A Qualitative Logic for Uncertain Evidence and Belief Comparison*

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Abstract

We introduce a qualitative logic for comparing strengths of belief and evidential support explicitly, discerning these two comparative notions both syntactically and semantically within a modal logical framework. More precisely, we employ Dempster-Shafer theory (DST) of belief functions to represent uncertain, possibly mutually inconsistent, and incomplete evidence, as well as evidence-based degrees of beliefs. We propose a bi-modal logic that compares propositions in two ways: (1) based on the strengths of belief an evidence-possessing agent has in them and (2) based on the degrees of certainty of the evidence supporting them. (2) is the novel component of the proposed logic, designed to capture a notion of *certainty-dominance* among sets of evidence, modeled via an Egli-Milner-like order lifting on individual pieces of evidence. We justify this modeling choice, provide key (in)validities of our logic, and establish links to existing modal logics of evidence and belief (functions).

Keywords

Uncertain evidence, Logic of evidence and belief, Dempster-Shafer Theory

1. Introduction

We constantly receive information from and about the world around us via various news channels, by observing our environment, taking measurements, and communicating with each other. Artificial agents, such as self-driving cars and smart surveillance systems, gather evidence from their environment via sensors that recognize the objects and movements in their surroundings. The effect of an agent's evidence on their beliefs can hardly be overestimated. Rational agents, artificial or human, are expected to merge the evidence they gather from various sources in "right ways" so that they can form *evidentially grounded, justified,* and *consistent* (degrees of) beliefs that guide their decisions and actions.

Interestingly though, depending on factors such as how an agent merges evidence and how uncertain, inconsistent or incomplete their body of evidence is, the agent's *direct, raw* evidence might provide a level of evidential support that diverges from their degrees of belief based on the combined evidence. That is, while the former notion might render, e.g., P evidentially more supported than Q based on a measure of evidential support derived from an agent's individual pieces of evidence, the latter might set the degree of belief in Q higher than that in P. To make this claim more vivid, consider a health institution that is preparing dietary recommendations for the prevention of a certain disease based on scientific studies. Some studies suggest that foods from group P are beneficial, others say they're harmful, and others conclude that foods from group Q are beneficial. Depending on how well the relevant study conditions replicate the real-world context, they end up having a set of highly certain evidence for P, highly certain evidence for $\neg P$, and less certain evidence for Q. In this case, a measure of evidential strength derived from the degrees of certainty of individual pieces of evidence supporting a proposition can suggests that P is more strongly supported than Q: after all, every piece of evidence that supports P is more certain that the ones supporting Q. However, given a measure of degrees of belief that also takes into account the mutual consistency of the evidence supporting a proposition, the institution might conclude that the food group Q should be recommended, since there is also highly

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certain evidence against P, but none against Q (akin to notions of belief formalized in (topological) evidence models [2, 3, 4, 5], and quantitative notions of belief based on uncertain evidence [6, 7, 8]).[1] In this paper, we explore how to model a comparative notion of evidential support based on a set of evidence with varying degrees of certainty, and how this relates to degrees of evidence-based belief, aiming to shed light on scenarios like the one described above.

We employ Dempster-Shafer theory (DST) of belief functions to represent and combine uncertain evidence and degrees of evidence-based belief. DST offers a quantitative framework that explicitly represents an agent's evidence and their uncertainties about this evidence, ways to combine uncertain evidence coming from various, independent sources via rules of evidence combination[2], and generates degrees of belief, represented by *belief functions*, based on possibly mutually inconsistent, incomplete, and uncertain evidence [6]. This rich framework provides at least two natural ways of comparing propositions with respect to the strengths of evidential support they receive: (1) one that compares resulting degrees of beliefs, as in [10]; and (2) another that compares propositions by directly comparing *sets* of evidence that supports them. While the former is a *cumulative* concept, measured by summing the degrees of certainty of pieces of evidence supporting a proposition; the other one is derived from comparing the degrees of certainty of the *individual* pieces of evidence in sets of evidence supporting propositions. Our goal is to define a bi-modal logic that compares propositions in these two ways. For (1), we simply use a Dempster-Shafer belief function, following, e.g., [10]. To the best of our knowledge, (2) is a novel component. To be further explained in Section 3, we define (2) following an Egli-Milner-like order-lifting of a total pre-order based on the degrees of certainty of the available pieces of evidence. This operator is intended to capture a notion of *certainty-domination*: it compares propositions based on the degrees of certainty of the most and least certain evidence supporting them.

In relation to other work, our logic extends the one in [10] by adding an explicit representation of evidence and a notion of evidence comparison in the semantics, and an evidence comparison operator in the syntax. Although there are many modal logics of belief functions, they represent degrees of belief in their semantics but lack an explicit representation of evidence [11, 12, 13, 14]. Our approach is also tightly linked to, and inspired by, logics for evidence-based belief interpreted on (topological) evidence models [2, 15, 5, 4, 3, 16, 17]. With the exception of [16, 17], these logics do not only have a belief modality but also evidence modalities in their syntax, making evidence (and its connection to belief) an explicit part of the logic. However, their evidence models formalize evidence purely qualitatively, so our models can be seen as a quantitative extension of them to represent uncertain evidence. An exception that extends evidence models of [2] with orders on sets of evidence are the justification models of [16, Ch. 4] and [17]. However, their order defined on bodies of weighted evidence differs from ours in many ways, e.g., in what it represents, how it is defined, and its formal properties[3]. Finally, another work that motivated the current project is our own [18], where we developed a so-called *multi-layer belief model* for measuring degrees of beliefs based on a body of possibly mutually inconsistent, incomplete, and uncertain evidence. This belief model combines DST and (the previously mentioned) topological models of evidence [4, 3]. In this sense, the proposal in this paper can also be seen as a first step toward developing comparative logics for the multi-layer belief model. The resulting logic of this paper also connects to and shares common (in)validities with many modal logics for comparing strengths of belief [19], convex order [20], preference [21], and justifications [17]. We will draw attention to such connections and common features throughout the paper.[4]

[1]We will return to a more detailed formalization of this example in Section 5.

[2]The first rule of evidence combination proposed within this framework is the so-called *Dempster rule of combination* (DRC) (see Definition 2) [9]. Since then many alternative evidence combination rules have been proposed. We refer the reader to [8] for a comprehensive overview of these evidence combination rules. In this paper we focus on DRC.

[3]To name one main difference, our evidence comparison operator is derived from degrees of certainty of individual pieces of evidence via order lifting, rather than by summing up the weights of pieces of evidence in a set, as in [16, Ch. 4]. The latter is more akin to our belief comparison operator.

[4]There are other logics of evidence that represent belief based on evidence. For example, [22, 23] study many-valued logics of evidence based on Belnap's four-valued logic [24]. Many-valued logics of evidence have been linked to the evidence logics of [2] in [25] and to Dempster-Shafer theory in [26]. There are also works that define qualitative logics of models based on other uncertainty theories, such as possibility theory [27, 28, 29]. Comparison to these references is left for future work.

The paper is organized as follows. Section 2 provides the required preliminaries of the DST and introduces an alternative presentation in terms of the uniform evidence models of [2]. Section 3 introduces and motivate the way we order propositions wrt the sets of uncertain evidence supporting them, justifying our semantics for (2). Section 4 presents the syntax and semantics of the proposed logic and lists its important (in)validities. In Section 5 we further develop the motivating example described in the introduction to show the value of our proposal in more practical terms. Section 6 concludes with a discussion of ongoing and future work.

2. Preliminaries of Dempster-Shafer Theory of Belief Functions

Throughout the paper, we assume that S is a *non-empty* and *finite* state space that represents the universe of possible elements to be observed. (For brevity, we skip the mention of finiteness and non-emptyness of S.) We call any subset $P \subseteq S$ a *proposition* (also denoted by capital letters such as A, B). *Pieces of basic evidence*, representing evidence directly obtained via, e.g., observation, measurement, testimony, are also formalized as subsets of S. We will denote a set of (basic) pieces of evidence by $\boldsymbol{\mathcal{E}} \subseteq 2^S$.

A core part of the current study is to define an order on propositions that is derived from comparing the degrees of certainty of the *individual* pieces of evidence supporting them. To do so, we first define a notion of evidential support, following the terminology in [4]. Given a state space S, a set of pieces of evidence $\boldsymbol{\mathcal{E}} \subseteq 2^S$, $E \in \boldsymbol{\mathcal{E}}$, and a proposition $P \subseteq S$, we say that evidence E *supports* proposition P iff $E \subseteq P$. In this case, we say E is evidence for P. Finally, we define the *basic evidence set* $\boldsymbol{\mathcal{E}}_P$ *for* P as $\boldsymbol{\mathcal{E}}_P = \{E \in \boldsymbol{\mathcal{E}} : E \subseteq P\}$. The latter will be of essential use in Section 3.

In DST, uncertain evidence is modeled via basic belief assignments (BBAs) defined on the set of subsets of a finite state space S.

Definition 1 (Basic belief assignment (BBA))**.** Given state space S, a *basic belief assignment* over the set S is a function $m : 2^S \to [0, 1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq S} m(A) = 1$.

For any $A \subseteq S$, $m(A)$ represents the degree of certainty that the agent has about that piece of evidence. Additionally, $m(S)$ represents the degree of uncertainty of the evidence modeled by the basic belief assignment m. Given a basic belief assignment m over S, any $A \subseteq S$ with $m(A) > 0$ is called a *focal element*. If, in addition, $A \neq S$, it is called a *proper focal element*. In this paper we will employ simple BBAs to represent evidence (due to their relevance for evidence models of [2], explained at the end of this section): m is a *simple BBA* if and only if there exists $A \subset S$ such that $m(A) > 0$ and $m(S) = 1 - m(A)$. That is, a simple BBA is a belief assignment with a unique proper focal element. Moreover, m is called a non-dogmatic BBA if $m(S) > 0$. Given a collection of BBAs, we can define a *combined* BBA by applying the so-called *Dempster's rule of combination*.

Definition 2 (Dempster's rule of combination (DRC))**.** Let m_1 and m_2 be BBAs over the same state space S and $A_1, \ldots, A_k$ and $B_1, \ldots, B_\ell$ all subsets of S such that $m_1(A_i) \neq 0$ and $m_2(B_i) \neq 0$, respectively. Moreover, suppose that $\sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j) < 1$. Then the following BBA m, also denoted by $m_1 \oplus m_2$, is the result of applying *Dempster's rule of combination* to m_1 and m_2: $m(\emptyset) = 0$ and $m(C) = \sum_{A_i \cap B_j = C} m_1(A_i)m_2(B_j)/K$, where K is the normalization factor $1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i)m_2(B_j)$, for all nonempty sets $C \subseteq S$.

While BBAs are used to represent uncertain evidence, DST represents degrees of belief for propositions through belief functions.

Definition 3 (Belief function)**.** Given state space S, a *belief function* is a function $\mathrm{Bel} : 2^S \to [0, 1]$ such that (1) $\mathrm{Bel}(\emptyset) = 0$, (2) $\mathrm{Bel}(S) = 1$, and (3) $\mathrm{Bel}(\bigcup_1^n A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \mathrm{Bel}(\bigcap_{i \in I} A_i)$.[5]

[5]Due to (3), belief functions are not probability functions as they are not necessarily finitely additive. They are super-additive: for any two disjoint $A, B \subseteq S$, $\mathrm{Bel}(A \cup B) \geq \mathrm{Bel}(A) + \mathrm{Bel}(B)$. This is motivated by allowing belief to be evidence based: an agent might have a piece of evidence $E \subseteq S$ supporting $A \cup B$, that is, $E \subseteq A \cup B$, without that piece of evidence supporting A or supporting B, that is, $E \not\subseteq A$ and $E \not\subseteq B$ (see, e.g., [30, ch. 2],[31, ch. 14.3] for motivating examples).

Besides directly defining belief functions without appealing to evidence as in Definition 3, DST provides a formula to get a belief function from any basic belief assignment m.

Definition 4 (Belief Function for m)**.** Given a state space S, propositions $A, B \subseteq S$, and a BBA m over S, we define the *belief function of* m, $\text{bel}_m : 2^S \to [0,1]$ as $\text{bel}_m(B) = \sum_{A \subseteq B} m(A)$.

The resulting bel_m given in Definition 4 is a Dempster-Shafer belief function, that is, it satisfies the conditions in Definition 3 [6, p. 51]. Belief functions that are obtained from combining BBAs are called *support functions*. In this work, we restrict our attention to support functions obtained from the combination of simple BBAs.

Definition 5 ((Separable) Support Function)**.** Let S be a set of possible states and $\text{Bel} : 2^S \to [0,1]$ be a belief function. If there are some BBAs $m_1, \dots, m_l$ such that Bel is the belief function of $(m_1 \oplus \cdots \oplus m_l)$, i.e., $\text{Bel} = \text{bel}_{(m_1 \oplus \cdots \oplus m_l)}$, then Bel is called a *support function*. If, in addition, all the BBAs $m_1, \dots, m_l$ are simple—i.e., they each have a unique proper focal element— Bel is said to be a *separable support function*.

In the rest of the paper, we adapt the previous terms to align more closely with established logical formalisms of evidence and belief. Specifically, we will represent basic belief assignments (BBAs), or evidence, in an alternative manner using quantitative evidence frames, as defined below.

Definition 6 (Quantitative Evidence Frame)**.** A *quantitative evidence frame* is tuple $(S, \boldsymbol{\mathcal{E}}^Q)$, where S is a (finite and non-empty) state space, $\boldsymbol{\mathcal{E}}^Q \subseteq \mathcal{P}(S) \times (0,1)$ is a non-empty set of uncertain evidence such that (i) $\emptyset \notin \boldsymbol{\mathcal{E}}$ and $S \notin \boldsymbol{\mathcal{E}}$; and (ii) if $(E,q), (E,r) \in \boldsymbol{\mathcal{E}}^Q$, then $q = r$. Moreover, $(S, \boldsymbol{\mathcal{E}})$ is called a *qualitative evidence frame* and $\boldsymbol{\mathcal{E}}$ a *qualitative evidence set*.

For any element $(E,p) \in \boldsymbol{\mathcal{E}}^Q$, E represents the propositional content of the evidence and p is its degree of certainty. We call a collection of evidence pieces $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{E}}$ *consistent* iff $\bigcap \boldsymbol{E} \neq \emptyset$, and *inconsistent* otherwise. As in DST, given a pair $(E,p) \in \boldsymbol{\mathcal{E}}^Q$, the value $1-p$ represents the uncertainty of the given piece of evidence (and not the certainty of $S \setminus E$). We hope that the connection to the Dempster-Shafer framework is clear: any $(E,p) \in \boldsymbol{\mathcal{E}}^Q$ is a more compact way of presenting a *simple BBA* $m : 2^S \to [0,1]$ such that $m(E) = p$ and $m(S) = 1 - m(E)$. Since for every $(E,p) \in \boldsymbol{\mathcal{E}}^Q$, $p > 0$, each (E,p) in fact represents a *non-dogmatic* simple BBA. Given a quantitative evidence frame $(S, \boldsymbol{\mathcal{E}}^Q)$, we define the collection of BBAs represented by $\boldsymbol{\mathcal{E}}^Q$ as $m_{\boldsymbol{\mathcal{E}}} = \{m : 2^S \to [0,1] \mid m(E) = p, m(S) = 1 - p \text{ for some } (E,p) \in \boldsymbol{\mathcal{E}}^Q\}$. In this particular setting, we sometimes use "degree of certainty" and "degree of uncertainty" of a piece of evidence interchangeably in our conceptual explanations since the latter is fully determined by the former.

Our qualitative evidence frame is a finite version of the so-called uniform evidence models introduced in [2], with the caveat that we impose $S \notin \boldsymbol{\mathcal{E}}$, as opposed to the constraint $S \in \boldsymbol{\mathcal{E}}$ for uniform evidence models.

3. Order Lifting for Uncertain Evidence

In this section, we introduce and motivate the way we order propositions by directly comparing *sets* of evidence that support them. This order will be used to interpret the evidence comparison operator in our bi-modal logic introduced in Section 4.

Given a quantitative evidence frame, it is easy to compare basic pieces of evidence with respect to the degrees of certainty: for any $(E, q_E), (E', q_{E'}) \in \boldsymbol{\mathcal{E}}^Q$, we define an order $\leq_e$ on $\boldsymbol{\mathcal{E}}$ as follows:

$$E \leq_e E' \text{ if and only if } q_E \leq q_{E'} \qquad \text{(order } \leq_e\text{)}$$

where $\leq$ is the standard total pre-order defined on $(0,1)$. We say E' *certainty-dominates* E when $E \leq_e E'$. When $q_E < q_{E'}$, we say E' *strictly certainty-dominates* E and denote it by $E <_e E'$. "Certainty-dominates" simply means "at least as certain as". We prefer the former reading as the latter leads to convoluted readings of the comparison operators we later define over sets of propositions. It is not difficult to see that $(\boldsymbol{\mathcal{E}}, \leq_e)$ a totally pre-ordered set ($\leq_e$ is not necessarily a partial order on $\boldsymbol{\mathcal{E}}$ since it could be that $q_E = q_{E'}$ but $E \neq E'$).

In order to compare propositions with respect to the degree of certainty of the evidential support they receive, we need to define an order on *sets* of pieces of evidence that support propositions. To this end, we need to *extend* or *lift* the order $\leq_e$ to an order $\preceq_e$ on $2^{\boldsymbol{\mathcal{E}}}$. We can then easily extend $\preceq_e$ to an order $\trianglelefteq_e$ on 2^S by simply linking sets of pieces of evidence to the propositions they support.

We now focus on defining $\preceq_e$ on $2^{\boldsymbol{\mathcal{E}}}$. We aim for an order lifting of $\leq_e$ that is non-trivially informative, i.e. we want an order lifting that can determine strict certainty-dominance even in cases where not all pieces of evidence supporting P is more certain than the pieces supporting Q, and vice versa. However, it is of course important that our definition does not introduce unintuitive assumptions, focusing too much on the most or the least certain evidence. To illustrate, let us return to example of the health institution and suppose that the evidence supporting P is $(E_1, 0.4)$ and $(E_2, 07)$ and the evidence supporting Q is $(E_1, 0.5)$ and $(E_2, 06)$. Neither $P \preceq_e Q$ nor $Q \preceq_e P$ seem to be quite reasonable, as the former seems to penalize having the lowest certainty, the latter seems to too much reward having the highest certainty. The following definition introduces an order lifting that satisfies both requirements, i.e., it is non-trivially informative and does not introduce unintuitive assumptions (see Figure 1 for a visual representation of the proposed order).

Definition 7 (Evidence Comparison Order)**.** Given a quantitative evidence frame $(S, \boldsymbol{\mathcal{E}}^Q)$, the *evidence comparison order* $\preceq_e$ on $2^{\boldsymbol{\mathcal{E}}}$ is defined as follows. For all $\boldsymbol{E}, \boldsymbol{E'} \in 2^{\boldsymbol{\mathcal{E}}}$:

1. $\emptyset \preceq_e \boldsymbol{E}$, and
2. for all $\boldsymbol{E} \neq \emptyset : \boldsymbol{E} \preceq_e \boldsymbol{E'}$ if and only if $(i)\ \forall E \in \boldsymbol{E}\ \exists E' \in \boldsymbol{E'} : E \leq_e E'$; and $(ii)\ \forall E' \in \boldsymbol{E'}\ \exists E \in \boldsymbol{E} : E \leq_e E'$.

Given any $\boldsymbol{E}, \boldsymbol{E'} \in 2^{\boldsymbol{\mathcal{E}}}$: $\boldsymbol{E} \preceq_e \boldsymbol{E'}$ says that the evidence set $\boldsymbol{E'}$ is *at least as certain as* $\boldsymbol{E}$ in the following sense: (i) every piece of evidence in $\boldsymbol{E}$ is certainty-dominated by some piece of evidence in $\boldsymbol{E'}$, and (ii) every piece of evidence in $\boldsymbol{E'}$ certainty-dominates some piece of evidence in $\boldsymbol{E}$, with the caveat that the empty set is always the least certain. We stipulate that the empty set is comparable according to the ordering $\preceq_e$ to enable our logic to distinguish between propositions supported by some evidence and those that are not. This creates a slight abuse of language, as the empty set is not truly certainty-dominated by every set of evidence, but rather 'evidence'-dominated by every set of evidence, in the sense that any set $\boldsymbol{E} \in 2^{\boldsymbol{\mathcal{E}}} \setminus \{\emptyset\}$ contains more pieces of evidence than the empty set. Proposition 1 lists important properties of $\preceq_e$.

Proposition 1. *Given a quantitative evidence frame $(S, \boldsymbol{\mathcal{E}}^Q)$, the ordered pair $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ defined as in Definition 7, and $\boldsymbol{E}, \boldsymbol{E}' \in 2^{\boldsymbol{\mathcal{E}}}$, we have:*

1. *The pair $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ is a pre-order which is not necessarily total.*
2. *If $\boldsymbol{E} \preceq_e \emptyset$, then $\boldsymbol{E} = \emptyset$.*
3. *$\emptyset$ is the unique minimal element of $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$.*
4. *$\boldsymbol{E} \subseteq \boldsymbol{E'}$ does not imply $\boldsymbol{E} \preceq_e \boldsymbol{E'}$ (i.e., $\preceq_e$ is not monotonic).*

Proof. Item 1 follows immediately by the definition of $\preceq_e$ that it is reflexive and transitive. To see that $\preceq_e$ is not total, consider $S = \{a, b, c\}$ and $\boldsymbol{\mathcal{E}}^Q = \big\{(\{a\}, 0.2), (\{b\}, 0.1)(\{c\}, 0.3)\big\}$, $\boldsymbol{E} = \big\{\{a\}\big\}$ and $\boldsymbol{E}' = \big\{\{b\}, \{c\}\big\}$. We then have $\boldsymbol{E} \not\preceq_e \boldsymbol{E'}$ since $\{b\} \in \boldsymbol{E'}$ does not certainty-dominate any element in $\boldsymbol{E}$, violating Definition 7.(ii). Moreover, $\boldsymbol{E'} \not\preceq_e \boldsymbol{E}$ since $\{c\} \in \boldsymbol{E'}$ is not certainty-dominated by any element in $\boldsymbol{E}$, violating Definition 7.(i).

To prove item 2, suppose, toward contradiction, that $\boldsymbol{E} \preceq_e \emptyset$ and $\boldsymbol{E} \neq \emptyset$. The former, by Definition 7.(i), means that for all $E \in \boldsymbol{E}$, there is $E' \in \emptyset$ such that $E \leq_e E'$, which cannot be the case as the empty set has no elements. Therefore, if $\boldsymbol{E} \preceq_e \emptyset$, then $\boldsymbol{E} = \emptyset$.

Item 2 and Definition 7.(i) together guarantee that $\boldsymbol{E} \preceq_e \emptyset$ if and only if $\boldsymbol{E} = \emptyset$, thus, $\emptyset$ is the unique minimal element of $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$.

Finally, consider $\boldsymbol{\mathcal{E}}^Q = \{(\{a\}, 0.2), (\{b\}, 0.3), (\{c\}, 0.1)\}$, $\boldsymbol{E} = \{\{a\}\}$, and $\boldsymbol{E'} = \{\{a\}, \{c\}\}$. In this case, $\boldsymbol{E} \subseteq \boldsymbol{E}'$ but $\boldsymbol{E} \not\preceq_e \boldsymbol{E}'$, since $\{c\} \in \boldsymbol{E}'$ violates Definition 7.(ii): it does not certainty-dominates any evidence in $\boldsymbol{E}$.

□

Notice that the non-monotonicity of $(\boldsymbol{\mathcal{E}}, \leq_e)$ (that is; $E \subseteq E'$ does not imply $E \leq_e E'$) is lifted to the order $\preceq_e$ (Proposition 1.4). This property of $(\boldsymbol{\mathcal{E}}, \leq_e)$ is at the core of Dempster-Shafer theory, since it distinguishes support functions from probability distributions.

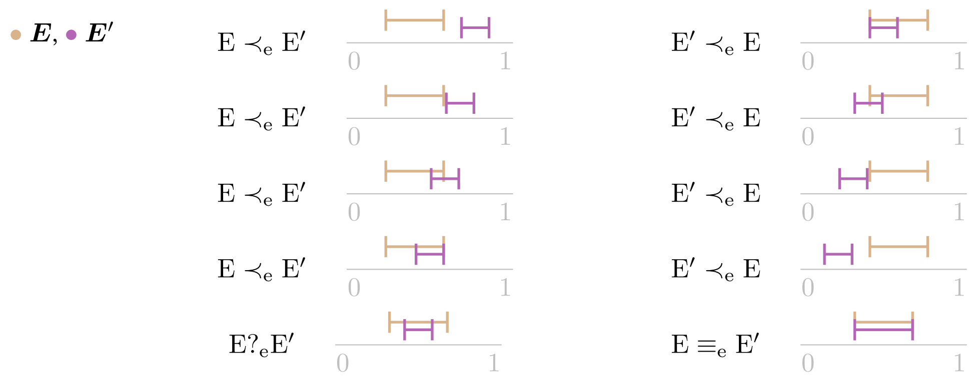

Figure 1: Visualization of $\preceq_e$. Brown segments represent the range of certainty values of the pieces of evidence that support $\boldsymbol{E}$, so the vertical lines in the extremes represent the minimum and maximum of these values. Purple segments represent the respective values for $\boldsymbol{E}'$. As for notation: $\boldsymbol{E} \prec_e \boldsymbol{E}'$ means $\boldsymbol{E} \preceq_e \boldsymbol{E}'$ and $\boldsymbol{E}' \not\preceq_e \boldsymbol{E}$; $\boldsymbol{E} \equiv_e \boldsymbol{E}'$ means $\boldsymbol{E} \preceq_e \boldsymbol{E}'$ and $\boldsymbol{E}' \preceq_e \boldsymbol{E}$; and $\boldsymbol{E} ?_e \boldsymbol{E}'$ means $\boldsymbol{E} \not\preceq_e \boldsymbol{E}'$ and $\boldsymbol{E}' \not\preceq_e \boldsymbol{E}$.

Modulo the stipulation in Definition 7.1 (i.e., restricted to non-empty sets of evidence pieces), $\preceq_e$ is the Egli-Milner lifting of $\leq_e$ on $\boldsymbol{\mathcal{E}}$ [20][6]. When defined over a finite set, as in our case, Egli-Milner lifting is equivalent to the order lifting *maxmin* defined in the context of preference lifting in Social Choice [32]. Below, we introduce the formal defintion of the maxmin extension and proof of this claim.

Definition 8. (Maxmin Extension) Given a total order $(X, \leq)$ over a finite set X and $Y_1, Y_2 \subseteq X$, we define the *maxmin extension* $(2^X, \preceq)$ as

$$Y_1 \preceq Y_2 \quad \text{if and only if} \quad \begin{array}{l} \max(Y_1) \leq \max(Y_2) \text{ and} \\ \min(Y_1) \leq \min(Y_2) \end{array}$$

where $\max(Y_i) \in Y_i$ and $\min(Y_i) \in Y_i$ such that $y' \leq \max(Y_i)$ and $\min(Y_i) \leq y'$ for all $y' \in Y_i$, $i \in \{1, 2\}$. For any $i \in \{1, 2\}$, both elements $\max(Y_i)$ and $\min(Y_i)$ exists since Y_i finite.

Proposition 2. *Given a finite quantitative evidence frame* $(S, \boldsymbol{\mathcal{E}}^Q)$, *the maxmin extension of* $(\boldsymbol{\mathcal{E}}, \leq_e)$ *is equivalent to* $(2^{\boldsymbol{\mathcal{E}}} \setminus \{\emptyset\}, \preceq_e)$.

Proof. Let $\boldsymbol{E}, \boldsymbol{E}' \in 2^{\boldsymbol{\mathcal{E}}} \setminus \{\emptyset\}$. If for all $E \in \boldsymbol{E}$ there exists $E' \in \boldsymbol{E}'$ such that $E \leq_e E'$, then there exists $E' \in \boldsymbol{E}'$ such that $\max(\boldsymbol{E}) \leq_e E' \leq_e \max(\boldsymbol{E}')$. And, if for all $E' \in \boldsymbol{E}'$ there exists $E \in \boldsymbol{E}$ such that $E \leq_e E'$, then there exists $E \in \boldsymbol{E}$ such that $\min(\boldsymbol{E}) \leq_e E \leq_e \min(\boldsymbol{E}')$. So Definition 7 implies maxmin extension. Conversely, if $\max(\boldsymbol{E}) \leq_e \max(\boldsymbol{E}')$ then $E \leq_e \max(\boldsymbol{E}')$ for every $E \in \boldsymbol{E}$. And if $\min(\boldsymbol{E}) \leq_e \min(\boldsymbol{E}')$ then $\min(\boldsymbol{E}) \leq_e E'$ for every $E' \in \boldsymbol{E}'$. □

[6] In this work the authors investigate a modal logic of Egli-Milner order as a preference lifting. Our logic shares with this logic a few validities for the evidence comparison, yet diverges from it as our evidence comparison operator is not interpreted exactly via the Egli-Milner lifting. We refer to this source for a comparison.

This connection between the Egli-Milner and maxmin liftings allows us to infer the following lifting properties of $\preceq_e$.

Proposition 3. *The following holds for any finite* $(\boldsymbol{\mathcal{E}}, \leq_e)$ *and the corresponding* $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ *(as given in Definition 7):*

1. $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ *satisfies the lifting rule: for all* $E, E' \in \boldsymbol{\mathcal{E}}$*, if* $E <_e E'$ *then* $\{E\} \prec_e \{E'\}$.
2. $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ *satisfies lifting dominance: for all* $\boldsymbol{E} \subseteq \boldsymbol{\mathcal{E}}$ *and all* $E \in \boldsymbol{\mathcal{E}} \setminus \boldsymbol{E}$:
   a) *if* $E <_e E'$ *for all* $E' \in \boldsymbol{E}$*, then* $\boldsymbol{E} \cup \{E\} \prec_e \boldsymbol{E}$*, and*
   b) *if* $E' <_e E$ *for all* $E' \in \boldsymbol{E}$*, then* $\boldsymbol{E} \prec_e \boldsymbol{E} \cup \{E\}$.
3. $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ *satisfies lifting independence: for all* $\boldsymbol{E}, \boldsymbol{E}' \subseteq \boldsymbol{\mathcal{E}}$ *and* $E \in \boldsymbol{\mathcal{E}} \setminus \boldsymbol{E} \cup \boldsymbol{E}'$*, if* $\boldsymbol{E} \prec_e \boldsymbol{E}'$*, then* $\boldsymbol{E} \cup \{E\} \preceq_e \boldsymbol{E}' \cup \{E\}$.
4. *Every order lifting* $(2^{\boldsymbol{\mathcal{E}}}, \preceq)$ *of* $(\boldsymbol{\mathcal{E}}, \leq_e)$ *such that (1)* $\emptyset \preceq \boldsymbol{E}$ *for every* $\boldsymbol{E} \in 2^{\boldsymbol{\mathcal{E}}}$*, (2) satisfies lifting dominance and (3) lifting independence, contains* $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$.

Proof. 1, 2, 3 follow from [32, Example 3.9]. 4 follows from the impossibility theorem of Kannai and Peleg about weak orders [33, p. 174], the impossibility theorem of Barberà and Pattanaik about binary relations [34, Proposition 3], and [32, Observation 3.1]. □

Remark 1. Notice that item 4 of Proposition 3 is a strong argument to support our evidence comparison order (Definition 7). In other words, the mathematical results cited in Proposition 3's proof imply that if we want our evidence comparison order to satisfy lifting dominance and lifting independence, then it will not be a total order [33, p. 174], will not satisfy strict lifting independence [34, Proposition 3], and will contain our order proposal of Definition 7 [32, Observation 3.1]. One direction to move towards a completely different certainty-dominance comparison order would be relaxing the lifting dominance and independence requirements. However, both properties are naturally suited to order sets of evidence based on the intended notion of certainty-dominance, without taking into account other properties of sets of evidence, e.g., the number of elements in a set and the mutual consistency of the pieces of evidence in an evidence set. To explain further, let's have a brief look at a situation where lifting independence is violated: take $E_1,\ E_2,\ E_3$ such that $\{E_1\} \prec_e \{E_2\}$ and $\{E_2, E_3\} \preceq_e \{E_1, E_3\}$. The certainty order between $\{E_1, E_3\}$ and $\{E_2, E_3\}$ can only be determined by the degrees of certainty of the elements that distinguish these two sets, namely, q_1 and q_2. Since $\{E_1\} \prec_e \{E_2\}$ and $(\boldsymbol{\mathcal{E}}, \leq_e)$ is a total pre-order, we know, by the lifting rule, that $q_1 < q_2$. Therefore, the only way to conclude $\{E_2, E_3\} \preceq_e \{E_1, E_3\}$ is taking into account some extra property that goes beyond the degrees of certainty of the evidence pieces and that makes E_3 to reinforce E_1 and weaken E_2 (for example, if E_3 is consistent with E_1 but not with E_2). Therefore, restricting our scope to certainty-dominance implies requiring an order lifting to satisfy independence or strict independence. Similarly, violating lifting dominance (a) implies that there will be cases where $\{E, E'\} \not\prec_e \{E\}$ in spite of the fact that $q_{E'} < q_E$. Consequently, an order lifting of $(\boldsymbol{\mathcal{E}}, \leq_e)$ that does not satisfy dominance does not necessarily order the elements of $2^{\boldsymbol{\mathcal{E}}}$ according to the degrees of uncertainty of the individual evidence pieces in $\boldsymbol{\mathcal{E}}$: in this particular case, the resulting order $\preceq_e$ seems to ignore the fact that the only extra element E' of $\{E, E'\}$ is strictly less certain than E (a similar argument can be made for lifting dominance (b)).

At this point of the discussion, two possibilities remain. For non-empty sets of pieces of evidence, we either adhere to Definition 7 or adopt an order that includes it. Extending $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ requires assumptions that are not directly justified by DST. Notice that $\boldsymbol{E}, \boldsymbol{E'} \in 2^{\boldsymbol{\mathcal{E}}}$ are such that $\boldsymbol{E} \not\preceq_e \boldsymbol{E'}$ and $\boldsymbol{E'} \not\preceq_e \boldsymbol{E}$ if and only if the least certainty value in $\boldsymbol{E}$ is smaller than the least certainty value in $\boldsymbol{E'}$ and the highest certainty value in $\boldsymbol{E}$ is greater than the highest certainty value in $\boldsymbol{E'}$ or vice versa. For example, given $\boldsymbol{E} = \{(E_1, 0.4), (E_2, 07)\}$, and $\boldsymbol{E'} = \{(E_3, 0.5), (E_4, 06)\}$, neither $\boldsymbol{E} \preceq_e \boldsymbol{E'}$ or $\boldsymbol{E'} \preceq_e \boldsymbol{E}$. Concluding $\boldsymbol{E} \preceq_e \boldsymbol{E'}$ penalizes having the lowest certainty, while concluding $\boldsymbol{E'} \preceq_e \boldsymbol{E}$ rewards having the highest certainty. Choosing between these assumptions is unjustified in general, at least from a DST perspective. This finalizes our justification for Definition 7.

Now, we will use $(2^{\boldsymbol{\mathcal{E}}}, \preceq_e)$ to compare propositions. To do so, we extend the order $\preceq_e$ to an order $\trianglelefteq_e$ on 2^S as:

$$P \trianglelefteq_e Q \text{ if and only if } \boldsymbol{\mathcal{E}}_P \preceq_e \boldsymbol{\mathcal{E}}_Q, \qquad \text{(order } \trianglelefteq_e)$$

where, recall that, $\boldsymbol{\mathcal{E}}_P = \{E \in \boldsymbol{\mathcal{E}} : E \subseteq P\}$. $(2^S, \trianglelefteq_e)$ inherits all properties listed in Proposition 1 and Proposition 3. We will use $\trianglelefteq_e$ to interpret the certainty-dominance, i.e., evidence comparison operator in our logic.

4. Logical Framework

This section is dedicated to defining the logic for comparing strengths of evidence and belief. We introduce its syntax and semantics, discuss the nature of the modal operator for comparing uncertain evidence, and list some of the important (in)validities of our logic.

We work with a bi-modal language $\mathcal{L}$ based on a countable set Prop of atomic formulas, defined recursively by the following grammar in BNF form:

$$A, B := p \mid \neg A \mid (A \wedge B)$$

$$\varphi, \psi := A \mid (A \trianglelefteq_e B) \mid (A \trianglelefteq_b B) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi$$

where $p \in \mathsf{Prop}$. We use $\rightarrow$ for the material conditional and $\vee$ for disjunction, defined in the usual manner as $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$ and $\phi \rightarrow \psi := \neg\phi \vee \psi$. The propositional constants for tautology and contradiction are denoted and defined standardly as $\top := p \vee \neg p$ and $\bot : \neg\top$, respectively. We will follow the usual rules for the elimination of the parentheses.

Notice that the first line of the BNF form defines the language of *classical propositional logic* $\mathcal{L}_{CPL}$. The binary comparison operators for evidence and belief, $\trianglelefteq_e$ and $\trianglelefteq_b$, respectively, connect only the sentences in $\mathcal{L}_{CPL}$. They are therefore intended to compare only first-order evidence and belief. We read $\varphi \trianglelefteq_e \psi$ as "*the evidence for ψ certainty-dominates the evidence for φ*;" and $\varphi \trianglelefteq_b \psi$ as "*belief in ψ is at least as strong as belief in φ*".[7] $\Box\varphi$ is the epistemic modality "It is *a priori* that φ" and will be interpreted as the global modality.

The following abbreviations will be useful in stating some of the important (in)validities. We define $\varphi \equiv_e \psi := (\varphi \trianglelefteq_e \psi) \wedge (\psi \trianglelefteq_e \varphi)$ and $\varphi \equiv_b \psi := (\varphi \trianglelefteq_b \psi) \wedge (\psi \trianglelefteq_b \varphi)$ for equivalence of φ and ψ with respects to evidential support and believability, respectively. The corresponding strong comparison operators $\triangleleft_e$ and $\triangleleft_b$, respectively, are defined as $\varphi \triangleleft_e \psi := (\varphi \trianglelefteq_e \psi) \wedge \neg(\psi \trianglelefteq_e \varphi)$ and $\varphi \triangleleft_b \psi := (\varphi \trianglelefteq_b \psi) \wedge \neg(\psi \trianglelefteq_b \varphi)$. We read $\varphi \triangleleft_e \psi$ as "*the evidence for ψ strictly certainty-dominates the evidence for φ*", and $\varphi \triangleleft_b \psi$ as "*belief in ψ is strictly stronger than belief in φ*".

We interpret the language $\mathcal{L}$ on *quantitative evidence-belief models.*

Definition 9 (Quantitative Evidence-Belief Models)**.** A *quantitative evidence-belief model* (in short, an *e-b model*) is a tuple $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$, where

1. $(S, \boldsymbol{\mathcal{E}}^Q)$ is quantitative evidence frame (as given in Definition 6),
2. $\text{Bel} : 2^S \rightarrow [0, 1]$ is the separable support function obtained by applying DRC to the BBAs in $m_{\boldsymbol{\mathcal{E}}}$, that is, $\text{Bel}(A) = \sum_{B \subseteq A} \oplus m_{\mathcal{E}}(B)$.
3. $V : \mathsf{Prop} \rightarrow 2^S$ is a standardly defined valuation map.

Our e-b model is a combination of uniform evidence models of [2] and qualitative belief models of [10]. The former, in our notation $\langle S, \boldsymbol{\mathcal{E}}, V\rangle$, is endowed with the quantitative components Q and Bel.

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[7] The analogous belief comparison operator $\varphi \triangleleft \psi$ in [10] is read as "*ψ is at least as believable as φ*" and interpreted with respect to a belief function in the same way we interpret $\trianglelefteq_b$ (see Definition 10). We here adopt the reading of the belief comparison operator $\succcurlyeq_B$ of [19] as we find it more intuitive and fitting to the formal interpretation and intended meaning of $\trianglelefteq_b$.

The latter, in our notation $\langle S, \text{Bel}, V\rangle$, is expanded by an explicit evidence set $\boldsymbol{\mathcal{E}}^Q$ with the following caveat: the belief functions of [10] are not necessarily support functions.

The semantics $\models$ for $\mathcal{L}$ in e-b models is defined recursively as in Definition 10. The *truth set* of $\varphi \in \mathcal{L}$ with respect to $\mathcal{M}$ is $[\![\varphi]\!]_{\mathcal{M}} := \{s \in S : \mathcal{M}, s \models \varphi\}$, namely, the set of all possible states that makes φ true. We omit the subscript $\mathcal{M}$ in $[\![\varphi]\!]_{\mathcal{M}}$ when the model is contextually clear. To simplify notation, instead of writing $\boldsymbol{\mathcal{E}}_{[\![\varphi]\!]_{\mathcal{M}}}$, we simply write $\boldsymbol{\mathcal{E}}_\varphi$ when the model is clear from context and call $\boldsymbol{\mathcal{E}}_\varphi$ the *evidence set for* φ.

Definition 10 (Semantics for $\mathcal{L}$ ($\models$)**.** Given an e-b model $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ and a state $s \in S$, the $\models$-semantics for the language $\mathcal{L}$ is defined recursively as follows, where $p \in \mathsf{Prop}$:

$$\begin{array}{lll}
\mathcal{M}, s \models p & \text{if and only if} & s \in V(p) \\
\mathcal{M}, s \models \neg\varphi & \text{if and only if} & \text{not } \mathcal{M}, s \models \varphi \\
\mathcal{M}, s \models \varphi \wedge \psi & \text{if and only if} & \mathcal{M}, s \models \varphi \text{ and } \mathcal{M}, s \models \psi \\
\mathcal{M}, s \models \varphi \trianglelefteq_e \psi & \text{if and only if} & \boldsymbol{\mathcal{E}}_\varphi \trianglelefteq_e \boldsymbol{\mathcal{E}}_\psi \\
\mathcal{M}, s \models \varphi \trianglelefteq_b \psi & \text{if and only if} & \text{Bel}([\![\varphi]\!]) \leq \text{Bel}([\![\psi]\!]) \\
\mathcal{M}, s \models \Box\varphi & \text{if and only if} & S \subseteq [\![\psi]\!].
\end{array}$$

We use $\mathcal{M}, s \not\models \varphi$ for "not $\mathcal{M}, s \models \varphi$". Note that $\trianglelefteq_e$ is used both in the syntax and semantics. In the syntax it represents the evidence comparison modality. In the semantics, we use it to *interpret* the evidence comparison modality. We believe the two uses of $\trianglelefteq_e$ will be clear from the context. The notions of logical consequence and validity are defined standardly as follows. Given a $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$, we say that φ is a *logical consequence* of Γ, denoted by $\Gamma \models \varphi$, if for all e-b models $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ and all $s \in S$: if $\mathcal{M}, s \models \psi$ for all $\psi \in \Gamma$, then $\mathcal{M}, s \models \varphi$. For single-premise entailment, we write $\psi \models \varphi$ for $\{\psi\} \models \varphi$. *Validity*, $\models \varphi$, is truth at all states of all e-b models. φ is called invalid, denoted by $\not\models \varphi$, if it is not a validity, that is, if there is an e-b model $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ and a state $s \in S$ such that $\mathcal{M}, s \not\models \varphi$ [35].

The semantic clauses for the Booleans are standard and $\Box$ is interpreted as the global modality. The semantic clause for the belief comparison operator $\trianglelefteq_b$ is the same as the one proposed in [10]. The novel component of the logic is the evidence comparison operator $\trianglelefteq_e$ and, therefore, it deserves further elaboration. According to our semantics, *evidence for* ψ *certainty-dominates the evidence for* φ if and only if the evidence set for ψ is *at least as certain as* the evidence set of φ, that is, either there is no evidence for φ or (i) every piece of evidence for φ is certainty-dominated by some piece of evidence for ψ, and (ii) every piece of evidence for ψ certainty-dominates some piece of evidence for φ. Next we elaborate on the principles (in)validated by the semantics.

The following table lists the important validities describing the behavior of $\trianglelefteq_e$, $\trianglelefteq_b$, and the connection between the two.

Theorem 1. *The principles in Groups (I)-(III) listed in Table 1 are valid in all e-b models. The principles in Group (IV) are invalid in e-b models.*

Proof of Theorem 1.

B1 follows from the properties of Bel such that $\text{Bel}([\![\top]\!]) = 1 \not\leq \text{Bel}([\![\bot]\!]) = 0$.

B2 follows from the fact that $([0,1], \leq)$ is a total order, thus, induces a total order on 2^S.

B3 follows from the fact that $([0,1], \leq)$ is a total order, thus, induces a transitive order on 2^S.

To prove B4, suppose $\mathcal{M}, s \models \Box(\varphi \to \psi)$. This means that $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Therefore, every subset of $[\![\varphi]\!]$ is a subset of $[\![\psi]\!]$. Hence, by the definition of Bel, we have $\text{Bel}([\![\varphi]\!]) \leq \text{Bel}([\![\psi]\!])$.

B5 follows from the axiom of partial monotonicity ($A \subseteq B$, $B \cap C = \emptyset$ and $\text{Bel}(B) > \text{Bel}(A)$ implies $\text{Bel}(B \cup C) > \text{Bel}(A \cup C)$) for all belief functions Bel used in [36] to characterize a belief function that fully agrees with a preference relation. If $\mathcal{M}, s \models (\Box(\varphi \to \psi) \wedge \Box\neg(\psi \wedge \chi) \wedge \neg(\psi \trianglelefteq_b \varphi))$, then $[\![\varphi]\!] \subseteq [\![\psi]\!]$, $[\![\psi]\!] \cap [\![\chi]\!] = \emptyset$ and $\text{Bel}([\![\varphi]\!]) < \text{Bel}([\![\psi]\!])$. By the partial monotonicity axiom, this implies that $\text{Bel}([\![\varphi]\!] \cup [\![\chi]\!]) < \text{Bel}([\![\psi]\!] \cup [\![\chi]\!])$. Since $[\![\varphi \vee \chi]\!] = [\![\varphi]\!] \cup [\![\chi]\!]$ and $[\![\psi \vee \chi]\!] = [\![\psi]\!] \cup [\![\chi]\!]$, we conclude $\mathcal{M}, s \models \neg((\psi \vee \chi) \trianglelefteq_b (\varphi \vee \chi))$.

	(I) Validities for $\trianglelefteq_b$		(II) Validities for $\trianglelefteq_e$
B1	$\neg(\top \trianglelefteq_b \bot)$	E1	$\bot \trianglelefteq_e \varphi$
B2	$(\varphi \trianglelefteq_b \psi) \vee (\psi \trianglelefteq_b \varphi)$	E2	$\neg(\top \trianglelefteq_e \bot)$
B3	$((\varphi \trianglelefteq_b \psi) \wedge (\psi \trianglelefteq_b \chi)) \rightarrow (\varphi \trianglelefteq_b \chi)$	E3	$\varphi \trianglelefteq_e \varphi$
B4	$\Box(\varphi \rightarrow \psi) \rightarrow (\varphi \trianglelefteq_b \psi)$	E4	$((\varphi \trianglelefteq_e \psi) \wedge (\psi \trianglelefteq_e \chi)) \rightarrow (\varphi \trianglelefteq_e \chi)$
B5	$(\Box(\varphi \rightarrow \psi) \wedge \Box\neg(\psi \wedge \chi) \wedge \neg(\psi \trianglelefteq_b \varphi)) \rightarrow$ $\neg((\psi \vee \chi) \trianglelefteq_b (\varphi \vee \chi))$	E5	$\Box(\varphi \leftrightarrow \psi) \rightarrow (\varphi \equiv_e \psi)$
B6	$(\varphi \trianglelefteq_b \psi) \rightarrow \Box(\varphi \trianglelefteq_b \psi)$	E6	$((\varphi \trianglelefteq_e \chi) \wedge (\psi \trianglelefteq_e \chi) \wedge \Box(\varphi \rightarrow \eta) \wedge$ $\Box(\eta \rightarrow \psi)) \rightarrow (\eta \trianglelefteq_e \chi)$
B7	$\neg(\varphi \trianglelefteq_b \psi) \rightarrow \Box\neg(\varphi \trianglelefteq_b \psi)$	E7	$((\chi \trianglelefteq_e \varphi) \wedge (\chi \trianglelefteq_e \psi) \wedge \Box(\varphi \rightarrow \eta) \wedge$ $\Box(\eta \rightarrow \psi)) \rightarrow (\chi \trianglelefteq_e \eta)$
B8	$(\Box\varphi \wedge \neg\Box\psi) \rightarrow \neg(\varphi \trianglelefteq_b \psi)$	E8	$(\varphi \trianglelefteq_e \psi) \rightarrow \Box(\varphi \trianglelefteq_e \psi)$
		E9	$\neg(\varphi \trianglelefteq_e \psi) \rightarrow \Box\neg(\varphi \trianglelefteq_e \psi)$
	(III) Validities connecting $\trianglelefteq_b$ and $\trianglelefteq_e$		**(IV) Invalidities**
C1	$\neg(\varphi \trianglelefteq_e \bot) \rightarrow \neg(\varphi \trianglelefteq_b \bot)$	I1	$\not\models (\varphi \trianglelefteq_e \psi) \vee (\psi \trianglelefteq_e \varphi)$
		I2	$\not\models \Box(\varphi \rightarrow \psi) \rightarrow (\varphi \trianglelefteq_e \psi)$
		I3	$\not\models \neg(\varphi \trianglelefteq_b \bot) \rightarrow \neg(\varphi \trianglelefteq_e \bot)$

Table 1
Important validities and invalidities

B6 and B7 follows from the fact that for any $\varphi, \psi \in \mathcal{L}_{CPL}$, $[\![\varphi \trianglelefteq_b \psi]\!] = S$ or $[\![\varphi \trianglelefteq_b \psi]\!] = \emptyset$.

To prove B8, suppose $\mathcal{M}, s \models \Box\varphi \wedge \neg\Box\psi$. This means that $S \subseteq [\![\varphi]\!]$ and $S \not\subseteq [\![\psi]\!]$. The former entails that $S = [\![\varphi]\!]$. As $\text{Bel}(P) \in [0,1]$ for all $P \subseteq S$ and $\text{Bel}(S) = 1$, we obtain that $\text{Bel}([\![\psi]\!]) \leq \text{Bel}([\![\varphi]\!]) = 1$. For our assumption of $q_E \neq 1$ for every $E \in \boldsymbol{\mathcal{E}}$, $\text{Bel}(P) < 1$ for every P such that $[\![P]\!] \subsetneq S$. Therefore, $\text{Bel}([\![\psi]\!]) < \text{Bel}([\![\varphi]\!]) = 1$ and $\mathcal{M}, s \models \neg(\varphi \trianglelefteq_b \psi)$.

E1 follows from the fact that $\emptyset \trianglelefteq_e \boldsymbol{E}$ for all $E \in 2^{\boldsymbol{\mathcal{E}}}$.

E2 follows from the fact that $\boldsymbol{\mathcal{E}} \neq \emptyset$.

E3 follows from the fact that $\trianglelefteq_e$ is a pre-order on $2^{\boldsymbol{\mathcal{E}}}$, thus, it is in particular reflexive.

E4 follows from the fact that $\trianglelefteq_e$ is a pre-order on $2^{\boldsymbol{\mathcal{E}}}$, thus, it is in particular transitive.

To prove E5, suppose $\mathcal{M}, s \models \Box(\varphi \leftrightarrow \psi)$. This means that $[\![\varphi]\!] = [\![\psi]\!]$. The latter means that $\boldsymbol{\mathcal{E}}_\varphi = \boldsymbol{\mathcal{E}}_\psi$, thus, $\mathcal{M}, s \models (\varphi \equiv_e \psi)$.

To prove E6, suppose that $\mathcal{M}, s \models \Box(\varphi \rightarrow \eta) \wedge \Box(\eta \rightarrow \psi)$. Therefore, $[\![\varphi]\!] \subseteq [\![\eta]\!] \subseteq [\![\psi]\!]$. This implies that $\boldsymbol{\mathcal{E}}_\varphi \subseteq \boldsymbol{\mathcal{E}}_\eta \subseteq \boldsymbol{\mathcal{E}}_\psi$. Therefore, $\min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\psi\}) \leq \min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\eta\}) \leq \min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\varphi\})$ and $\max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\varphi\}) \leq \max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\eta\}) \leq \max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\psi\})$. Now, suppose that $\mathcal{M}, s \models (\varphi \trianglelefteq_e \chi) \wedge (\psi \trianglelefteq_e \chi)$. This means, following the maxmin definition of $\trianglelefteq_e$ (Appendix **??**, Definition 8), that $\min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\varphi\}) \leq \min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\})$ and $\max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\psi\}) \leq \max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\})$. Merging these two facts, we get that $\min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\eta\}) \leq \min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\})$, $\max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\eta\}) \leq \max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\})$, thus, $\mathcal{M}, s \models \eta \trianglelefteq_e \chi$.

E7 follows from a similar reasoning as before, noticing that $\mathcal{M}, s \models (\chi \trianglelefteq_e \varphi) \wedge (\chi \trianglelefteq_e \psi)$ implies that $\min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\}) \leq \min(\{q_E : E \in \boldsymbol{\mathcal{E}}_\psi\})$ and $\max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\chi\}) \leq \max(\{q_E : E \in \boldsymbol{\mathcal{E}}_\varphi\})$.

E8 and E9 follows from the fact that for any $\varphi, \psi \in \mathcal{L}_{CPL}$, $[\![\varphi \trianglelefteq_e \psi]\!] = S$ or $[\![\varphi \trianglelefteq_e \psi]\!] = \emptyset$.

For C1, notice that $0 < q_E < 1$ for every $E \in \boldsymbol{\mathcal{E}}$. This implies that $m_{\boldsymbol{\mathcal{E}}}(E) > 0$ for all $E \in \boldsymbol{\mathcal{E}}$. Now suppose that $\mathcal{M}, s \models \neg(\varphi \trianglelefteq_e \bot)$. This means that $\boldsymbol{\mathcal{E}}_\varphi \neq \emptyset$. Therefore, there is $E \in \boldsymbol{\mathcal{E}}_\varphi$ such that $m_{\boldsymbol{\mathcal{E}}}(E) > 0$. Then, by the definition of Bel, we obtain that $\text{Bel}([\![\varphi]\!]) > 0$, that is, $\mathcal{M}, s \models \neg(\varphi \trianglelefteq_b \bot)$.

To show I1, take $\boldsymbol{\mathcal{E}}^Q = \{(\{a\}, 0.3), \{b\}, 0.6), (\{c\}, 0.8)\}$ and $[\![p]\!] = \{a, c\}$ and $[\![q]\!] = \{b\}$. It is then easy to see that $\mathcal{M}, a \not\models (p \trianglelefteq_e q) \vee (q \trianglelefteq_e p)$.

To prove I2, consider the e-b model $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ such that such that $S = \{a, b, c\}$ and $\boldsymbol{\mathcal{E}}^Q = \big\{(\{a\}, 0.7), (\{a, b\}, 0.4)\big\}$, $V(p) = \{a\}$ and $V(q) = \{a, b\}$. Then, obviously $\mathcal{M}, a \models \Box(p \rightarrow q)$. However, $\mathcal{M}, a \not\models p \trianglelefteq_e q$ since $\{a, b\} \in \boldsymbol{\mathcal{E}}_q$ and there is no $E \in \boldsymbol{\mathcal{E}}_p$ such that $E \leq_e \{a, b\}$.

To prove I3, consider the e-b model $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ such that $S = \{a, b, c\}$, $\boldsymbol{\mathcal{E}}^Q = \big\{(\{a, b\}, 0.7), (\{b, c\}, 0.6)\big\}$, and $V(p) = \{b\}$. It holds that $\text{Bel}([\![p]\!]) > 0$ but there is no $E \in \boldsymbol{\mathcal{E}}$

such that $E \subseteq \{b\}$. Therefore, $\mathcal{M}, a \not\models \neg(p \trianglelefteq_b \bot) \to \neg(p \trianglelefteq_e \bot)$.

□

Here we elaborate on the interrelations of the principles in the table. Validities from B1 to B7 of Group (I) (together with the inference rules Modus Ponens and Necessitation for □) form a complete axiomatization of the qualitative logic for belief functions presented in [10]. The principles in this group together form a binary relation, $\succ$, on the set of propositions that generates a belief function such that $A \succ B$ iff $\text{Bel}(A) > \text{Bel}(B)$ (as shown in [36] and used in the relevant completeness proof in [10]). We refer the reader to these sources for the technical details. As for the intuitive readings of the axioms, B1 says that belief in a contradiction cannot be as strong as belief in a tautology, and is validated by the first two properties of a belief function given in Definition 3. B4 states that weaker propositions are at least as strongly believed as the stronger ones. It in particular entails $\varphi \trianglelefteq_b \varphi$, thus, together with B2 & B3, state that belief comparison relation forms a total preorder on propositions. B5 states that when a weaker proposition is not more strongly believed that a stronger, the strict weakening of these propositions do not change the comparative strengths of belief in them. B6 & B7 just state that the belief comparison relation is world independent. B8 expresses that assuming non-dogmatic BBAs is a sufficient condition (although not necessary) to guarantee that Bel will return degree of belief 1 only for the total set S. This is the main difference between the belief comparison operator of [10] and ours, as it is not a validity in the logic of [10].

Group (II) describes the properties of $\trianglelefteq_e$. E1 states that evidence for any proposition certainty dominates the evidence for any contradiction. It is validated due to Definition 7.1. E2 is analogous to B2: evidence for a contradiction cannot certainty dominate evidence for a tautology. Together they capture the intuition that contradictions are never evidentially supported and tautologies are always evidentially supported (our agents do not have the empty set as a piece of evidence and they always have some contingent evidence with non-zero degree of certainty). E3 & E4 state that the relation for certainty domination forms a preorder on propositions. This relation is not necessarily total though: see I1 in Table 1. E5 says that necessarily equivalent propositions are equally evidentially supported. Unlike $\trianglelefteq_b$, certainty-domination is not preserved under necessary implications: see I2 in Table 1 . This is because for any two propositions P, Q such that $P \subseteq Q$, we have $\boldsymbol{\mathcal{E}}_P \subseteq \boldsymbol{\mathcal{E}}_Q$; and $\boldsymbol{\mathcal{E}}_Q \setminus \boldsymbol{\mathcal{E}}_P$ might have an element that is strictly less or strictly more certain than any element of $\boldsymbol{\mathcal{E}}_P$, that is, $\boldsymbol{\mathcal{E}}_Q$ and $\boldsymbol{\mathcal{E}}_P$ might not certainty-dominate each other. Even though $\trianglelefteq_e$ does not track strength of propositions in the sense explained above, validities E6 & E7 bring some uniformity to certainty-domination within nested propositions. These validities show that the strongest and weakest propositions within a chain of propositions ordered according their logical strength determine the upper and lower bounds for the certainty-dominance of the proposition of intermediate strength. Finally, E7 & E8 are analogous to B6 & B7, stating that certainty-domination too is world independent.[8]

Finally, principle C1 shows the connection between the operators $\trianglelefteq_b$ and $\trianglelefteq_e$. To be able to properly interpret this principle, let us first state what the antecedent and the consequent express. It is not difficult to see that, given a model $\mathcal{M} = \langle S, \boldsymbol{\mathcal{E}}^Q, \text{Bel}, V\rangle$ and $s \in S$, $\mathcal{M}, s \models \neg(\varphi \trianglelefteq_e \bot)$ iff $\boldsymbol{\mathcal{E}}_\varphi \neq \emptyset$, that is, there is a basic piece of evidence for φ.[9] Similarly, $\mathcal{M}, s \models \neg(\varphi \trianglelefteq_b \bot)$ iff $\text{Bel}([\![\varphi]\!]) > 0$, that is, φ is believed to some non-zero degree. Therefore, C1 states whenever a proposition is supported by some evidence, it is believed to a non-zero degree. That is, the agent takes into account every piece of basic evidence they have in forming their beliefs, in line with the way degrees of belief are defined based on evidence combined via the DRC. On the other hand, the converse of the principle is not valid, see I3 in Table 1: having a non-zero degree of belief in a proposition does not mean that the agent has a *basic* piece of evidence for that proposition. The belief may be supported by combined evidence obtained via the DRC.

[8]The world-independence of $\trianglelefteq_e$ and $\trianglelefteq_b$ do not bear on any substantive conceptual points we want to make in this work. It is simply a result of taking the agent's evidence set to be uniform across all states. To make these orders world-dependent, we can modify the framework such that our models contain world-dependent evidence sets, $\{\boldsymbol{\mathcal{E}}^Q_s\}_{s\in S}$, instead of one $\boldsymbol{\mathcal{E}}^Q$; and Bel and $\trianglelefteq_e$ in the semantics are accordingly defined in a world-dependent way.

[9]This is the modality E_0 for "having a basic piece of evidence" in [4].

5. Small Illustrative Example

Now that we have introduced and justified all the elements of our language, let us further explore the example of the health institution. As mentioned in the introduction, consider that a health institution is preparing dietary recommendations for the prevention of a certain disease based on scientific studies. Suppose there is a finite set of foods $\{a, b, c, d, e, f\}$ and some studies suggest that foods from group $P = \{a, b\}$ are beneficial, others say they are harmful (we will represent these as $\neg P = \{c, d, e, f\}$), others conclude foods from group $Q = \{b, c, d\}$ are beneficial, and others conclude that foods from group $T = \{d, e\}$ are beneficial. Depending on how well the relevant experimental conditions replicate the real-world context, the certainty degree for the direct evidence obtained from these studies vary. Let us assume that the institution ends up having a set of evidence $\boldsymbol{\mathcal{E}}^Q = \{(\{a, b\}, 0.8), (\{c, d, e, f\}, 0.83), (\{b, c, d\}, 0.6), (\{d, e\}, 0.57)\}$. When the DRC is applied to combine these pieces of evidence (as described in Definition 2), we obtain a degree of belief 0.22 for recommending P, 0.45 for recommending T, 0.58 for recommending Q, and 0.73 for *not recommending* P. In the notation of our logic, we can write:

$$P \trianglelefteq_b T \trianglelefteq_b Q \trianglelefteq_b \neg P$$

If we understand how DRC balances inconsistencies and uncertainty degrees, we accept that these are rational conclusions. However, a lobby for the group of foods P could also reasonably argue that, considering the degrees of certainty of individual pieces of evidence supporting each proposition, P is more strongly supported than Q: after all, every piece of evidence that supports P (i.e., every $E \in \boldsymbol{\mathcal{E}}$ such that $E \subseteq P$) is more certain that the ones supporting Q. In our notation, what this lobby is arguing is following:

$$P ?_e \neg P \quad \text{and} \quad Q \prec_e P,$$

where $P ?_e \neg P$ comes from the fact that proposition $\neg P$ is supported by the direct pieces of evidence $T = \{d, e\}$ and $\neg P = \{c, d, e, f\}$ with degree of certainty 0.57 and 0.83 respectively; while proposition P is only supported by the direct piece of evidence $P = \{a, b\}$ with degree of certainty 0.8. Employing elaborate combination methods, such as DRC, that integrate different dimensions of the available information (e.g., inconsistencies, number of supportive pieces of evidence, uncertainty, etc.) and return a normalized total order is undoubtedly useful for those situations where an answer must be given and a decision must be taken (e.g., in the context of autonomous agents). However, the previous situation is an example of when it can be safer and fairer to answer "there is not enough conclusive evidence". In these cases, a logic that can express the certainty-dominance provided by the direct evidence and the degrees of belief based on the combined evidence, such as the one proposed in this work, paves the way to a more nuanced modeling of decision-making, recommendation, and discussion processes.

6. Conclusions and Future Work

We introduced a qualitative logic for comparing strengths of belief and certainty of sets of evidence explicitly, providing a modal logic that distinguishes these two comparative notions both syntactically and semantically. Our bi-modal logic extends existing modal logics of belief functions by allowing for two distinct forms of comparison of proposition: (i) based on degrees of belief and (ii) a novel evidence-based comparison that captures certainty domination among sets of evidence. We established key validities and invalidities of the logic.

There remains several directions for future research. An important open question is the sound and complete axiomatization of the proposed logic. Variations of the proposed models and their logics, where the belief functions are not necessarily separable support functions or are defined based on different notions of evidence, such as combined evidence, argument, justification, afforded by the topological models of evidence [3, 5, 4] and explored in our previous work via the so-called multi-layer belief model [18] are topics of current on-going research.

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Declaration on Generative AI

During the preparation of this work, the author(s) used DeepL Write for grammar and spelling check. After using these tool(s)/service(s), the author(s) reviewed and edited the content as needed and take(s) full responsibility for the publication's content.

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