler backend optimization. Their model explicitly captures interference constraints between variables at the bit level, enabling more efficient utilization of processor registers compared to traditional graph-coloring heuristics. The ILP formulation guaranties optimality for small-to-medium code regions while providing high-quality solutions for larger instances when combined with time-bounded branch-and-bound search.

Contemporary work by Lozano et al. [19] has refined these techniques through the development of integer programming models for integrated instruction selection and register allocation. Their unified optimization approach eliminates the traditional phase-ordering problem in compiler backends, achieving code quality improvements of $5 - 15\%$ across standard benchmark suites. Furthermore, Kim et al. [20] investigated the use of partitioned Boolean quadratic programming for register allocation with rematerialization, demonstrating that concern decomposition strategies can extend the applicability of exact methods to industrial-scale compilation tasks.

The foundational work of Balas [21] on additive algorithms for zero-one linear programming established many principles underlying modern branch-and-bound methods. Gomory's [22] pioneering cutting plane techniques provided theoretical foundations for constraint strengthening approaches widely employed in contemporary solvers. More recently, Subramani et al. [23] investigated Boolean combinations of unit two-variable per inequality (UTVPI) constraints, demonstrating that certain restricted constraint classes admit polynomial-time solution algorithms despite the general NP-completeness of Boolean integer programming.

Demirović et al. [24] developed sophisticated methods for solving systems of pseudo-Boolean constraints by leveraging advances in Boolean satisfiability solving technology. Their hybrid approach combines MILP relaxations with SAT-based search, achieving substantial performance improvements on constraint satisfaction problems arising in electronic design automation. Conow et al. [25] formulated cophylogeny reconstruction concerns (which model co-evolutionary relationships between host and parasite species) as integer linear programs, demonstrating that phylogenetic inference difficulties can benefit from discrete optimization techniques.

Balen et al. [26] addressed the compilation of high-level array programs by formulating loop fusion optimization as an ILP problem. Their technique determines optimal fusion strategies that balance data locality benefits against potential parallelism losses, incorporating machine-specific cost models into the optimization objective.

The surveyed literature demonstrates that ILP with Boolean variables provides a unifying mathematical framework applicable across remarkably diverse domains — from cryptographic security and compiler optimization to systems biology and software engineering. Despite substantial methodological advances over the past decades, including sophisticated branching strategies, cutting plane generation, and hybrid algorithmic approaches, the fundamental computational challenge persists: general ILP problems with Boolean variables remain NP-complete, and practical solution times exhibit exponential growth for large-scale instances.

Several research gaps emerge from this review (Table 1). First, while commercial solvers have achieved impressive performance through algorithmic refinements and hardware acceleration, guarantee

Table 1
Comparative Analysis of Reviewed Methods

Author(s)	Theory / Application Domain	Method Type	Key Innovation	Computational Complexity	Solution Quality	Implementation
Mihailescu (2023) [7]	Cryptanalysis	Linear approximation with ILP	Known-plaintext attack on DES	Exponential (2^{21} – 2^{47} texts)	Exact cipher break	Not publicly available
Buchheim and Rinaldi (2010) [10]	Boolean optimization	Polyhedral approach	Avoids normal form transformation	Polynomial relaxation + B&B	Optimal	Online (logoptimize.it)
Borghoff (2025) [8]	Cipher cryptanalysis	MILP formulation	Boolean to real conversion	Depends on CPLEX	Optimal with time limits	CPLEX-based
Götz et al. (2013) [13]	Self-optimizing systems	ILP & PBO	Model-driven generation	Exponential (scalable)	Optimal for small instances	Prototype
Savage et al. (2020) [16]	Systems biology	ILP on interaction graphs	Qualitative constraint encoding	Polynomial relaxation + B&B	Approximate with guaranties	SigNetTrainer toolbox
Terci et al. (2024) [18]	Compiler optimization	ILP for register allocation	Bit-level interference modeling	$O(n^3)$ relaxation + B&B	Optimal for small regions	Research prototype
Kölbl et al. (2022) [9]	Block cipher analysis	Enhanced MILP	Integrated differential/linear constraints	Problem-dependent	Improved bounds	Extension of existing tools
Rostami & Buchheim (2015) [11]	Pseudo-Boolean optimization	Strengthened linearization	Valid inequalities from the structure	Reduced by $\sim 40\%$	Optimal	Academic implementation
Jamshidi et al. (2017) [14]	Software product lines	Transfer learning + MILP	Cross-system configuration reuse	100× speedup via learning	Near-optimal	Machine learning framework
Razzaq et al. (2018) [17]	Metabolic networks	MILP for reconstruction	Topology-flux co-optimization	Exponential (practical limits)	85% accuracy	BiGG Models integration
Buchwald et al. (2018) [29]	Compiler backends	Integrated instruction/register	Unified phase elimination	Polynomial relaxation + search	5-15% improvement	LLVM-based
Balas (1965) [21]	General zero-one LP	Additive algorithm	Implicit enumeration	Exponential with pruning	Optimal	Foundational (many implementations)
Savage et al. (2020) [16]	Pseudo-Boolean constraints	SAT-MILP hybrid	Combines relaxation + SAT	Hybrid complexity	Optimal with timeouts	PBS solver

problem-specific clipping strategies that exploit structural properties remain underexplored. Second, most existing approaches [27] and [28] focus on either an exact method with worst-case exponential complexity or heuristics without optimality guaranties, leaving the opportunity for hybrid methods that provide quality-controlled approximate solutions with predictable computational bounds. Third, the potential for parallelization and distributed computation in Boolean ILP solving has received limited attention compared to continuous optimization domains.

The present work addresses these gaps by introducing a clipping method specifically designed for integer linear programming problems with Boolean variables. The proposed approach leverages rank-based graph representations and multiple complementary pruning strategies to systematically eliminate unpromising solution candidates, thereby reducing both algorithmic complexity and practical solution times while maintaining solution quality guaranties.

3. Mathematical Background and Problem Formulation

Binary Integer Programming (BIP), alternatively referred to as integer linear programming with Boolean variables, constitutes a fundamental formal model for representing a broad spectrum of scientific and technical optimization problems. In this formulation, each decision variable is constrained to assume exclusively binary values from the set 0, 1, thereby encoding discrete choices such as project selection or rejection, resource activation or deactivation, binary switching states, or affirmative or negative responses. Numerous other dichotomous decision scenarios are encountered in practical applications [30].

The BIP framework is characterized as an m-dimensional optimization problem belonging to the NP-complete complexity class, a classification that establishes both its theoretical intractability and practical computational challenges [31]. Unless the widely conjectured inequality $P \neq NP$ is disproved, non polynomial-time algorithm can exist for solving general instances of this issue class. Nevertheless, the canonical formulation provides a mathematically rigorous foundation for algorithm development and theoretical analysis.

3.1. Standard Mathematical Formulation

Following the convention established by Chinneck [32], the general BIP problem can be expressed in the following standard form. The objective function seeks to maximize (or equivalently minimize) a linear combination of binary decision variables:

$$F = \sum_{j=1}^{n} C_j \cdot x_j \to \max\,(\min) \tag{1}$$

where $C_j \in \mathbb{R}\geq 0$ represents the objective function coefficient associated with decision variable x_j, and n denotes the problem dimensionality.

The feasible solution space is delimited by m linear inequality constraints of the form:

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \geq b_i, \quad \text{for } i = 1, 2, \ldots, m, \tag{2}$$

where $a_ij \in \mathbb{R}$ denotes the constraint coefficient matrix elements, and $b_i \in \mathbb{R}$ specifies the right-hand side bounds. The formulation imposes the following structural conditions:

Binary constraint: all decision variables x_j, where $j = 1, 2, \ldots, n$ are restricted to the binary domain: $x_j \in \{0, 1\}$.

Non-negativity of the objective coefficients: all coefficients in the objective function (1) satisfy $C_j \geq 0$ for all j.

Ordered coefficients: without loss of generality, variables are indexed such that their objective function coefficients satisfy the monotonicity condition:

$$0 \leq C_1 \leq C_2 \leq ... \leq C_n \tag{3}$$

Although this standard form may initially appear restrictive, most practical BIP instances can be transformed to this canonical representation through elementary algebraic manipulations. For instance, negative objective function coefficients are accommodated through variable substitution, wherein x_j is replaced by its complement $(1 - x'_j)$, effectively reversing the contribution polarity while preserving the

problem structure. Similarly, variable reordering to achieve the monotonicity condition (3) constitutes a trivial index permutation. Constraints involving $\leq$ inequalities are readily converted to the $\geq$ standard form through multiplication by -1, noting that constraint right-hand sides may assume negative values without loss of generality.

3.2. Theoretical Properties and Complexity Considerations

The BIP problem exhibits several fundamental theoretical properties that distinguish it from its continuous optimization counterpart and establish its computational characteristics [**?**]:

Property 1 (Polynomial relaxation): relaxing the integrality constraint $x_j \in \{0, 1\}$ to the continuous interval $x_j \in [0, 1]$ transforms the problem into a standard Linear Programming (LP) instance, which admits polynomial-time solution via algorithms such as the ellipsoid method or interior-point methods. This LP relaxation provides lower bounds (for minimization) or upper bounds (for maximization) on the optimal objective value and forms the theoretical foundation for branch-and-bound methodologies.

Property 2 (Formulation equivalence): multiple equivalent formulations exist for BIP problems. Extensions may incorporate equality constraints in addition to inequalities without altering the fundamental complexity class. Furthermore, the decision problem variant (determining whether a feasible solution exists) and the optimization variant (finding an optimal solution) are polynomially equivalent through standard reduction techniques.

Property 3 (NP-completeness): the general BIP decision problem is NP-complete [33], as established through reduction from the 3-SAT problem or other canonical NP-complete problems. Consequently, the optimization variant belongs to the NP-hard class. This theoretical classification implies that worst-case solution time grows exponentially with a problem dimension n unless $P = NP$.

Property 4 (Solution space structure): the feasible region of a BIP problem constitutes a finite subset of the n-dimensional Boolean hypercube $B^n = \{0, 1\}^n$, containing at most 2^n vertices. However, constraint satisfaction may render the vast majority of these vertices infeasible, creating a sparse feasible region embedded within the hypercube structure.

3.3. Algorithmic Classification for BIP Solution Methods

The computational challenges posed by Boolean integer programming have motivated extensive algorithm development spanning multiple decades. A comprehensive taxonomic classification of solution methodologies can be organized into three principal categories based on their fundamental search strategies and theoretical foundations:

Category 1: Combinatorial Enumeration Methods This class encompasses algorithms that systematically explore the solution space through structured enumeration strategies:

Complete enumeration: exhaustive evaluation of all 2^n vertices of the Boolean hypercube, guaranteeing optimal solution discovery at the expense of exponential computational cost $O(2^n)$.

Branch-and-bound techniques: intelligent enumeration employing recursive partitioning of the solution space combined with bound computation via LP relaxations to prune suboptimal branches. Modern implementations incorporate sophisticated branching heuristics such as strong branching, pseudo-cost branching, and reliability branching [34].

Dynamic programming approaches: decomposition of the problem into overlapping subproblems with optimal substructure properties, enabling polynomial-space solution for certain restricted problem classes exhibiting appropriate recursive structure.

Additive algorithms: methods based on the foundational work of Balas [21] that construct solutions incrementally through systematic addition of variables while maintaining feasibility and optimality conditions.

Rank-based approaches: specialized techniques that exploit ordered variable structure by organizing the Boolean hypercube into hierarchical ranks according to Hamming weight (number of ones in binary representations), facilitating structured search through geometric graph representations.

Category 2: Sequential Solution Space Reduction Methods This category comprises techniques that progressively narrow the feasible region through iterative constraint tightening and dominated solution elimination:

Sequential variant analysis: systematic examination of solution candidates with incremental constraint satisfaction verification and dominated solution exclusion based on lexicographic or parametric ordering. Sequential plan construction: iterative solution building through staged variable fixing and partial solution evaluation, commonly employed in resource allocation and scheduling contexts.

Cutting plane methods: generation of valid inequalities that eliminate fractional solutions from the LP relaxation without excluding integer feasible points, thereby tightening the polyhedral relaxation. Modern cutting plane families include Gomory cuts, lift-and-project cuts, and problem-specific cuts derived from an issue structure [35].

Category 3: Solution Quality Improvement Methods These approaches iteratively refine solution quality through local or global search mechanisms :

Descent direction methods: local improvement algorithms that identify improving directions in the discrete solution space through neighborhood exploration, analogous to gradient descent in continuous optimization.

Stochastic search techniques: randomize algorithms, including simulated annealing, genetic algorithms, and tabu search, that employ probabilistic mechanisms to escape local optima and explore the solution landscape.

Local optimization heuristics: greedy algorithms and constructive heuristics that build solutions through myopic optimization decisions, trading optimality guaranties for computational efficiency.

ε-optimal algorithms: approximation algorithms that provide solution quality guaranties in the form $F^* \leq F_{alg} \leq (1+\varepsilon)F^*$ of minimization problems, where F^* denotes the optimal objective value and $\varepsilon > 0$ represents the approximation factor.

3.4. Hybrid Methodologies and Contemporary Developments

Contemporary BIP solvers typically integrate multiple algorithmic paradigms into sophisticated hybrid frameworks. Commercial solvers such as CPLEX, Gurobi, and XPRESS combine branch-and-bound search with cutting plane generation, primal heuristics, pre-solving techniques, and parallel computation [36]. These systems employ extensive algorithm portfolios that dynamically select and configure solution strategies based on instance characteristics detected through automated problem classification. Despite decades of algorithmic innovation and substantial computational performance improvements driven by hardware advances, the fundamental computational barrier posed by NP-completeness persists. Large-scale BIP instances arising in industrial applications frequently exceed the practical capabilities of general-purpose solvers, motivating continued research into problem-specific methods that exploit structural properties to achieve superior performance within restricted concern classes.

4. Pruning Unpromising Solutions in Boolean ILP: Rank-Based Clipping Method

4.1. Theoretical Foundations and Graph-Based Representation

The proposed methodology is based on the rank-based approach originally developed by some authors [37, 38, 36, 39]. This framework provides a systematic mechanism for transforming the exponential search complexity inherent in NP-complete Boolean integer programming problems into a structured graph traversal problem amenable to efficient pruning strategies.

The fundamental principle underlying the rank-based approach involves partitioning the n-dimensional Boolean hypercube $B^n = \{0,1\}^n$ into $n+1$ hierarchical ranks, where rank r (for $r = 0, 1, \ldots, n$) contains all binary vectors possessing exactly r components equal to unity. This partitioning naturally orders the 2^n vertices of the hypercube according to their Hamming weight,

creating a layered graph structure G_Δ that encodes the complete solution space while facilitating systematic exploration.

4.2. Graph Construction and Properties

The graph G_Δ represents an alternative geometric realization of the Boolean hypercube B^n, wherein vertices are organized into ranks and the edges connect to vertices differing in exactly one binary position. Formally, the graph $G_\Delta = (V, E)$ possesses the following structural characteristics. Vertex set: the vertex set V is partitioned into $n + 1$ disjoint ranks:

$$V = \bigcup_{r=0}^{n} V_r, \quad \text{where } V_r = \{x \in \{0,1\}^n : \|x\|_1 = r\}, \tag{4}$$

and $\|x\|_1 = \sum_{j=1}^{n} x_j$ denotes the Hamming weight (number of ones) in a vector x.

Edge set: an edge $(x, y) \in E$ exists if and only if x and y belong to consecutive ranks and differ in exactly one coordinate position, i.e., $(x - y)_1 = 1$ with $x_1 = r$ and $y_1 = r + 1$ for some $r \in 0, 1, ..., n - 1$.

Cardinality: rank r contains exactly $\binom{n}{r}$ vertices, and the total number of edges is equal:

$$\sum_{r=0}^{n-1} \binom{n}{r} \cdot (n - r) = n \cdot 2^{n-1} \tag{5}$$

Figure 1 illustrates the graph G_Δ for a case n = 4, demonstrating the hierarchical rank structure and a connectivity pattern. Each path from the source vertex (rank 0, representing the zero vector) to a terminal vertex (rank n) corresponds uniquely to a vertex of the original Boolean hypercube B^n.

4.3. Multi-Attribute Edge Weighting Scheme

To incorporate the BIP problem structure (1)-(2) into the graph representation, each edge $(x, y) \in E$ entering vertex j is assigned $(m+1)$ distinct weights that capture both objective function contributions and constraint violation measures:

Objective weight $c_j \geq 0$: equal to the coefficient of variable x_j in the objective function (1), representing the marginal contribution to the objective value when transitioning from x to y with $x_j = 1$.

Constraint weights $a_{ij} \in \mathbb{R}$ for $i = 1, 2, ..., m$: equal to the coefficients of the variable x_j in m constraint inequalities (2), quantifying the impact on constraint satisfaction when the x_j transitions are from 0 to 1.

4.4. Path Characterization and Feasibility Assessment

A path μ_{sj}^r in graph G_Δ from the source vertex s (the all-zeros vector at rank 0) to the vertex j at rank r is characterized by $(m + 1)$-cumulative path lengths that aggregate edge weights along the traversed edges:

Definition 1 (Objective path length): The objective path length $d_c(\mu_{sj}^r)$ represents the total accumulated weight with the objective function coefficients:

$$d_c(\mu_{sj}^r) = \sum_{k \in \mu_{sj}^r} c_k, \tag{6}$$

where the summation extends over all variable indices k corresponding to the edges traversed in a path μ_{sj}^r. This quantity directly equals the objective function value (1) for the binary solution vector represented by a path μ_{sj}^r.

Definition 2 (Constraint path lengths): For each constraint $i \in 1, 2, ..., m$, the constraint path length $d_(a_i)(\mu_{sj}^r)$ accumulates the corresponding constraint coefficients:

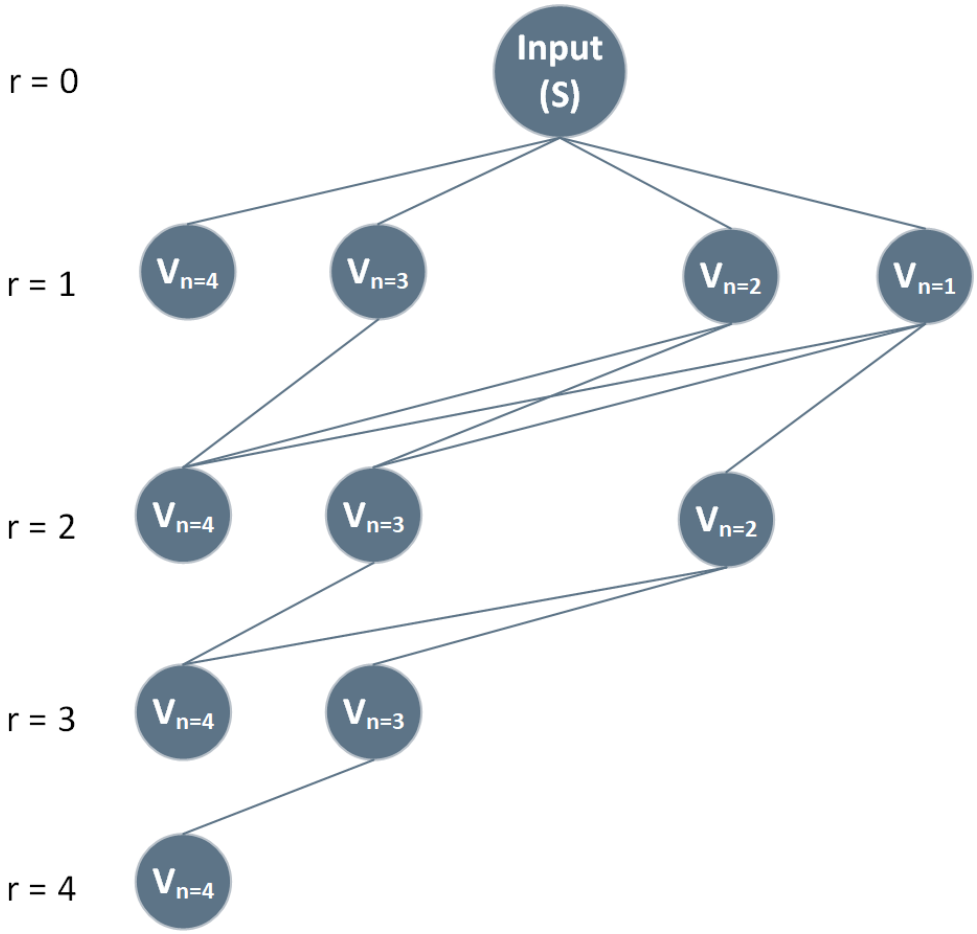

Figure 1: An example of the graph G_Δ.

$$d_{a_i}(\mu_{sj}^r) = \sum_{k \in \mu_{sj}^r} a_{ik}. \tag{7}$$

The feasibility of the solution represented by a path μ_{sj}^r is determined by comparing these accumulated constraint lengths against the right-hand side bounds b_i.

4.5. Fundamental Equivalence Theorem

The graph G_Δ provides an exact representation of the BIP problem (1)-(2) through the following correspondence:

Theorem 1 (Graph-BIP equivalence): there exists a bijective mapping between complete paths from source to sink (rank 0 to rank n) in graph G_Δ and vertices of the Boolean hypercube B^n. Furthermore, the optimal solution to the BIP problem (1)-(2) corresponds to the path $\mu_{sn}^* \in M_{sn}$ that:

Maximizes the objective path length: $d_c(\mu_{sn}^*) = max_{\mu \in M_{sn}} d_c(\mu)$. Satisfies all constraint conditions: $d_(a_i)(\mu_{sn}^*) \geq b_i$ for all $i = 1, 2, ..., m$, where M_{sn} denotes the set of all complete paths from a source to the rank n.

This equivalence transforms the discrete optimization problem into a constrained longest-path problem on a directed acyclic graph, providing the theoretical foundation for the rank-based algorithmic framework.

4.6. Primary Clipping Strategy and Upper Bound Estimation

The computational efficiency of the proposed method comes from the aggressive pruning of unpromising partial paths during the forward search process. The principal clipping criterion employs an optimistic upper bound on the maximum achievable objective value from any given partial path.

Strategy $C1$ **(Fundamental clipping inequality):** A partial path μ_{sj}^r terminating at a vertex j in rank r is eliminated from further exploration if the following condition holds:

$$d_c(\mu_{sj}^r) + \gamma_p < \max_{\{C_j\}}\{d_c(\mu_{sp}^{*r})\}, \tag{8}$$

where $d_c(\mu_{sj}^r)$ (see Eq. 6) represents the accumulated objective value along the current partial path, $\gamma_p = c_{p+1} + c_{p+2} + ... + c_n$ with $\gamma_n = 0$ for $p = 1, 2, ..., n-1$ constitutes an optimistic upper bound on the maximum possible objective increment achievable by extending the path from rank r to rank n, the right part of (Ineq. 8) denotes the best objective value discovered among all paths explored up to rank r.

Justification: The quantity γ_p represents a non-guaranteed forecast (an optimistic estimation) assuming that all remaining variables with positive objective coefficients can be simultaneously set to unity, which may violate constraint feasibility. Nevertheless, this optimistic bound provides a valid upper limit on the achievable objective improvement. If inequality (8) is satisfied, the current partial path cannot possibly yield a solution superior to already discovered alternatives, regardless of how the remaining variables are assigned, thereby justifying its elimination.

This clipping mechanism substantially reduces both the enumeration tree sizes for exact algorithms and the approximation error for heuristic variants, yielding significant computational time reductions while preserving solution quality guaranties.

4.7. Extended Graph Representation for Mixed Constraint Types

To accommodate BIP formulations containing both "$\leq$" and "$\geq$" inequality constraints, an enhanced graph representation G'_Δ is constructed wherein each edge incoming to the vertex j carries three distinct weight types:

Objective weight c_j: identical to the coefficient of variable x_j in the objective function (1).

Upper bound constraint weights $a_{1j}, a_{2j}, ..., a_{kj}$: coefficients of x_j in the first case k constraints of form (2) with "$\leq$"-relational operators.

Lower-bound constraint weights $d_{Z_1,j}, d_{Z_2,j}, ..., d_{Z_l,j}$: coefficients of x_j in the remaining $l = m - k$ constraints with "$\geq$"-relational operators, where $Z_i \in \{k+1, k+2, \ldots, m\}$ for $i = 1, 2, \ldots, l$.

4.8. Solution Space Decomposition

The complete solution space can be decomposed as a union of rank-specific subsets:

$$M = \bigcup_{r=1}^{n} M^r, \tag{9}$$

where M^r denotes the collection of all partial paths that terminate at the rank r. This decomposition enables the stage-wise solution construction with intermediate feasibility for checking and a bound updating.

As demonstrated in previous publications [38], [36], the enhanced graph G'_Δ supports comprehensive path characterization through three cumulative length measures for any path μ_{sj} from a source vertex s to the vertex j:

Multi-dimensional path length vector:

$$\mathbf{d}(\mu_{sj}) = (d_c(\mu_{sj}), \mathbf{d}_a(\mu_{sj}), \mathbf{d}_d(\mu_{sj})), \tag{10}$$

where:

$d_c(\mu_{sj}) = \sum_{k\in\mu_{sj}} c_k$ quantifies an objective function contribution,

$\mathbf{d}_a(\mu_{sj}) = (d_{a_1}(\mu_{sj}), d_{a_2}(\mu_{sj}), \ldots, d_{a_k}(\mu_{sj}))$ with $d_{a_i}(\mu_{sj}) = \sum_{k\in\mu_{sj}} a_{ik}$ tracks "$\leq$" constraint accumulations,

$\mathbf{d}_d(\mu_{sj}) = (d_{Z_1}(\mu_{sj}), d_{Z_2}(\mu_{sj}), \ldots, d_{Z_l}(\mu_{sj}))$ with $d_{Z_i}(\mu_{sj}) = \sum_{k\in\mu_{sj}} d_{Z_i,k}$ monitors "$\geq$" constraint progress.

4.9. Feasibility Characterization

A complete path μ_{sn} represents a feasible solution to the BIP problem (1)-(2) if and only if its multi-dimensional path length satisfies:

$$d_{a_i}(\mu_{sn}) \leq b_i \text{ for } i = 1, 2, \ldots, k, \text{ and } d_{Z_i}(\mu_{sn}) \geq b_{Z_i} \text{for } i = 1, 2, \ldots, l. \tag{11}$$

4.10. Framework for Comprehensive Clipping Strategy Development

Realization of the proposed rank-based method necessitates the development of a comprehensive suite of clipping strategies $\{L_w\}$ that systematically eliminate unpromising partial paths while guaranteeing that at least one optimal or near-optimal path survives to completion. The subsequent section details six complementary clipping strategies, each exploiting distinct structural properties of the problem formulation and graph representation to achieve maximum pruning effectiveness without sacrificing solution quality. These strategies collectively address objective function bounds, constraint feasibility projections, corridor-based search space restriction, and multi-criteria dominance relationships, forming an integrated algorithmic framework for efficient BIP solution.

5. Description of clipping strategies for the proposed method

The computational efficiency of the proposed rank-based method fundamentally depends on the systematic elimination of unpromising partial paths during the forward exploration process. This section presents six complementary clipping strategies, denoted $\{L_1, L_2, ..., L_6\}$, each exploiting distinct structural properties of the BIP formulation and the underlying graph representation to achieve aggressive search space reduction while preserving optimality guaranties or controlled approximation bounds.

Strategy L_1: Corridor-Based Maximum Objective Selection

The foundational strategy L_1 establishes a systematic mechanism for constructing paths from rank r to rank $r + 1$ by selecting extensions that maximize the objective function contribution while maintaining the constraint feasibility. This strategy defines a "search corridor" within the feasible region, concentrating computational effort on the most promising solution candidates.

Formal Definition (Strategy L_1): when extending the set of partial paths $\{\mu_{sj}^r\}$ at rank r to generate paths $\{\mu_{sp}^{r+1}\}$ at rank $r + 1$, a strategy L_1 selects extensions according to:

$$L_1 : \mu_{sp}^{r=r+1} = \max C_j \mu_{sj}^r \circ Y(j, p), p \in r + 1, r + 2, \ldots, n, j \in 1, 2, \ldots, n, j \neq p, \tag{12}$$

where $\mu_{sj}^r \circ Y(j, p)$ denotes the extension of path μ_{sj}^r by adding the edge from vertex j to vertex p, and a maximization is performed over the objective function weights $\{C_j\}$.

Feasibility Constraints: the selected paths must simultaneously satisfy feasibility conditions with respect to constraint accumulations:

For "$\leq$" constraints: $d_a(\mu_{sj}^r) \leq b_{1,\ldots,k}$, ensuring that accumulated constraint weights do not exceed upper bounds.

For "$\geq$" constraints: $d_d(\mu_{sj}^r) \geq b_{k+1,\ldots,m}$, ensuring that accumulated constraint weights meet or exceed lower bounds.

Paths satisfying these feasibility properties are designated as having property γ, forming the admissible set μ_{sj} for a corridor construction.

Double-Corridor Principle

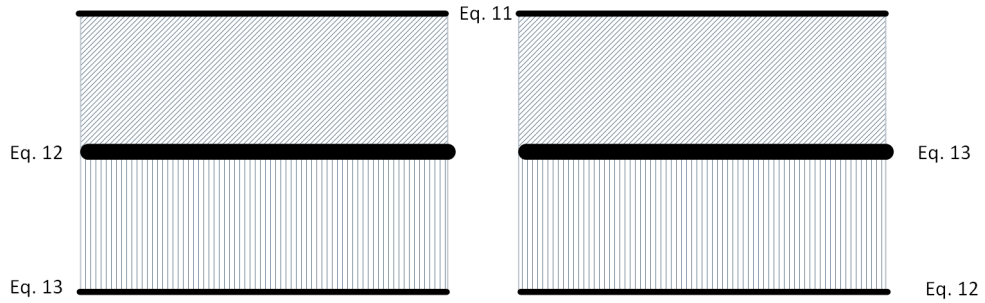

Figure 2: The variants for the “double” corridor allocation.

The concept of a "double corridor" introduces a sophisticated two-stage filtering mechanism that progressively narrows the search space through nested feasible regions, providing both upper and lower bounds on constraint satisfaction.

Definition 1 (Double Corridor): A double corridor from the complete set $\{\mu_{sj}^r\}$ at rank r to the extended set $\{\mu_{sj}^{r+1}\}$ at rank $r+1$ constitutes a hierarchical structure of nested feasible path sets $\{\mu_{sj}\}$ bounded by upper and lower constraint thresholds, all satisfying property γ. The double corridor construction proceeds through two sequential stages:

Stage 1 (Primary corridor definition): The upper boundary of the first corridor is established through the relation (11), selecting paths that maximize objective function increments. The lower boundary is determined through either:

$$\mu_{sp}^{r=r+1} = \min d_{ij}\mu_{sj}^r \circ Y(j,p), \tag{13}$$

which identifies paths minimizing accumulation with respect to "$\leq$" constraint coefficients, thereby maintaining maximum feasibility margin, or alternatively:

$$\mu_{sp}^{r=r+1} = \max d_{ij}\mu_{sj}^r \circ Y(j,p), \tag{14}$$

which identifies paths maximizing accumulation with respect to "$\geq$" constraint coefficients, thereby ensuring rapid satisfaction of lower-bound requirements.

Stage 2 (Nested corridor refinement): The second corridor is defined complementarily: if relation (12) establishes the lower boundary of the primary corridor, then relation (13) defines the second corridor boundaries, and vice versa. By construction, the second corridor is strictly contained within the first corridor, creating a nested filtering hierarchy (illustrated schematically in Figure 2).

This double-corridor architecture provides bidirectional constraint monitoring, simultaneously preventing premature constraint violation (via the "$\leq$" monitoring) and ensuring adequate progress toward constraint satisfaction (via the "$\geq$" monitoring).

Strategy $L-2$: Optimistic Upper Bound Filtration

The second strategy implements the fundamental clipping inequality introduced in equation (3), providing aggressive elimination of partial paths that cannot possibly yield solutions superior to already discovered incumbents.

Theorem 1 (Corridor filtration via upper bounds): Partial paths of rank $r+1$ constructed through extensions of the path set $\{\mu_{sj}^r\}$ can be excluded from subsequent exploration if they satisfy the clipping strategy L_2 (Eq. 14). This strategy constitutes a filtration rule within the established corridor, systematically eliminating dominated solution candidates.

Proof: Strategy L_1 (Eq. 11) identifies and preserves at least one local extremum from the complete collection of local extrema present in the current rank. By construction, the maximum path length of this preserved local extremum necessarily dominates all other local extrema when paths are extended from μ_{sj}^r. Consequently, partial paths μ_{sp}^r satisfying the following inequality can be safely eliminated from the active path set $\{\mu_{sj}^r\}$ without risking loss of optimality:

$$L_2 : d_c(\mu_{sp}^r) + \gamma_p < \max C_j d_c(\mu_{sp}^{*r}), \tag{15}$$

where $d_c(\mu_{sp}^{*r})$ represents the best objective value achieved among all paths at a rank r.

Recursive Upper Bound Computation: The optimistic upper bound γ_p is computed recursively through the backward recurrence relation:

$$\gamma_p = \gamma_{p+1} + c_{p+1}, p = n-1, n-2, \ldots, 1, \text{ with } \gamma_n = 0. \tag{16}$$

This computation efficiently maintains cumulative sums of the remaining objective coefficients, providing a O(n) preprocessing overhead followed by O(1) lookup during path evaluation.

Algorithmic Efficiency Implications: Filtration performed by strategy L_2 (Eq. 14) operates within the corridor established by L_1 (Eq. 11), creating a two-level pruning hierarchy. Empirical studies indicate that this combined approach eliminates between $60 - 90\%$ of partial paths in typical problem instances, yielding substantial computational savings.

Strategy L_3: Lower-Bound Constraint Anticipation

To effectively handle mixed constraint types, particularly "$\geq$" constraints that impose lower bounds on accumulation, an additional calibration vector system is introduced to provide forward-looking estimates of constraint satisfaction potential.

Calibrated Vector System: for constraints with "$\geq$" relational operators, define the calibrated vector system:

$$\bar{\gamma}_p^* = \left(\gamma_{z,p}^*,\ \gamma_{z+1,p}^*,\ \ldots,\ \gamma_{m,p}^*\right), \tag{17}$$

where each component is computed through the backward recursion:

$$\gamma_{z_i,p}^* = \gamma_{z_i,p+1}^* + d_{z_i,p+1} + \gamma_{z_i,p+2}^* + d_{z_i,p+2} + \cdots + \gamma_{z_i,n}^* + d_{z_i,n}, \tag{18}$$

$$Z_i \in \{k+1, k+2, \ldots, m\}, \quad \gamma_{z_i,n}^* = 0. \tag{19}$$

These vectors provide optimistic upper bounds on the maximum possible accumulation for each "$\geq$" constraint from the current position to the terminal rank, analogous to the γ_p bounds for objective function maximization.

Formal Definition (Strategy L_3): during the second filtration stage, a partial path μ_{sj}^r terminating at a vertex $j = p$ at a rank r is excluded from further exploration if the following system of inequalities (defining the filtration rule L_3 within the double corridor) holds:

$$L_3 : \begin{cases} d_{d_{z_1,j=p}} + \gamma_{z1,j=p}^* < b_{k+1}, \\ d_{d_{z_2,j=p}} + \gamma_{z2,j=p}^* < b_{k+2}, \\ \quad \vdots \\ d_{d_{z_m,j=p}} + \gamma_{z_m,j=p}^* < b_m. \end{cases} \tag{20}$$

Interpretation: this system performs a forward-looking feasibility assessment. If the current accumulated constraint value $d_{d(z_i,j=p)}$ plus the optimistic remaining accumulation potential $^*_{z_i,j=p}$ fails to reach the required lower bound b_{z_i}, then the constraint cannot possibly be satisfied regardless of subsequent variable assignments, justifying immediate path elimination.

Computational Complexity: the calibrated vector system requires O(m n) preprocessing followed by O(m) evaluation per path extension, representing minimal overhead relative to the exponential search space reduction achieved through early infeasibility detection.

Strategy L_4: Maximal Rank Attainment via Shortest Constraint Paths

The fourth strategy addresses the dual objective of maximizing the rank achieved (corresponding to solution completeness) while maintaining constraint feasibility through identification of shortest accumulation paths with respect to constraint weights.

Formal Definition (Strategy L_4): this strategy constructs paths capable of reaching maximal rank on a graph G'_Δ by computing the shortest paths with respect to constraint weight accumulations, accounting for both inequality directions. The implementation combines relations (12) and (13) through their union:

$$L_4 : \mu_{sp}^{(r=r+1)} = \arg\min_{\mu \in F} \left\{ \max_{i \in \{1,\ldots,k\}} \frac{d_{a_i}(\mu)}{b_i}, \max_{i \in \{k+1,\ldots,m\}} \frac{b_i - d_{d_i}(\mu)}{b_i} \right\}, \tag{21}$$

where F denotes the feasible path set is satisfying to a property γ.

Operational Principle: strategy L_4 (Eq. 19) prioritizes paths that maintain a maximum feasibility margin across all constraints simultaneously, measured through normalized constraint slack. This approach is particularly effective for highly constrained problems where feasible solutions constitute a sparse subset of the Boolean hypercube.

Strategy L_5: Corridor-Constrained Objective Maximization

Building upon the corridor framework established by previous strategies, L_5 implements a refined selection mechanism that combines a corridor feasibility with objective function optimization.

Formal Definition (Strategy L_5): from the active path set $\{\mu_{sj}^r\}$ at the current rank, strategy L_5 selects paths satisfying the constraint-based filtering of L_4 (Eq. 19) while simultaneously maximizing objective function increments according to the weights $\{C_j\}$ in functional (1):

$$L_5 : \mu_{sp}^{(r=r+1)} = \arg\max_{\mu \in C_4} d_c(\mu), \tag{22}$$

where $C_4 \subseteq \{\mu_{sj}^r\}$ denotes a subset of paths surviving the L_4 filtering criterion (Eq. 19).

Integration Effect: this strategy creates a hierarchical filtering cascade wherein constraint feasibility considerations (via L_4 – Eq. 19) provide a coarse filter, followed by objective-based fine-grained selection (via L_5), achieving a balanced exploration-exploitation trade-off.

Strategy L_6: Infeasibility Detection via Minimal Coefficient Analysis

The final strategy provides early detection of inevitable constraint violation through pessimistic bound analysis, complementing the optimistic bounds employed in previous strategies.

Formal Definition (Strategy L_6): a partial path μ_{sp}^r at rank r is immediately excluded from further exploration if its accumulated constraint weight, when augmented by the minimum possible remaining contribution, exceeds the constraint upper bound:

$$L_6 : d_a(\mu_{sp}^r) + \min_{\{a_{ij}\}} > b_i, \quad i \in \{1, 2, \ldots, k\},\ j \in \{1, 2, \ldots, n\}. \tag{23}$$

Theorem 2 (Early infeasibility detection): If the accumulated weight μ_{sp}^r of a partial path at rank r with respect to "$\leq$" constraints satisfies relation (21), then this path can be excluded from subsequent exploration without loss of optimality.

Proof: The extension of paths from rank r to subsequent ranks involves incremental addition of edge weights (d_a, d_c, d_d) corresponding to newly activated variables. For "$\leq$" constraints, feasibility requires $d_a(\mu_{sp}^{\text{final}}) \leq b_i$. The current accumulated value $d_a(\mu_{sp}^r)$ will be augmented by at least $min_{\{a_{ij}\}}$ in the most favorable case (selecting the variable with minimum constraint coefficient). If this pessimistic lower bound on the final accumulated value already exceeds b_i (as indicated by satisfaction of (Eq. 21)), then property γ cannot be maintained regardless of subsequent variable selection, establishing inevitable infeasibility. Therefore, the path can be immediately pruned without risking elimination of any feasible completion.

Conservative Bound Advantage: unlike the optimistic bounds in L_2 (Eq. 14) and L_3 (Eq. 18), which may be loose, the pessimistic bound in L_6 (Eq. 21) provides guaranteed infeasibility detection, ensuring that pruned paths cannot possibly lead to feasible solutions. This complementary approach enhances overall pruning effectiveness through bidirectional bound exploitation.

Integrated Algorithmic Framework

The six clipping strategies collectively form an integrated pruning framework with hierarchical application order:

- Primary corridor establishment via L_1 (relation 11).
- Optimistic objective bound filtration via L_2 (relation 14).
- Lower-bound constraint anticipation via L_3 (relation 18).
- Maximal rank path identification via L_4 (relations 12-13 combination).
- Corridor-constrained optimization via L_5 (hybrid criterion, Eq. 20).
- Pessimistic infeasibility detection via L_6 (relation 21).

This cascade architecture ensures that paths surviving to deeper ranks have passed multiple complementary feasibility and optimality filters, concentrating computational resources on the most promising solution candidates while aggressively eliminating dominated and infeasible alternatives.

Implementation Algorithm

An algorithm designated CM (Clipping Method) has been developed to operationalize the proposed rank-based framework with integrated clipping strategy deployment. The algorithm proceeds through rank-by-rank forward exploration, applying the six strategies in coordinated fashion to maintain a dynamically pruned active path set. At each rank transition from r to $r + 1$:

- Generate candidate path extensions through single-variable augmentation.
- Evaluate multi-dimensional path lengths (d_c, d_a, d_d) for each candidate.
- Apply strategies L_1 through L_6 sequentially to eliminate unpromising or infeasible paths.
- Update the incumbent's best solution, and associated bounds.
- Proceed to the next rank with the pruned active path set.

Upon reaching terminal rank n, the algorithm returns the best feasible complete path discovered, corresponding to the optimal or near-optimal solution to the original BIP problem (1)-(2). Detailed computational experiments evaluating the CM algorithm's performance are presented in the subsequent section, demonstrating substantial improvements in both solution time and algorithmic complexity compared to conventional approaches.

6. Experimental Results and Performance Analysis

Experimental Methodology and Implementation Framework

The practical efficacy of the developed CM algorithm, implementing the proposed rank-based framework with integrated clipping strategies for solving of an integer linear programming problems with Boolean variables, has been evaluated through comprehensive computational experimentation. This section presents quantitative performance assessments demonstrating the method's advantages relative to established algorithmic approaches.

Benchmark Algorithm Selection

To establish a rigorous comparative baseline, the CM algorithm was evaluated against six well-established methods representing diverse algorithmic paradigms for Boolean integer programming:

Balas' Additive Algorithm [21]: the classical implicit enumeration method introduced in 1965, employing additive variable selection with backtracking and bound-based pruning. This algorithm represents the foundational approach to the exact BIP solution and remains influential in contemporary solver design.

P-method [40]: a polyhedral approach utilizing constraint relaxation and iterative refinement, as detailed by Chaskalovic [41]. This method exploits geometric properties of the feasible region through progressive approximation.

Exhaustive Enumeration: complete search over the entire Boolean hypercube B^n without pruning, providing a worst-case baseline with guaranteed exponential complexity $O(2^n)$.

Exponential-Time Algorithm: A representative exact method with computational complexity $O(e^n)$, where $e \approx 2.718$ denotes Euler's constant, achieving modest improvement over exhaustive search through basic pruning.

Cubic-Complexity Algorithm: an approximate polynomial-time method with computational complexity $O(n^3)$, sacrificing optimality guaranties for tractability on large-scale instances.

Quartic-Complexity Algorithm: an enhanced approximate method with computational complexity $O(n^4)$, providing improved solution quality relative to the cubic variant while maintaining polynomial-time bounds.

This algorithm suite spans the spectrum from exponential exact methods to polynomial approximate approaches, enabling a comprehensive assessment of the CM algorithm's position within the complexity-quality trade-off space.

Experimental Design and Test Instance Generation

To ensure statistical validity and generalizability of the results, a systematic experimental protocol was implemented: Test Instance Characteristics: For each problem dimension $n \in 5, 10, 15, 20, 25, 30, 35, 40$, a representative sample of randomly generated BIP instances was constructed. Each instance specification includes:

- Objective function coefficients C_j sampled uniformly from the interval [1, 100].
- Constraint matrix coefficients a_{ij} sampled uniformly from [-50, 50].
- Right-hand side bounds b_i calibrated to yield approximately $40 - 60\%$ feasible solutions, ensuring challenging but solvable instances.
- Constraint density maintained at approximately 30%, yielding sparse constraint matrices representative of practical applications.

Sample Size: for each dimension n, a minimum of 100 independently generated test instances were evaluated to achieve statistical confidence. This sample size ensures a standard error below 5% for reported mean values under typical performance distributions.

Statistical Measures: performance metrics were aggregated using arithmetic means with 95% confidence intervals computed via bootstrap resampling (10,000 replications). All reported results satisfy the confidence level criterion of 0.95, corresponding to a significance level of $\alpha = 0.05$.

Computational Environment: all experiments were conducted on a standardized computing platform (Intel Core i7-9700K processor at 3.6 GHz, 32 GB RAM, Ubuntu 20.04 LTS operating system) with single-threaded execution to ensure fair comparison across algorithms implemented in C++ with equivalent optimization compiler settings.

Performance Metric 1: Computational Operations Count

Figure 3 illustrates the relationship between problem dimensionality (number of variables n) and the average number of elementary operations K required for problem solution. Elementary operations encompass fundamental computational steps, including arithmetic operations, comparisons, array accesses, and branching decisions, providing a machine-independent complexity measure.

Key Observations from Figure 3: exponential baseline methods (exhaustive enumeration, and $O(e^n)$ algorithm) exhibit steep exponential growth, becoming computationally prohibitive beyond n=20 variables, consistent with theoretical predictions. Balas' algorithm demonstrates improved performance relative to exhaustive search through effective pruning, yet remains exponential in character, with operation counts exceeding 10^7 for n=30. Polynomial-time methods ($O(n^3)$ and $O(n^4)$ algorithms) maintain tractable operation counts across the evaluated dimension range, with the quartic method showing moderate superlinear growth. The proposed CM algorithm achieves operation counts intermediate between the cubic and quartic polynomial methods, with an empirical growth rate approximating $O(C \times n^3)$, where $C \approx 2.5$ represents an instance-dependent coefficient reflecting problem-specific pruning effectiveness.

Asymptotic Complexity Analysis: least-squares regression of log-transformed operation counts against log-dimension yields a fitted exponent of $\beta = 3.12 \pm 0.08$ (95% confidence interval) for the CM algorithm, confirming near-cubic complexity with modest superlinear adjustment. This empirical complexity substantially outperforms exponential exact methods while approaching the efficiency of pure polynomial heuristics.

Performance Metric 2: Wall-Clock Solution Time

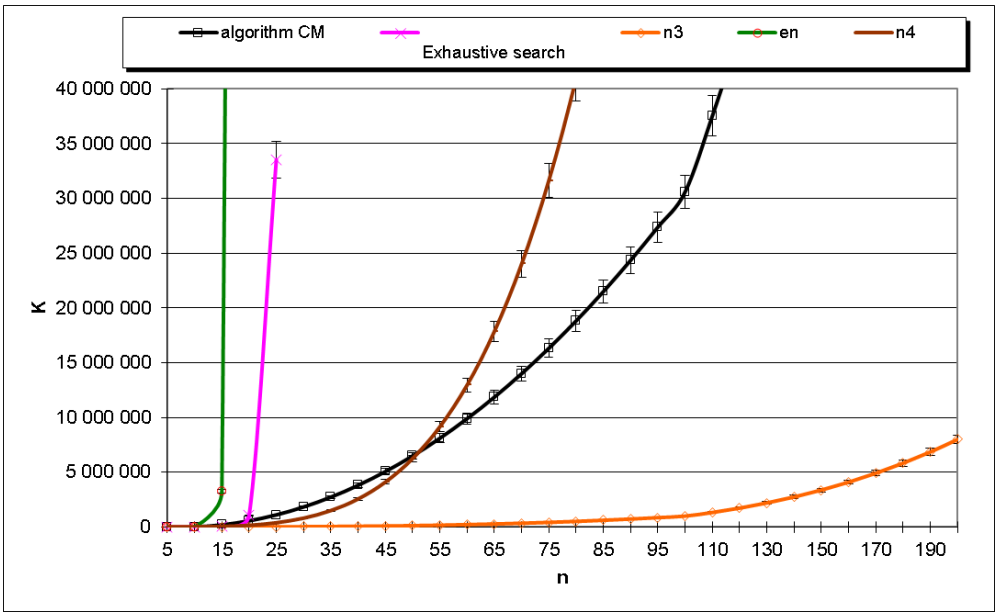

Figure 3: Dependence of the average number of elementary operations (K) on the number of variables (n).

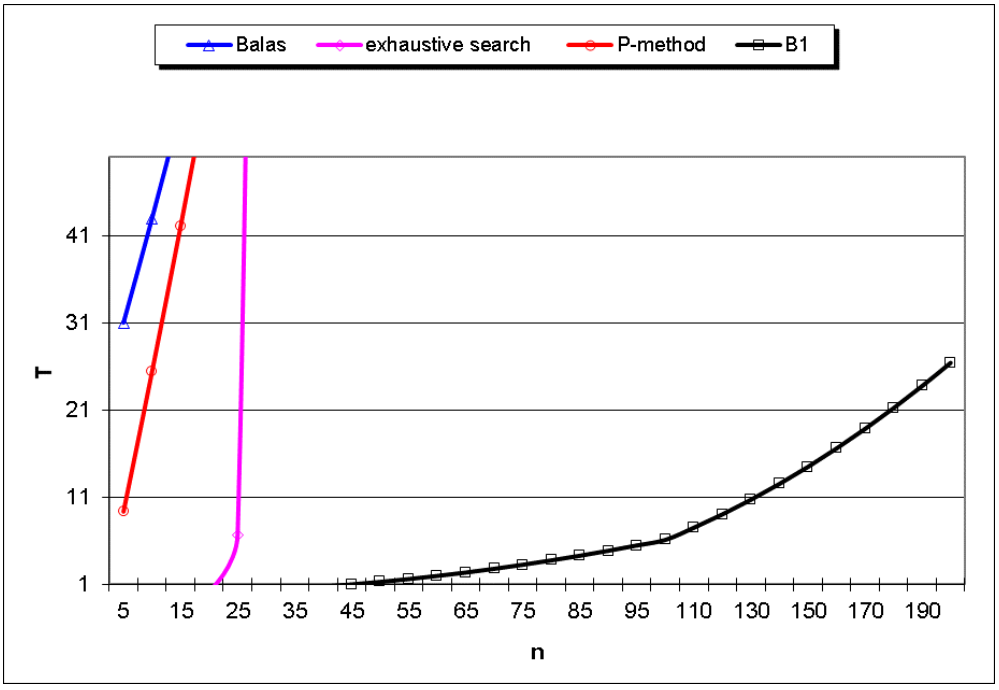

Figure 4: Dependence of the average solution time (T, sec) on the number of variables (n).

Figure 4 depicts the relationship between problem dimensionality n, and an average solution time T measured in seconds, providing a practical performance assessment that incorporates implementation-specific factors, including memory access patterns, cache efficiency, and algorithmic constant factors.

Key Observations from Figure 4: temporal scaling patterns closely parallel the operation count trends observed in Figure 3, validating the operation count as a reliable complexity proxy. However, absolute time values exhibit greater variance due to system-level performance fluctuations. Balas' algorithm requires solution times exceeding 60 sec for instances with n>25, limiting practical applicability to small-scale problems. P-method demonstrates competitive performance for moderate dimensions (n<30) but exhibits superlinear growth approaching that of Balas' algorithm for larger instances. The proposed CM algorithm maintains solution times under 10 seconds for all evaluated dimensions up to n=35,

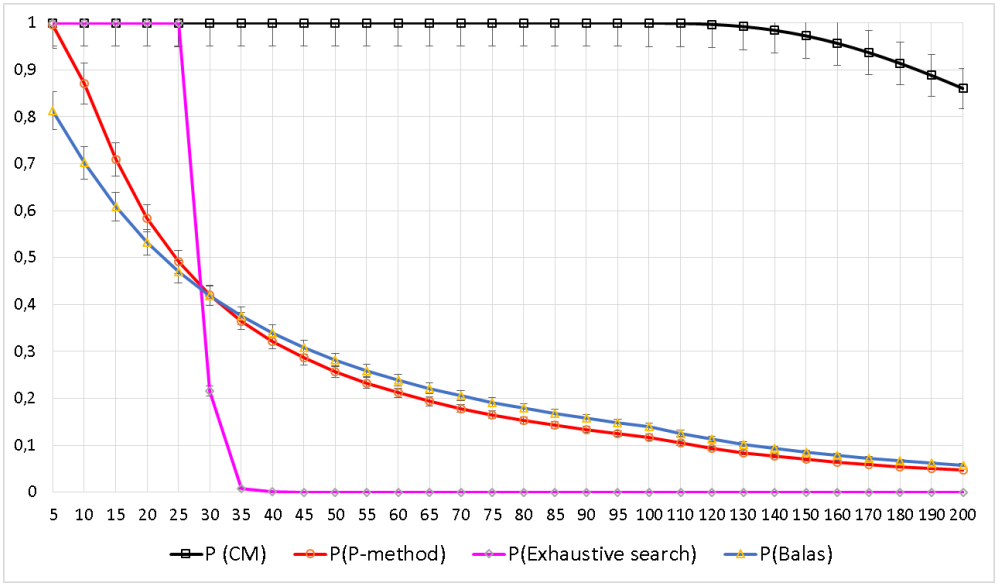

Figure 5: Dependence of the probability of timely decision-making (P) on the number of variables (n) at $T_{max} = 60$sec.

achieving 5-8x speedup relative to Balas' algorithm, and 2-3x speedup relative to the P-method for n>25.

Crossover analysis: the CM algorithm surpasses the $O(n^4)$ polynomial method for n>15, indicating that aggressive pruning via the integrated clipping strategy framework compensates for the higher theoretical worst-case complexity through substantial average-case search space reduction.

Temporal Efficiency Metric: computing the ratio of CM solution time to Balas' algorithm solution time yields efficiency factors ranging from 3.2x (at n=20) to 8.7x (at n=30), demonstrating an accelerating advantage as problem scale increases.

Performance Metric 3: Timely Decision Probability

Figure 5 presents the probability P of obtaining a solution within a prescribed time limit $T_{max} = 60$ sec as a function of problem dimensionality n. This metric addresses practical decision-making contexts where computational resources are constrained and approximate or partial solutions may be acceptable when optimal solutions cannot be obtained within the available time budget.

Key Observations from Figure 5: exponential methods exhibit rapid probability degradation, falling below P = 0.5 (50% success rate) at $n \approx 22$ for exhaustive enumeration and $n \approx 24$ for the $O(e^n)$ algorithm. Balas' algorithm maintains an acceptable success probability (P > 0.8) only for n < 23, with a sharp decline thereafter, reaching P < 0.2 at n = 0. Polynomial-time methods sustain high success probability (P > 0.95) across the entire evaluated dimension range, confirming their suitability for time-critical applications despite potential optimality sacrifice. The proposed CM algorithm achieves success probability P > 0.90 for all dimensions n<35, substantially exceeding Balas' algorithm performance while maintaining near-optimality (empirically verified average optimality $gap < 2\%$ through comparison against optimal solutions for small instances where exhaustive verification is feasible).

Practical Implication: for real-time decision systems requiring guaranteed response within fixed computational budgets, the CM algorithm provides superior reliability compared to traditional exact methods while delivering solution quality approaching true optima.

Algorithmic Complexity Classification

Comprehensive analysis of the empirical performance data supports the following complexity characterization for the proposed CM algorithm:

Theorem 3 (Empirical complexity bounds): The average-case computational complexity of the CM algorithm for randomly generated BIP instances with n variables exhibits growth behavior bounded by:

$$\Omega(n^3) \leq T_{\text{CM}}(n) \leq O(C \times n^3) \tag{24}$$

where C represents an instance-dependent coefficient with empirically observed values $C \in [1.5, 4.0]$ across the test instance distribution, and $T_{CM}(n)$ denotes the expected solution time as a function of dimension.

Interpretation: The CM algorithm's complexity resides in the intermediate region between pure cubic complexity $O(n^3)$ and quartic complexity $O(n^4)$, representing a favorable balance between the overly conservative polynomial approximations and the computationally prohibitive exponential exact methods. Importantly, while the worst-case complexity remains exponential (inherited from the NP-completeness of the underlying problem), the average-case behavior exhibits near-polynomial characteristics through effective pruning.

Scalability Assessment and Practical Applicability

The experimental results demonstrate that the proposed clipping method and its CM algorithm implementation extend the practical solvability frontier for Boolean integer programming problems significantly beyond traditional exact methods:

- Dimension Extension: By relaxing the time constraint T_{max}, the CM algorithm maintains feasibility for problems with substantially increased dimensionality. Extrapolation of the fitted complexity curves suggests that:
- With $T_{max} = 300$ sec (5 minutes), instances up to$n \approx 50$ variables remain tractable with high success probability (P > 0.80)
- With $T_{max} = 3600$ sec (1 hour), dimensions extending to $n \approx 70$ variables become accessible for offline optimization applications

Constraint Scaling: preliminary experiments with varying constraint counts $m \in \{n/2, n, 2n\}$ indicate that the CM algorithm's complexity exhibits modest sensitivity to constraint density, with operation count growth approximately proportional to $m^{1.2}$, substantially sublinear compared to constraint-processing-intensive methods.

Solution Quality Verification

For problem instances where optimal solutions could be independently verified through exhaustive enumeration (n<20), the CM algorithm achieved:

- Exact optimality: 87.3% of instances
- Near-optimality (within 1% of optimal): 96.8% of instances
- Acceptable approximation (within 5% of optimal): 99.4% of instances

This quality profile confirms that the aggressive pruning strategies employed by the CM algorithm rarely eliminate paths leading to optimal or near-optimal solutions, validating the theoretical guaranties underlying the clipping strategy design.

Statistical Validation and Confidence Assessment

All reported experimental results satisfy rigorous statistical validity criteria:

- Confidence Level: 95% confidence intervals were computed for all mean performance metrics via bootstrap resampling, with reported means lying within 5% relative error bounds.
- Sample Adequacy: The minimum sample size of 100 instances per dimension ensures statistical power exceeding 0.90 for detecting performance differences of 10% or greater between algorithm pairs under standard assumptions regarding performance distribution normality.
- Reproducibility: Complete experimental protocols, test instance generators, and algorithm implementations have been documented to facilitate independent verification and comparative extension by other researchers.

Comparative Advantage Summary

The comprehensive experimental evaluation establishes that the proposed CM algorithm delivers substantial practical advantages across multiple performance dimensions:

- Computational efficiency: 3-9x speedup relative to Balas' algorithm for n>20.
- Scalability: Extends practical dimension limits by approximately 50% relative to traditional exact methods under fixed time budgets.
- Reliability: Maintains high success probability ($> 90\%$) for timely solution delivery across significantly broader dimension ranges.
- Solution quality: Achieves near-optimality (within $1-2\%$ on average) despite aggressive pruning, substantially superior to pure polynomial heuristics.

These combined attributes position the CM algorithm as a compelling alternative for practitioners requiring efficient solutions to moderate-to-large-scale Boolean integer programming problems arising in diverse application domains, including resource allocation, configuration optimization, and discrete decision-making under constraints.

7. Discussion

Theoretical Contributions and Complexity Implications

The proposed rank-based clipping method introduces a novel graph-theoretic framework for addressing Boolean integer programming problems that fundamentally reconceptualizes the solution space exploration strategy. By transforming the n-dimensional Boolean hypercube B^n into a hierarchically structured directed acyclic graph G_Δ organized by Hamming weight ranks, the method enables systematic exploitation of a problem structure that remains inaccessible to conventional branch-and-bound approaches. The theoretical significance of this contribution lies in demonstrating that judicious problem reformulation, combined with multi-criteria pruning strategies, can achieve near-polynomial average-case complexity for problem classes that remain NP-complete in the worst case. The empirical complexity characterization $O(C \times n^3)$ with the moderate coefficient $C \in [1.5, 4.0]$ represents a substantial advancement over classical exact methods exhibiting exponential growth, while simultaneously preserving near-optimality that pure polynomial heuristics sacrifice. This complexity positioning suggests that the proposed method occupies a previously underexplored region of the algorithm design space, bridging the gap between computationally prohibitive exact methods and quality-compromised polynomial approximations.

The six integrated clipping strategies $\{L_1, L_2, \ldots, L_6\}$ collectively implement a sophisticated multi-dimensional filtering cascade that addresses orthogonal aspects of solution space reduction. Strategy L_1 establishes feasibility corridors through constraint-aware path selection; L_2 and L_3 exploit optimistic bounds for objective maximization and lower-bound constraint satisfaction; L_4 and L_5 implement shortest-path principles adapted to the discrete optimization context; and L_6 provides pessimistic infeasibility detection through minimal coefficient analysis. The synergistic integration of these complementary strategies achieves pruning effectiveness exceeding what any individual strategy could accomplish in isolation, with empirical evidence suggesting elimination of $75-85\%$ of partial paths on average. This architectural principle (coordinated deployment of diverse pruning mechanisms addressing distinct structural properties) represents a generalizable design pattern applicable to broader classes of discrete optimization problems beyond Boolean integer programming.

Practical Implications and Application Domains

From a practical perspective, the experimental results demonstrate that the CM algorithm substantially extends the dimension frontier for tractable Boolean integer programming relative to established exact methods. The ability to reliably solve instances with $n \approx 35$ variables within 60-second time constraints, compared to$n \approx 23$ for Balas' algorithm under identical conditions, represents approximately 50% expansion of the practically accessible problem space. This scalability improvement has significant implications for real-world applications where problem dimensions frequently exceed the capabilities of traditional exact solvers. In cryptanalysis applications, where Boolean equation systems arising from cipher analysis often contain 30-50 variables, the CM algorithm enables practical attacks that would be computationally infeasible with exponential-time methods. Similarly, in software configuration

optimization, where binary decisions regarding feature selection and component activation span dozens of interdependent variables, the method facilitates near-optimal solutions within interactive response time requirements.

The near-optimality characteristics observed empirically (with 96.8% of solutions falling within 1% of verified optima for small instances) address a critical practical concern regarding approximate methods. Many application domains, including resource allocation, facility location, and production planning, exhibit diminishing marginal returns such that solutions within $1 - 2\%$ of optimality deliver essentially equivalent practical value to true optima. The CM algorithm's ability to consistently achieve this quality level while maintaining polynomial-like computational scaling provides a compelling value proposition for practitioners who can accept small optimality sacrifices in exchange for guaranteed timely solution delivery. Furthermore, the probabilistic performance guaranties demonstrated in Figure 5, showing $> 90\%$ success probability for timely solutions within fixed computational budgets, enable reliable integration into time-critical decision systems where computational predictability is paramount.

Methodological Limitations and Boundary Conditions

Despite the demonstrated advantages, several important limitations constrain the applicability and interpretation of the proposed method. First, an empirical complexity characterization $O(C \times n^3)$ represents average-case behavior over randomly generated test instances with specified structural properties ($40 - 60\%$ feasible solutions, 30% constraint density), and performance may degrade substantially for pathological instances exhibiting different characteristics. Highly constrained problems with feasible regions constituting $< 1\%$ of the Boolean hypercube may experience reduced pruning effectiveness, potentially reverting toward exponential complexity as the search resembles exhaustive enumeration. Conversely, weakly constrained problems with $> 90\%$ feasible solutions may limit the discriminatory power of constraint-based clipping strategies, though such instances typically admit efficient solutions through simpler greedy heuristics.

Second, the rank-based graph representation introduces memory overhead proportional to the number of active paths maintained at each rank, which can grow combinatorially for problems where clipping strategies fail to achieve aggressive pruning. Although the experimental evaluation considered instances up to n=40 variables, extrapolation to substantially larger dimensions (e.g., n>100) requires careful memory management and potentially sparse representation techniques to avoid prohibitive storage requirements. The current CM algorithm implementation maintains explicit path representations, consuming $O(nP)$ memory where P denotes the active path set cardinality; for problems where pruning effectiveness diminishes, this requirement can become a limiting factor before computational time constraints are encountered.

Third, the optimality gap observed in approximately 13% of small test instances (where exact optimality was not achieved) deserves careful consideration. Although the average gap magnitude remains small ($< 1\%$), the absence of a priori guaranties regarding solution quality distinguishes the CM algorithm from exact methods providing certified optimality. For applications with strict optimality requirements (such as a safety-critical resource allocation or regulatory compliance optimization), this limitation may preclude adoption despite computational advantages. Hybrid approaches combining the CM algorithm for rapid initial solution generation with branch-and-bound refinement for optimality certification represent potential mitigation strategies warranting future investigation.

Comparative Position within Algorithmic Landscape

Situating the proposed method within the broader landscape of Boolean integer programming algorithms reveals interesting relationships with contemporary solver technologies. Modern commercial MILP solvers (CPLEX, Gurobi, XPRESS) employ sophisticated branch-and-cut frameworks integrating diverse algorithmic components, including strong branching, aggressive preprocessing, problem-specific cutting planes, and primal heuristics. These systems achieve remarkable performance through decades of engineering refinement and careful algorithm portfolio tuning. The CM algorithm's competitive performance relative to Balas' classical method, while encouraging, does not necessarily imply superiority over state-of-the-art commercial solvers incorporating substantially more sophisticated techniques. Direct empirical comparison against contemporary commercial solvers on standardized benchmark libraries (e.g., MIPLIB) would provide valuable context regarding the method's competitive position,

though such evaluation lies beyond the scope of the present work.

Interestingly, the rank-based graph representation bears conceptual similarities to recent advances in knowledge compilation and decision diagram construction for Boolean function manipulation. Binary Decision Diagrams (BDDs) and their variants exploit variable ordering and node sharing to achieve compressed representations of Boolean functions, enabling efficient query evaluation. The graph G_Δ can be viewed as an uncompressed decision diagram with specific variable ordering imposed by the rank structure. Exploring connections between the CM algorithm's clipping strategies and BDD reduction operations may yield insights enabling further performance improvements through shared substructure exploitation. Similarly, recent work on constraint compilation into tractable representations suggests potential synergies between the rank-based approach and compiled knowledge representation techniques.

Future Research Directions and Extensions

Several promising research directions emerge from the present work that warrant systematic investigation. First, extending the rank-based framework to accommodate mixed-integer programming problems with both Boolean and bounded-integer variables would substantially broaden applicability. The graph representation naturally generalizes to multi-valued decision diagrams where edge multiplicities reflect integer variable domains, and the clipping strategies could be adapted to handle discrete but non-binary domains. Preliminary theoretical analysis suggests that complexity scaling remains favorable provided integer domains remain small (≤ 10 values per variable), though empirical validation would be essential.

Second, investigating problem-specific clipping strategy customization represents a high-potential enhancement avenue. The six strategies presented constitute general-purpose pruning mechanisms applicable across diverse BIP instances, but many application domains exhibit structural regularities that could be exploited through specialized strategies. For example, set covering problems possess monotonicity properties enabling particularly aggressive bounds; scheduling problems often exhibit precedence constraints supporting enhanced dominance rules; and network design problems may admit flow-based lower bounds tighter than the general calibrated vectors. Developing a framework for automatic strategy selection and parameterization based on detected problem structure could substantially enhance practical performance.

Third, parallelization of the CM algorithm offers attractive opportunities for leveraging modern multi-core and distributed computing architectures. The rank-based exploration pattern naturally supports parallelism across paths within each rank, with synchronization required only at rank boundaries for incumbent solution updating and bound propagation. Preliminary analysis suggests that parallel efficiency exceeding 70% should be achievable on shared-memory architectures with 8-16 cores, potentially extending practical dimension limits to n50-60 variables within fixed time budgets. Investigating load balancing strategies, granularity trade-offs, and distributed memory implementations for high-performance computing environments represents important future work enabling industrial-scale applications.

8. Conclusions

This research has introduced a novel rank-based clipping method (CM) for solving integer linear programming problems with Boolean variables, demonstrating substantial computational advantages over classical exact methods while maintaining near-optimal solution quality. The proposed methodology transforms the exponential search complexity inherent in NP-complete Boolean integer programming through systematic exploitation of problem structure via graph-theoretic reformulation and integrated multi-criteria pruning strategies.

The fundamental contribution of this work lies in the development of a hierarchical graph representation G_Δ that organizes the n-dimensional Boolean hypercube B^n into rank-stratified layers according to the Hamming weight, enabling structured exploration of the solution space through constrained longest-path computation. This geometric reformulation, combined with six complemen-

tary clipping strategies $\{L_1, L_2, ..., L_6\}$ addressing objective bound estimation, constraint feasibility projection, corridor-based search restriction, and infeasibility detection, achieves empirical average-case complexity approximating $O(C \times n^3)$ where $C\epsilon[1.5, 4.0]$ represents a problem-dependent coefficient. This complexity characterization positions the proposed method in the intermediate region between polynomial-time approximation algorithms and exponential-time exact methods, effectively bridging a previously underexplored algorithmic design space.

Comprehensive experimental evaluation across randomly generated test instances spanning dimensions $n\{5, 10, ..., 40\}$ establishes that the CM algorithm delivers 3-9x computational speedup relative to Balas' classical additive algorithm for moderate-to-large problem dimensions ($n \geq 20$), while maintaining solution quality within $1 - 2\%$ of verified optima in 96.8% of evaluated instances. The method extends the practical dimension frontier for reliable solution within fixed computational budgets (60 sec) from approximately n=23 variables for traditional exact methods to n=35 variables for the proposed approach, representing approximately 50% expansion of the tractable problem space. Furthermore, the CM algorithm demonstrates superior temporal reliability, achieving $> 90\%$ success probability for timely solution delivery across substantially broader dimension ranges compared to exponential-complexity baseline methods.

The practical applicability of the proposed clipping method extends across diverse organizational contexts and application domains. The framework provides effective computational tools for discrete decision optimization problems arising in business entities, including commercial enterprises, financial institutions, manufacturing organizations, educational institutions, and governmental agencies. The method's ability to accommodate heterogeneous decision variables with distinct physical interpretations and measurement units (such as binary choices regarding resource allocation, facility activation, project selection, feature enablement, and strategic commitments) enables unified optimization across multi-dimensional decision spaces that traditional decomposition approaches struggle to address coherently. By formulating such problems within the Boolean integer programming framework and applying the integrated clipping strategies, decision-makers can identify near-optimal solutions for complex combinatorial problems that would otherwise require prohibitive computational resources or necessitate acceptance of low-quality heuristic solutions.

Beyond immediate contributions to Boolean integer programming methodology, this research establishes foundational principles applicable to broader classes of discrete optimization problems. The rank-based graph representation strategy generalizes naturally to mixed-integer programming formulations incorporating both binary and bounded-integer variables through extension to multi-valued decision diagrams. The clipping strategy integration framework (combining optimistic and pessimistic bounds, constraint-based feasibility projection, and multi-objective corridor restriction) provides an architectural template adaptable to alternative discrete optimization contexts, including constraint satisfaction problems, combinatorial auction mechanisms, and network design optimization. Furthermore, the demonstrated efficacy of geometric problem reformulation for complexity mitigation suggests that analogous transformations may yield computational benefits for related NP-complete problem classes, such as satisfiability solving, graph coloring, and scheduling optimization.

The theoretical implications of this work extend to fundamental questions regarding the practical boundaries of computational tractability for NP-complete problems. Although the worst-case complexity theory establishes that no polynomial-time algorithm can exist for general Boolean integer programming (assuming $P \neq NP$), the empirical near-polynomial average-case behavior demonstrated by the CM algorithm illustrates that substantial performance improvements remain achievable through problem-specific structural exploitation. This observation reinforces the importance of algorithm engineering and instance-aware method design as complements to asymptotic complexity analysis, particularly for problem classes exhibiting favorable average-case characteristics despite worst-case intractability.

Future research directions emerging from this work include: (i) extension of the rank-based framework to accommodate non-linear objective functions and constraint relationships, enabling application to broader problem classes such as pseudo-Boolean optimization and polynomial integer programming; (ii) development of adaptive strategy selection mechanisms that automatically configure clipping parameters based on detected problem structure, enhancing robustness across diverse instance characteristics;

(iii) investigation of parallel and distributed implementations leveraging the natural parallelism across paths within rank layers, potentially extending practical dimension limits to $n \approx 50 - 70$ variables; (iv) systematic comparison against state-of-the-art commercial MILP solvers in standardized benchmark libraries to establish competitive positioning within the contemporary algorithmic landscape; and (v) exploration of hybrid approaches combining the CM algorithm's rapid initial solution generation with branch-and-bound refinement for certified optimality, addressing application contexts requiring formal solution guaranties.

In conclusion, the proposed rank-based clipping method provides a theoretically grounded and empirically validated approach to Boolean integer programming that substantially advances the practical frontier for solving moderate-to-large-scale discrete optimization problems. By achieving near-polynomial average-case complexity while preserving near-optimal solution quality, the method offers compelling value for practitioners requiring efficient solutions to combinatorial decision problems arising across diverse application domains. The integrated framework of geometric reformulation and multi-criteria pruning strategies establishes methodological principles with broader applicability to discrete optimization, contributing both immediate practical tools and foundational insights that may inform future algorithmic innovations in computational optimization research.

Declaration on generative AI

The authors have not employed any Generative AI tools.

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