Computational modeling of nonlinear processes in thin elastic plates subjected to electromagnetic fields*

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Abstract

This paper presents a comprehensive study on the mathematical modeling of nonlinear processes in electromagnetic elastic thin plates of complex configuration. The work addresses the coupled interaction between electromagnetic and elastic fields within thin structural elements, considering nonlinear material behavior and intricate boundary geometries. A system of partial differential equations is derived using Hamilton's variational principle, incorporating both mechanical and electromagnetic energy contributions. Analytical techniques and numerical simulations are utilized to examine stress distributions, deformation patterns, and field interactions under various boundary and loading conditions. The research offers insights applicable to smart materials, MEMS devices, and adaptive structures operating under multiphysical environments.

Keywords

Mathematical modeling, Hamilton variation principle, Maxwell's electromagnetic tensor, magneto elastic thin plate of complex configuration.

1. Introduction

In the modern era of science and technology, the mathematical modeling of complex physical phenomena has become essential for the analysis, design, and optimization of advanced engineering systems. One such class of problems involves nonlinear processes in electromagnetic elastic thin plates—structural elements that are simultaneously subjected to mechanical, electromagnetic, and often thermal effects. These thin plates are integral to aerospace, micro-electromechanical systems (MEMS), piezoelectric sensors, and flexible electronics.

When these plates are of complex geometrical configuration, additional challenges emerge, such as the appearance of stress singularities, boundary layer effects, and difficulties in defining appropriate boundary conditions. Moreover, nonlinear material behavior and coupled field interactions between mechanical and electromagnetic domains necessitate the development of refined mathematical models that go beyond classical linear elasticity.

The present study focuses on the formulation and analysis of nonlinear coupled partial differential equations (PDEs) describing the behavior of electromagnetic elastic thin plates of arbitrary shape. Using variational principles such as Hamilton's principle, energy-based formulations, and finite element approximation methods, the paper aims to provide accurate representations of the physical behavior of such systems.

2. Related Work

This research is particularly relevant to the design of smart materials, adaptive structures, and active control mechanisms in intelligent systems. It contributes to the theoretical foundation necessary for developing software tools used in structural simulation under multiphysics conditions.

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A number of prominent scientists around the world have carried out investigations related to the theory of electrical conductivity in electromagnetic fields. Among them are S.A. Ambartsumyan, G.E. Bagdasaryan, M.V. Belubekyan, V.L. Rvachev, L.V. Kurpa, L.V. Molchenko, I.T. Selezov, and M.R. Korotkina, S.A. Nazarov and V.A. Kozlov, who have made substantial contributions in this field. In addition, such notable figures as Academician V.Q. Qobulov, Academician X.A. Rakhmatulin, Professor Sh.A. Nazirov, T. Yuldashev, R. Indiaminov, and F.M. Nuraliev have also conducted extensive research and attained noteworthy achievements in the study of electromagnetic field interactions with conductive media.

3. Methodology

During the research process, analytical and approximate computational methods (Bubnov–Galerkin, Iteration, Gauss–Quadrature, Gauss, Newmark), variational principles for mathematical and numerical modeling, iterative methods for solving equations, algorithm development techniques, and object-oriented programming technologies with relevant frameworks were utilized.

4. Development of a mathematical model

A mathematical formulation describing the nonlinear deformation behavior of a magnetoelastic plate is developed based on Hamilton's variational principle. The model incorporates the Kirchhoff-Love plate theory, Cauchy stress relations, the Lorentz force interpreted within the framework of Hooke's law, and utilizes Maxwell's formulation of the electromagnetic field tensor [1].

$$\int_t (\delta K - \delta \Pi + \delta A) dt = 0, \tag{1}$$

where: t - time, the product of variation, K – kinetic energy, Π– potential energy, A - the work done by external volumetric and surface forces.

In accordance with the Kirchhoff-Love hypothesis, deformation in the thickness direction (along the Z-axis) of a thin plate is neglected, and the displacement components of the mid-surface are represented as follows:

$$u_1 = u(x,y,t) - z\frac{\partial w}{\partial x}, u_2 = v(x,y,t) - z\frac{\partial w}{\partial x}, u_3 = w(x,y,t) \tag{2}$$

where: the displacement of the middle plane of the thin plate along the coordinate (x, y, z) axes.

In order to compute the variation of the potential energy, the geometrically nonlinear representation of the body's deformation is initially determined. In this process, the Cauchy strain tensor in conjunction with the Kirchhoff–Love hypothesis is employed to describe the nonlinear strain components, which are formulated as follows:

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial u}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{2}\dfrac{\partial w}{\partial x}\cdot\dfrac{\partial w}{\partial x}, \\ \varepsilon_{yy} = \dfrac{\partial v}{\partial y} - z\dfrac{\partial^2 w}{\partial y^2} + \dfrac{1}{2}\dfrac{\partial w}{\partial y}\cdot\dfrac{\partial w}{\partial y}, \\ \varepsilon_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} - 2z\dfrac{\partial^2 w}{\partial x \partial y} + \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial w}{\partial y}. \end{cases} \tag{3}$$

where the deformation coefficients of the plate [8].

Now, the work variations performed by kinetic energy, potential energy, and external forces are reduced to the Hamilton-Ostrogradsky variation principle (1), resulting in a special derivative equilibrium equation with initial and limit conditions.

$$
\begin{cases}
-\rho h\frac{\partial^2 u}{\partial t^2}+\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}+N_x+R_x+q_x+T_{zx}=0\\
-\rho h\frac{\partial^2 v}{\partial t^2}+\frac{\partial N_{yy}}{\partial y}+\frac{\partial N_{xy}}{\partial x}+N_y+R_y+q_y+T_{zy}=0\\
-\rho h\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+\\
+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}+N_{xy}\frac{\partial^2 w}{\partial x\partial y}+\left(\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\frac{\partial w}{\partial x}+\left(\frac{\partial N_{yy}}{\partial y}+\frac{\partial N_{xy}}{\partial x}\right)\frac{\partial w}{\partial y}+\\
+N_z+R_z+q_z+T_{zz}=0,
\end{cases}
\tag{4}
$$

where $X,Y,Z,\rho K_x,\rho K_y,\rho K_z$ - the generating volume forces, $q_x,q_y,q_z,T_{zx},T_{zy},T_{zz}$ – the surface forces, $P_x,P_y,P_z,T_{xx},T_{xy},T_{xz},F_x,F_y,F_z,T_{yy},T_{yz},T_{zx}$ - the generating contour forces. Initial conditions:

$$
\rho h\frac{\partial u}{\partial t}\delta u\Big|_t=0,\ \rho h\frac{\partial v}{\partial t}\delta v\Big|_t=0,\ \rho h\frac{\partial w}{\partial t}\delta w\Big|_t=0,
\tag{5}
$$

Natural limit conditions:

$$
\begin{cases}
(N_{xx}+N_{Px}+N_{Tx})\,\delta u\Big|_x=0,\left(N_{xy}+N_{Py}+N_{Txy}\right)\delta v\Big|_x=0,\\
M_{xx}\delta\frac{\partial w}{\partial x}\Big|_x=0, M_{xy}\delta\frac{\partial w}{\partial y}\Big|_x=0,\\
\left(N_{yy}+N_{Fy}+N_{Tyy}\right)\delta\, v\Big|_y=0,\left(N_{xy}+N_{Fx}+N_{Tyx}\right)\delta\, u\Big|_y=0,\\
M_{yy}\delta\frac{\partial w}{\partial y}\Big|_y=0, M_{xy}\delta\frac{\partial w}{\partial x}\Big|_y=0,\\
\left[N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}-\frac{\partial M_{xx}}{\partial x}-\frac{\partial M_{xy}}{\partial y}+N_{Pz}+N_{Txz}\right]\delta\, w\Big|_x=0,\\
\left[N_{yy}\frac{\partial w}{\partial y}+N_{xy}\frac{\partial w}{\partial x}-\frac{\partial M_{yy}}{\partial y}-\frac{\partial M_{xy}}{\partial x}+N_{Fz}+N_{Tyz}\right]\delta\, w\Big|_y=0.
\end{cases}
\tag{6}
$$

5. Computational algorithm of numerical solution of the problem

An algorithm for computing the geometrically nonlinear deformation behavior of electromagnetic thin plates [4] includes the following steps:

1. Formulation of solution functions that satisfy the prescribed boundary conditions
2. Discretization of the governing equations with respect to spatial coordinates
3. Solving the resulting discrete system to determine the unknown variables within the solution framework.
4. Evaluation of the normal (transverse) displacements of the plate’s mid-surface.

To compute the unknown variables in the motion equation based on the proposed algorithm, the displacement coefficients are determined by employing a combination of methods, including the variational Bubnov-Galerkin approach, Gaussian quadrature, the Newmark integration scheme, and iterative solution techniques.

The limit equation of a magnetoelastic plate of complex configuration is constructed by the R-function [3].

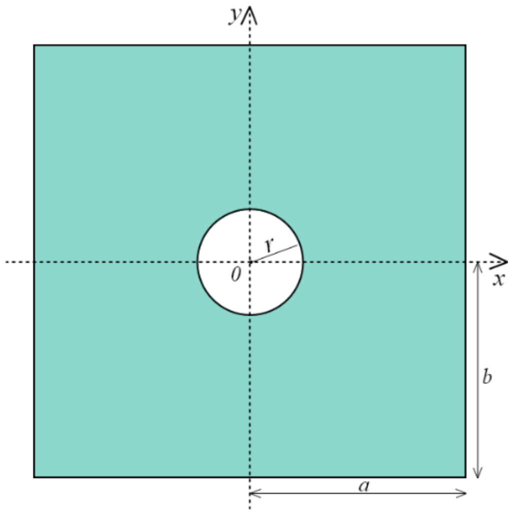

Figure 1: Magnetic plate with complex configuration

$$\mathrm{f}_1 = \frac{(\mathrm{a}^2 - \mathrm{x}^2)}{2\mathrm{a}} \geq 0, \mathrm{f}_2 = \frac{(\mathrm{b}^2 - \mathrm{y}^2)}{2\mathrm{b}} \geq 0, \tag{7}$$

$$f_3 = \frac{(x^2 + y^2 - r^2)}{2r} \geq 0, \omega = (f_1 \wedge f_2) \wedge f_3. \tag{8}$$

where f_1, f_2, f_3- the functions representing the field. ω - a normalized function representing the field.

$$u\Big|_{\Gamma} = 0,\ v\Big|_{\Gamma} = 0,\ w\Big|_{\Gamma} = 0, \frac{\partial w}{\partial n}\Big|_{\Gamma} = 0, \tag{9}$$

where n - the external normal falling on the surface of the plate, Γ - the limit of the sphere. The structure of solutions built according to a given limit condition [4]:

$$u{=}\omega\Phi_1,\ v{=}\omega\Phi_2,\ w{=}\omega^2\Phi_3. \tag{10}$$

Limit conditions for rigidly configured plate configurations [5]:

Φ_1, Φ_2, Φ_3 - unknown components.

$$\Phi_1 = \sum_{i=1}^{N} c_i(t)\phi_i(x,y),\ \Phi_2 = \sum_{j=1}^{N} c_j(t)\phi_j(x,y), \tag{12}$$

$$\Phi_3 = \sum_{k=1}^{N} c_k(t)\chi_k(x,y),\ u = \omega \sum_{i=1}^{N} c_i(t)\varphi_i(x,y), \tag{13}$$

$$v = \omega \sum_{j=1}^{N} c_j(t)\phi_j(x,y),\ w = \omega^2 \sum_{k=1}^{N} c_k(t)\chi_k(x,y). \tag{14}$$

where c_i - the unknown coefficients of the structure of the solutions, and $\varphi_i, \phi_j, \chi_k$ - some known polynomial functions [6].

The iteration method is used to find the displacements $u_i(x,y,t), v_i(x,y,t), w_i(x,y,t)$ of the middle surface of the plate Y from the system of equations formed, and a numerical solution is obtained [8].

The effect of electromagnetic field forces on the process of geometric nonlinear deformation of a thin plate was analyzed by obtaining numerical values of $w_i(x, y, t)$ the displacement function when the thin plate is exposed to electromagnetic field forces and without taking into account the effect of field forces (Table 1). At the same time, the difference between the maximum displacement points was 19%. This graph is shown through Figure 2 [7].

Table 1
Numerical results of the deflection of a thin plate with a complex structural shape along the OZ coordinate axis

x	The effect of electromagnetic field forces has not been considered	When exposed to electromagnetic field forces
-1	0	0
-0.9	0.000013	0.00002
-0.8	0.00010	0.000143
-0.7	0.000228	0.000309
-0.6	0.000285	0.000373
-0.5	0.000232	0.000295
-0.4	0.000119	0.000149
-0.3	0.000028	0.000035
-0.2	0	0
0.2	0	0
0.3	0.000028	0.000035
0.4	0.000119	0.000149
0.5	0.000232	0.000295
0.6	0.000285	0.000373
0.7	0.000228	0.000309
0.8	0.00010	0.000143
0.9	0.000013	0.00002
1	0	0

In the subsequent stage of the research, a numerical investigation was conducted to examine the influence of the thickness parameter h on the deformation behavior of a thin plate with a complex geometric configuration. The simulation results, along with the corresponding graphical illustration (see Figure 3), clearly demonstrate that as the plate thickness h decreases, the magnitude of bending deformation significantly increases.

This behavior is consistent with the physical understanding that thinner plates exhibit greater flexibility and are more susceptible to deflection under external loads and field influences. The computational analysis confirms that geometric nonlinearity becomes more pronounced in thinner structures due to reduced structural stiffness.

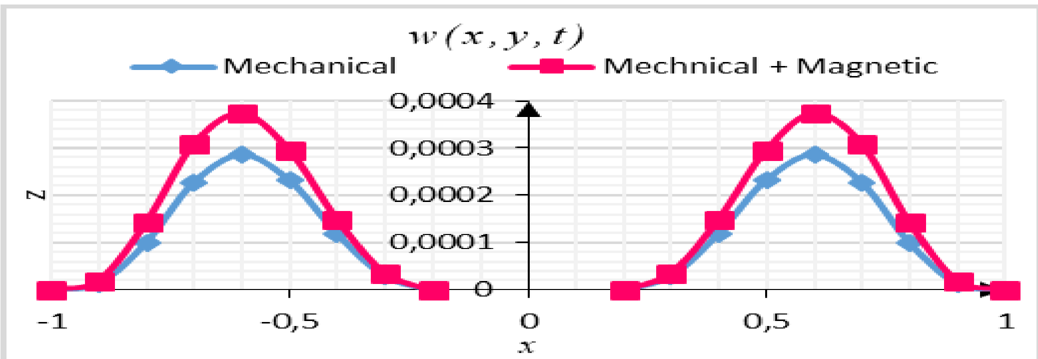

Figure 1: The effect of electromagnetic field forces on a thin plate of complex shape.

The geometric and mechanical parameters utilized in this experiment — including plate dimensions, material properties, boundary conditions, and electromagnetic field characteristics — are systematically presented below as part of the modeling setup for transparency and reproducibility of the results.

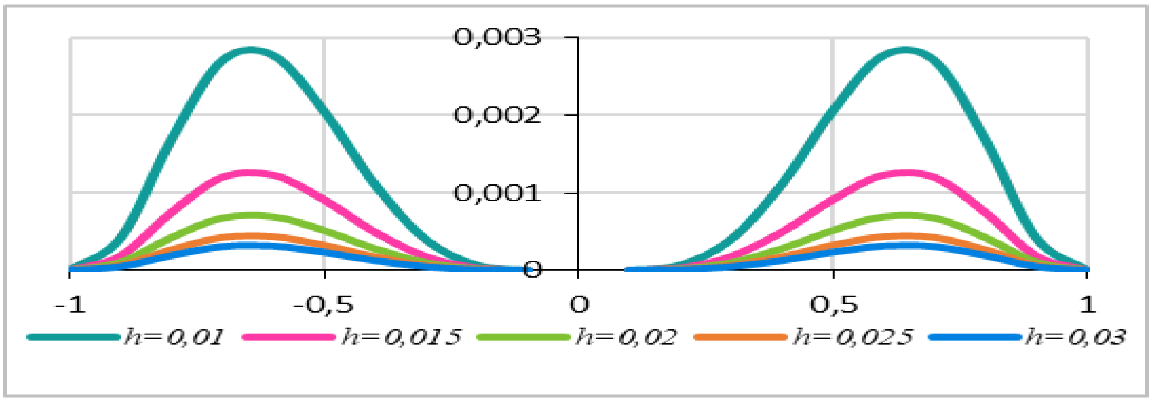

Figure 3: Experiment on plate thickness.

Numerical simulations and corresponding graphical illustrations (Figure 4) were carried out to investigate the effect of the inner radius *r* on the deformation behavior of a thin plate with a finely detailed complex geometry (see Figure 1). The computational experiments revealed that as the internal cut-out radius *r* decreases, the bending deformation of the plate under external loading conditions increases significantly [9].

This outcome highlights the sensitivity of geometrically intricate thin plates to changes in internal geometry, indicating that smaller internal radii lead to reduced structural stiffness and greater deflection. The results provide important insights into the mechanical behavior of thin-walled magnetoelastic structures with cut-outs or openings.

The deformation behavior of a magnetoelastic thin plate with a complex geometry was analyzed under the influence of a second symmetric electromagnetic field configuration, as illustrated in Figure 5. Within this framework, comprehensive numerical simulations were carried out to evaluate the plate's nonlinear response to the applied field conditions.

As a result of the computations, quantitative data were compiled and presented in Table 2, while the corresponding deformation patterns and trends were visualized graphically in Figure 5. These findings provide valuable insight into the structural response characteristics of magnetoelastic plates subjected to symmetric field distributions, and they serve as a basis for validating the effectiveness of the proposed modeling approach [10].

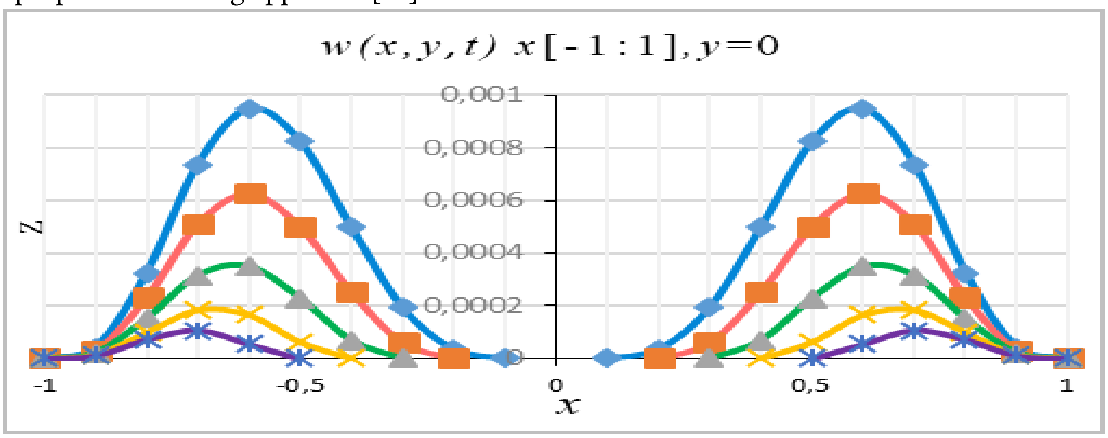

Figure 4: Numerical experiment illustrating the influence of internal and external radii on a complex-shaped plate

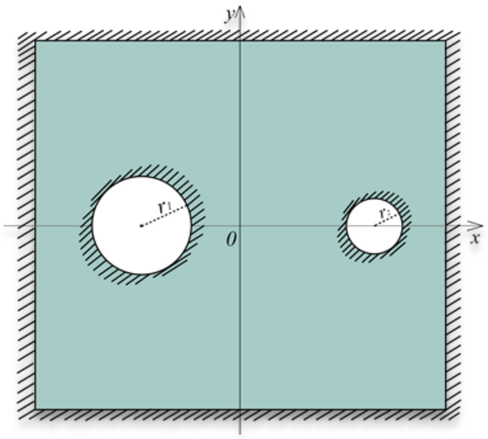

Figure 5: Magnetoelastic plate possessing an asymmetrical and geometrically complex structure

An analytical expression was derived for the applied asymmetrical complex field defined by the R-function method (see Figure 5), and it is represented by Equation (4). The resulting formulation was used to analyze the bending behavior of a magnetoelastic plate with an asymmetric and complex configuration. The corresponding bending distribution along the coordinate axis is illustrated in Figure 6, which visually demonstrates the deformation characteristics under the influence of the given field.

$$\omega = (f_1 \wedge f_2) \wedge f_3 \wedge f_4, \tag{15}$$

where

$$\mathrm{f}_1 = \frac{(\mathrm{a}^2 - \mathrm{x}^2)}{2\mathrm{a}} \geq 0, \mathrm{f}_2 = \frac{(\mathrm{b}^2 - \mathrm{y}^2)}{2\mathrm{b}} \geq 0, \tag{16}$$

$$f_3 = \frac{((x-a_1)^2+y^2-r^2)}{2r} \geq 0, f_4 = \frac{((x+a_2)^2+y^2-r^2)}{2r}. \tag{17}$$

Geometric-mechanical parameters in computational experiments [11]:

$$a = 1, b = 1, a_1 = 0.5, a_6 = 0.55, h = 0.01, r_1 = 0.1, r_2 = 0.1,$$
$$v = 0.3, q = 1, H_x = H_y = H_z = 10к Э, E = 10^{11} H/m^2$$

Figure 5: presents the conical deformation results corresponding to the bending behavior along the axis of a magnetoelastic plate with complex geometry. Additionally, a graphical representation of these results is provided in Figure 7, offering a visual interpretation of the deformation patterns observed under the given loading and field conditions.

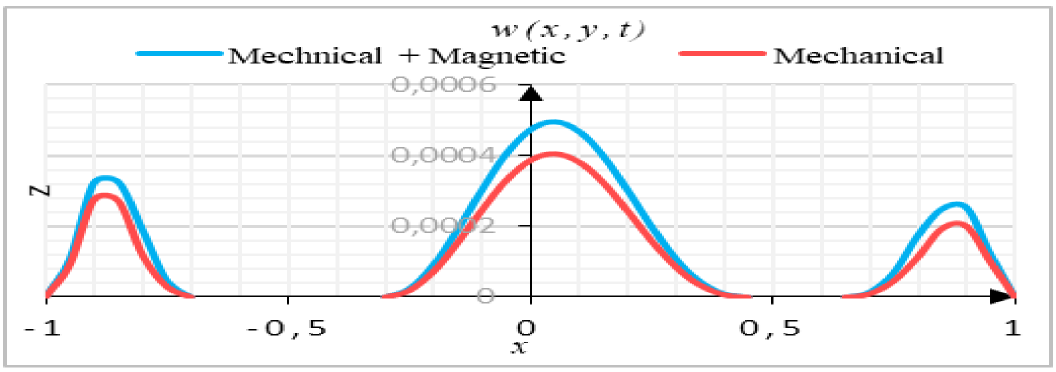

Figure 6: Bending behavior of a magnetoelastic thin plate with an asymmetrical complex geometry

6. Conclusions

This study investigates the geometrically nonlinear deformation processes of thin plates with complex configurations under the influence of electromagnetic field forces. Special attention is given to analyzing how the electromagnetic field affects the nonlinear deformation and stress state of magnetoelastic plates with nontrivial geometry.

A new mathematical model has been developed to describe the vibration behavior of thin plates with structurally complex shapes immersed in an electromagnetic field. Furthermore, computational algorithms have been formulated to solve the governing equations efficiently. Based on these methods, a dedicated software tool was created to conduct numerical experiments.

During the modeling process, two types of complex-shaped plates—one symmetric and one asymmetric—were constructed to test the methodology. The corresponding numerical results were presented in tabular form, accompanied by graphical visualizations to facilitate interpretation and analysis. The outcomes of this research provide a robust foundation for solving similar coupled nonlinear problems in the future and can be applied in engineering practice for the analysis of smart materials and multifunctional structural systems.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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