Computation and Size of Interpolants for Hybrid Modal Logics (Extended Abstract)*

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Abstract

Recent research has established complexity results for the problem of deciding the existence of interpolants in logics lacking the Craig interpolation property (CIP). The proof techniques developed so far are non-constructive, and no meaningful bounds on the size of interpolants are known. Hybrid modal logics (or modal logics with nominals) are a particularly interesting class of logics without CIP: in their case, CIP cannot be restored without sacrificing decidability and, in applications, interpolants in these logics can serve as definite descriptions and separators between positive and negative data examples in description logic knowledge bases. In this contribution we show, using a new hypermosaic elimination technique, that in many standard hybrid modal logics Craig interpolants can be computed in fourfold exponential time, if they exist. On the other hand, we show that the existence of uniform interpolants is undecidable, which is in stark contrast to modal or intuitionistic logic where uniform interpolants always exist.

Keywords

Interpolation, Modal Logic, Hybrid Logic, Separator Computation

1. Introduction

Consider formulae φ and ψ such that $\varphi \to \psi$ is valid. Then a *Craig interpolant* for φ and ψ is any formula χ using only the shared non-logical symbols of φ and ψ such that both $\varphi \to \chi$ and $\chi \to \psi$ are valid. The formula χ often provides a useful explanation of why $\varphi \to \psi$ is valid. Craig interpolants have found many applications, including the inference of loop invariants and program abstractions in program verification [2], query rewriting in databases [3, 4], and the construction of separators for positive and negative data examples in concept learning [5]. In all of these settings, it is crucial that interpolants can be computed efficiently, given φ and ψ.

A logic is said to enjoy the Craig interpolation property (CIP) if a Craig interpolant exists whenever $\varphi \to \psi$ is valid. In such cases, interpolants can often be extracted directly from proofs of $\varphi \to \psi$ in an appropriate proof system [6, 7, 8]. By contrast, if a logic does not have CIP, then the existence of a proof does not guarantee the existence of an interpolant, and it remains unclear how interpolants could be extracted even when they do exist. Although recent work has made significant progress on the problem of deciding whether interpolants exist in logics without CIP, very little has been achieved on the construction of interpolants or on estimating their size. This problem arises because the algorithms developed so far are non-constructive: rather than providing explicit interpolants, they rely on variants of Robinson's joint consistency, which in turn rely on compactness.

The aim of this paper is to establish the first elementary upper bounds for constructing Craig interpolants in modal and description logics that lack CIP. We focus on the family of decidable hybrid modal logics, or equivalently, description logics with nominals. The computation of Craig interpolants in these logics is particularly significant for two reasons.

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First, in hybrid modal logics, CIP cannot be restored merely by adding further logical connectives without sacrificing decidability [9]. Consequently, the natural strategy of repairing CIP by moving to a more expressive logic is effectively ruled out, at least for applications where effective reasoning is essential.

Second, when hybrid modal logics are viewed as description logics with nominals, Craig interpolants have numerous potential applications in knowledge representation and reasoning [10]. They can explain modularity violations between ontologies or knowledge bases [11, 12], act as separators between positive and negative data examples in description logic knowledge bases [5, 13], provide definitions of concept names [8], enable alignments between ontologies [14], and supply definite descriptions (or referring expressions) for individuals in description logic knowledge bases [15].

We now present our main contributions. We study hybrid modal logic $\mathcal{H}$, which extends modal logic with *nominals* interpreted as singleton sets, together with its extensions $\mathcal{H}_{@}$ and $\mathcal{H}_{@,U}$, incorporating the *@-operator*, $@_a\varphi$, stating that φ is true in the node denoted by nominal a, and the *universal modality*, $[U]\varphi$, stating that φ is true everywhere, respectively. In addition, we examine the further extensions of these logics with *graded modalities* (also known as qualified number restrictions in description logics) [16, 17]. Let HL and GHL denote the respective collections of these logics.

For logics in $\mathsf{HL} \cup \mathsf{GHL}$, satisfiability is PSpace-complete in the absence of the universal modality, and ExpTime-complete when the universal modality is present [18, 19, 20]. No logic in $\mathsf{HL} \cup \mathsf{GHL}$ enjoys the CIP, as witnessed by the validity of $(a \wedge \langle R \rangle a) \rightarrow (b \rightarrow \langle R \rangle b)$, with a, b nominals, for which no interpolant exists.

Interpolant existence has previously been investigated [13], where it was shown to be coNExpTime-complete for $\mathcal{H}$ and 2ExpTime-complete for $\mathcal{H}_{@,U}$. Building on this, one can easily show that for all logics in $\mathsf{HL} \cup \mathsf{GHL}$ interpolant existence is coNExpTime-complete in the absence of the universal modality and 2ExpTime-complete otherwise. Our interest here is not in deciding the existence of interpolants but in understanding their complexity. We state our main result which gives an elementary upper bound on the size of Craig interpolants.

Theorem 1. *Let $\mathcal{L} \in \mathsf{HL} \cup \mathsf{GHL}$. Given φ, ψ in $\mathcal{L}$ such that a Craig interpolant for $\varphi \rightarrow \psi$ exists, one can construct such an interpolant in fourfold exponential time (and of fourfold exponential size) in the size of φ and ψ.*

Before turning to the proof, we note that a standard strategy for bounding the size of interpolants for $\varphi \rightarrow \psi$, namely, bounding the size of the logically strongest consequence of φ over the symbols shared by φ and ψ, is not applicable here, as there is no elementary upper bound on the size of this consequence.

Example 1. *Let $n \geq 2$ and $\sigma = \{R, S, p\}$ with R and S modal accessibility relations and p a propositional variable. Let*

$$\varphi_n = [R]a \wedge [S]a \wedge \bigwedge_{1 \leq i \leq n-1} [R]^i[S]\bot \wedge [R]^n\bot \wedge [S]^2\bot$$

with a a nominal. Then, up to logical equivalence, there are only finitely many formulae in $\mathcal{H}$ over σ that are consistent with φ_n. Their disjunction, χ_n, is the logically strongest consequence of φ_n in $\mathcal{H}$ using symbols in σ only. Hence, if there is an interpolant between φ_n and some formula ψ with shared symbols in σ, then χ_n is such an interpolant. However, using the fact that $\varphi_n \rightarrow (\langle R \rangle \chi \rightarrow \langle S \rangle \chi)$ is valid for all formulae χ one can easily show that there is no elementary upper bound on the size of formulae in $\mathcal{H}$ equivalent to χ_n. □

Instead, our proof of Theorem 1 builds on the following two ideas. Given formulae φ and ψ sharing a set of non-logical symbols σ:

- **(bisim)** for modal and guarded logics, the non-existence of an interpolant for $\varphi \rightarrow \psi$ is often equivalent to the existence of σ-bisimilar models satisfying φ and $\neg\psi$ [21, 22];

(elim) for modal and guarded logics with CIP, interpolants can sometimes be constructed using the standard type elimination procedure, a technique commonly employed to establish complexity upper bounds for satisfiability [23, 24]. The procedure eliminates pairs of types that cannot be jointly satisfied, and inductively computes interpolants between them.

We illustrate (elim) by showing that one can always construct at most exponential size interpolants for $\varphi \rightarrow \psi$ in $\mathcal{H}$ *if* all nominals in φ, ψ are shared. To apply the elimination procedure to logics without the CIP, it must be lifted to *mosaics*, pairs of sets of types rather than pairs of individual types. In this form, the method (elim) has already been successfully employed to obtain tight complexity bounds for the existence of interpolants in a variety of logics lacking the CIP [25, 13, 26]. The underlying idea is that sets of types in a mosaic represent conceivable types of points from a bisimilarity class, which may contain more than two elements.[1]

The transition from pairs of types to mosaics, however, comes at a cost. Making mosaic elimination constructive is substantially more difficult, and has so far been achieved only in very restricted settings [27]. We argue that for hybrid modal logics, mosaic elimination is actually too local to compute interpolants at each elimination step. To capture global interactions between mosaics, we extend the elimination method further by eliminating unrealizable *sets* of mosaics, called hypermosaics.

It remains open whether the fourfold exponential upper bound is tight. However, we show a triply exponential (doubly exponential for DAG representation) lower bound on the size of Craig interpolants for languages in $\mathsf{HL} \cup \mathsf{GHL}$ with universal modality. Thus, interpolants are at least one exponential larger than in modal logic.

Finally, we investigate uniform interpolants in hybrid modal logics. In description logic, uniform interpolants have been extensively investigated and applied to forget symbols in ontologies or knowledge bases [28, 29]. Here we further demonstrate the complexity of constructing interpolants in hybrid modal logics by proving that the existence of uniform interpolants for a formula and set of non-logical symbols is undecidable. We show this for a broader class of logics, which in particular subsumes $\mathsf{HL} \cup \mathsf{GHL}$.

Theorem 2. *Let $\mathcal{L}$ be a logic containing $\mathcal{H}$ and contained in* FO. *The existence of a uniform $\mathcal{L}(\sigma)$-interpolant for given $\varphi \in \mathcal{L}$ and signature σ is undecidable.*

This result is in stark contrast to modal and intuitionistic logic, where uniform interpolants always exist [30, 31] and can, at least for modal logic, even be constructed in exponential time [32].

Related Work. We discuss related work not yet mentioned above. An overview of what is currently known about the size of (uniform) interpolants in modal logic is given in [33]. The CIP has been investigated extensively for hybrid modal logics and corresponding description logics with nominals. The main positive results are that by adding the binding operator $\downarrow$ to $\mathcal{H}(@)$ one obtains a logic with CIP (also if restricted to popular classes of frames) [34], and that CIP holds for languages in HL if arbitrary nominals can be used in interpolants (called CIP for propositional variables) [35, 36]. However, satisfiability, and hence also interpolant existence, are undecidable for $\mathcal{H}(\downarrow, @)$. This makes its use in knowledge representation applications problematic. Likewise, allowing arbitrary nominals in interpolants undermines the very purpose of interpolants in description logic. The decision problem of whether an interpolant exists for a given implication in hybrid modal logics lacking CIP is studied in [13]. The principal question left open there—how to effectively construct interpolants when they do exist, and what bounds can be established on their size—is the focus of the present contribution.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

[1]We note an unfortunate clash of terminology: in [23], mosaics are a generalization of types tailored to guarded logics, and are always realized within a single model. In [25, 13] and our work, mosaics represent sets of types realized by bisimilar nodes.

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