Craig Interpolation Theorem in the Logic of Russellian Definite Descriptions

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Abstract

In this paper we focus on the intersection of two important results: Russellian theory of definite descriptions (RDD) and Craig interpolation theorem (CIT). RDD provides the most recognisable approach to formalisation of complex terms. CIT is one of the most important metalogical results hence it is crucial to show that it holds for a theory like RDD. One of the possible ways of proof-theoretic characterisation of RDD was provided by means of a cut-free sequent calculus satisfying the subformula property. In the paper we apply this system to obtain a Maehara-style constructive proof of CIT for RDD.

Keywords

Definite Descriptions, Russell, Craig, Maehara, Sequent Calculus, Interpolation Theorem

1. Introduction

In formal languages terms are usually treated as the elements of language which only refer to the objects in the domain of discourse. In contrast, in natural languages, naming expressions are used very often not only for referring but also for conveying information. One of the most important class of such expressions are *definite descriptions*, expressions of the form 'the so and so', such as 'the author of «On Denoting»'. Definite descriptions are ubiquitous not only in natural languages but also in mathematics and science ('the sum of 7 and 5', 'the square root of n'). A definite description (DD) aims to denote an individual object by virtue of a property that only that object possess, and if it succeeds it is called proper. Otherwise – if nothing or more than one thing satisfies it – it is called improper. In artificial languages the iota-operator $\imath$, due to Peano, is usually applied to formalise definite descriptions: 'the φ' is represented formally as $\imath x\varphi$.

The applicability of formal devices like the $\imath$ operator and other term-forming operators was for many years rather underestimated. But recently their usefulness has been better recognised in proof theory and automated deduction. In particular, some natural deduction, sequent and tableau systems have been formulated with suitably constructed sets of rules for DDs, usually in some kind of free logic. One may mention here for example natural deduction systems of Tennant [39] and Kürbis [27], sequent systems proposed by Orlandelli [35], Kürbis [28] and Indrzejczak [16, 17] or tableau calculi due to Indrzejczak and Zawidzki [24, 25]. In automated deduction the usefulness of the application of DDs instead of functional terms was also noticed. One may mention *KeYamera* X theorem prover [7] for differential dynamic logic, or the work by Oppenheimer and Zalta who used a free logic with DDs to formalise Anselm's ontological argument in *PROVER9* [34], or Scott and Benzmüller's implementation of free logic using proof assistant *Isabelle/HOL* [5] (applied also by Blumson to encode Oppenheimer and Zalta's formalisation of Anselm's ontological argument [6]). Interesting applications of DDs are also in works on description logics (see e.g. [2], [32]).

In this paper we focus on the Russellian theory of DD ([36] and [41]), briefly called RDD, which plays a central role in this area. Despite some weak points of RDD it is still a standard point of reference of almost all works devoted to the analysis of definite descriptions. Moreover, it is still widely accepted by formal logicians as a proper way of handling descriptions; the great number of textbooks that use it

CI-BD-SOQE 2026: Workshop on Craig Interpolation, Beth Definability, and Second-Order Quantifier Elimination, at FLoC 2026: The 9th Federated Logic Conference, July 24–25, 2026, Lisbon, Portugal
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as their official theory of definite descriptions count as witnesses for this claim. Russell's theory has also strong affinities to logics closely connected with applications in constructive mathematics and computer science like the logic of the existence predicate by Scott [37] or the definedness logic (or the logic of partial terms) of Beeson [4] and Feferman. [12]. These connections were elaborated in [18].

This paper is not a place to discuss the advantages and drawbacks of the Russellian approach to DD. Our aim is to prove that the Craig interpolation theorem (CIT), originally proved by Craig for classical first-order logic (FOL), holds for RDD. Since this theorems, as well as their consequences like Beth definability theorem or Robinson's joint consistency theorem, belong to the most significant results in logic, it is important to show that it holds for RDD. Satisfaction of interpolation property is usually treated as a witness of a well-behaved logic. Moreover, Beth definability shows that the syntax and semantics of a logic are well-balanced; what is implicitly determined by semantics, can be explicitly defined syntactically. The failure of interpolation for several non-classical logics is a sign of their problems with expressivity, but sometimes it is possible to fix the situation by extending the language (see e.g. Areces, Blackburn and Marx [1], Indrzejczak and Zawidzki [25] or Indrzejczak [21]). Additionally, there is a close relationship of interpolation with important algebraic properties, like amalgamation or filter extension (see e.g. Czelakowski [9] and [10]). Also in computer science the interpolation theorems found important applications in restricting the proof-search space in automated deduction and verification (cf. D'Silva, Kroening and Wessenbacher [11], McMillan [31] or Bonacina and Johansson [8]). Hence, it is not surprising that several attempts were made to prove their counterparts for the extensions of classical logic and several non-classical logics (e.g. Fitting [13], Gherardi, Maffezzioli and Orlandelli [15], Maffezzioli and Orlandelli [30], Muskens and Wintein [42], Indrzejczak and Petrukhin [23], Indrzejczak [22]).

Interpolation theorems can be proved either by means of model-theoretic methods, or constructively, by means of proof-theoretic methods. Even if we restrict attention to constructive proof-theoretic approaches one may find several techniques which were applied for this aim (a useful comparison may be found in [19]). However, one of the most general, useful and popular method, in particular in the setting of sequent calculus, is due to Maehara [29] (see Takeuti [38], Ono [33] or Troelstra and Schwichtenberg [40]). In this paper we follow this tradition and apply Maehara's method to obtain the Craig's interpolation theorem for RDD.

One should note that what we call simply RDD can be made precise in several ways. In the original version contextual definitions provided by Russell are supposed to eliminate DDs from the formal discourse, showing that their treatment as names in natural languages is based on the wrong analysis. However, the rules of elimination are not without their own problems and it is possible to treat Russellian principles as a kind of axioms characterising DDs rather, not ways of their elimination from the discourse. So we can save the spirit of the Russellian analysis and save the status of DDs as genuine names. We follow here a formal characterisation of such an approach provided by Kalish, Montague and Mar [26] and later developed in terms of sequent calculus in [20]. It uses a standard form of first-order language with only iota-operator added. An alternative formalisation may be based on the additional introduction of lambda operator which avoids the scoping difficulties of the original RDD. The earlier approaches to combination of lambda- and iota-operator were provided in Fitting and Mendelsohn [14]. The formalisation of RDD in this fashion, with alternative proof of the interpolation theorem was provided by Indrzejczak and Zawidzki [25]. Here we focus on the examination of RDD following the lines of [26] and [20]. In section 2 we briefly recall the essential features of this approach and then in section 3 we present the sequent calculus for RDD. The main result of this paper, the proof of the interpolation theorem, is presented in section 4. In the conclusion we point out that the result can be extended to intuitionistic variant of RDD.

2. Preliminaries

We will be using a standard first-order predicate language with quantifiers $\forall, \exists$, identity predicate $=$ and iota-operator $\imath$ forming definite descriptions from formulae of the language. In accordance with

Gentzen's custom we divide individual variables into bound $VAR = \{x, y, z, \ldots\}$ and free variables (parameters) $PAR = \{a, b, c, \ldots\}$. It makes easier an elaboration of some technical issues concerning substitution and proof transformations. In the metalanguage φ, ψ, χ denote any formulae and $\Gamma, \Delta, \Pi, \Sigma$ their multisets.

The definition of a term and formula is standard, by simultaneous recursion on both categories. In the presented system the only terms are variables and definite descriptions since functions may be always represented as descriptions. Following Russell, we also do not consider individual constants although the system from [26] has them. In fact, the system which we present in section 3 can be easily extended to cover constants so it is not necessary to consider them separately. Descriptions are written as $\imath x\varphi$ where φ is a formula in the scope of respective operator. Metavariables $t, t_1, \ldots$ denote arbitrary terms, moreover, we will use a metavariable d to denote any description if its structure is not essential in the context. $\varphi[t_1/t_2]$ is officially used for the operation of correct substitution of a term t_2 for all occurrences of a term t_1 (a variable or parameter) in φ, and similarly $\Gamma[t_1/t_2]$ for a uniform substitution in all formulae in Γ. Note that $\varphi[x/b]$, for any b, is counted as a subformula of $\forall x\varphi, \exists x\varphi$ but also of $\imath x\varphi$. Also in order to save the complexity-reducing character of the notion of subformula we admit that $\varphi[x/b]$ is counted as a subformula of $\varphi[x/d]$ but $\varphi[x/d]$ is not counted as a subformula of $\varphi[x/b]$.

It will be useful to introduce several distinctions concerning formulae which are usually called atomic or elementary. We will keep the latter name for any formula which consists of a predicate (including =) with its arguments. We will call atomic only those elementary formulae where all arguments are variables. If at least one argument is a DD such a formula is quasi-atomic (q-atomic). So elementary formulae are divided into atomic and q-atomic. Moreover, if the predicate is not an identity, such a formula is proper atomic or proper q-atomic (both are called proper elementary). Quasi-atomic identities will be additionally divided into d-identities (both arguments are descriptions) and mixed identities with one argument a parameter and one a DD. Briefly:

(a) elementary formulae are either atomic (no DD) or q-atomic (with DD);

(b) formulae in both categories are proper (proper elementary, proper atomic, proper q-atomic), if their predicate is not an identity;

(c) identities (i.e. non-proper elementary formulae) are either atomic (no DD) or q-atomic, and the latter are either mixed identities (one argument DD) or d-identities (both arguments DD).

Before we introduce a sequent calculus for RDD a brief explanation of its essential features is necessary. Originally, Russell treated descriptions as a kind of incomplete signs and showed how to get rid of them by means of contextual definitions of the form:

$$\psi[x/\imath y\varphi] \ := \ \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$$

This way of doing things leads to scoping difficulties if ψ is not elementary. In fact it may even run into contradition if ψ is a valid formula but φ is a contradictory one. To avoid problems Russell [41] introduced additional devices allowing formal control over the scope of description. On the other hand, in [41] one may find several theorems formulated with descriptions occurring in term position and it seems at least reasonable to treat them as genuine terms in accordance with natural language practice and the earlier Fregean tradition. Kalish, Montague and Mar [26] proposed a natural deduction system characterising RDD with DD treated as the first-class citizens. This shows that it is possible to save some features of the Russellian approach to descriptions yet to treat them as genuine terms. In what follows we provide a sequent system which is equivalent to the system from [26] and in this sense provide a proof theoretic formalisation of RDD.

What features are central for saving the spirit of RDD, if we do not try to eliminate descriptions from every context but rather to develop a suitable logic of descriptions as terms? In the Russellian aproach they can be summarised in the following four points:

1. fixed evaluation (as false) of all elementary formulae with nondenoting terms;

2. quantification rules of universal instantiation UI and existential generalization EG restricted to variables;

3. identity with reflexivity restricted to variables;
4. taking object language counterpart of Russellian definition as an axiom characterising descriptions (we will call it R):

R $\psi[x/\imath y\varphi] \leftrightarrow \exists x(\forall y(\varphi \leftrightarrow y = x) \wedge \psi)$

where ψ must be elementary. This restriction is important since unrestricted version can lead to contradiction as we noticed above.

The first three features concern the background logic whereas the last one is a specific characterisation of descriptions. In fact the most important feature is expressed in point 1; points 2 and 3 are its consequences. 1 is expressed in [26] by the following two denotation principles:

$(DP\text{-}1)$ $R^n t_1 ... t_n \vdash \exists x x = t_1 \wedge ... \wedge \exists x x = t_n$
$(DP\text{-}2)$ $t_1 = t_2 \vdash \exists x x = t_1$

which claim that the atomic predicates (including identity) are strict in the sense that if elementary formulae are true, then all arguments must denote. Since we consider as terms only variables and descriptions, and variables denote by definition, it comes to the claim that all DDs must be proper.

Point 2 means that when a universal quantifier is eliminated we cannot unrestrictidly substitute quantified variable with an arbitrary term satisfying the conditions of proper substitution. In case of nondenoting term we can deduce a formula which is false because of point 1. Therefore the rules of universal elimination and existential generalization must be restricted to such terms of which it was earlier established that they denote. In [26] both rules are simply restricted to variables as instantiated terms since it is assumed that all variables are denoting. However, one may also easily show that the restricted rules of universal instantiation and existential generalization are sufficient to provide instantiation also for descriptions, if they are proper. Namely, we can prove that these two rules are derivable in KM:

$(\forall E_d)$ $\forall x\varphi, \exists x x = d \vdash \varphi[x/d]$
$(\exists I_d)$ $\varphi[x/d], \exists x x = d \vdash \exists x\varphi$

Similarly, point 3 states that if t is nondenoting, then even $t = t$ is false according to point 1. Therefore reflexivity must be restricted to denoting terms. On the other hand one may prove that $\exists x x = d \vdash d = d$.

Summing up these three features call for a logic which is currently identified as a kind of negative free logic which in [18] is called a quasi negative free logic since in free logics in the strict sense all kinds of terms may lack denotation, whereas here variables are always denoting. This kind of logic was investigated by Feferman [12] under the name definedness logic.

Point 4. is also sensitive to the choice of background logic. R cannot be just added to FOL with unrestricted UI and EG, since it leads to inconsistency. Restricting logic according to the points 1–3 makes possible to use R without problems. However, the fact that underlying logic is a negative free logic allows us to replace R with another formula which is easier to deal with. It is a well known Lambert axiom:

LA $\forall x(x = \imath y\varphi \leftrightarrow \forall y(\varphi \leftrightarrow y = x))$

LA and R are interderivable on the basis of the system from [26].

3. The Sequent Calculus for RDD

In [20] we have introduced several sequent formalisations of RDD. One of them, called there LKID2, was shown to be both cut-free and satisfying the subformula property. Here we introduce a calculus GRDD, a slightly modified version of LKID2. Sequents of the form $\Gamma \Rightarrow \Delta$ are ordered pairs of finite multisets of formulae called the antecedent and the succedent, respectively. The part of this calculus

$$(AX)\ \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad (\top)\ \Gamma \Rightarrow \Delta, \top$$

$$(Cut)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\bot)\ \bot, \Gamma \Rightarrow \Delta$$

$$(W\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow W)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(C\Rightarrow)\ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow C)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \qquad (\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta}$$

$$(\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}$$

$$(\leftrightarrow\Rightarrow)\ \frac{\Gamma\Rightarrow \Delta, \varphi, \psi \qquad \varphi, \psi, \Gamma\Rightarrow \Delta}{\varphi \leftrightarrow \psi, \Gamma\Rightarrow \Delta} \qquad (\forall\Rightarrow)\ \frac{\varphi[x/b], \Gamma\Rightarrow \Delta}{\forall x\varphi, \Gamma\Rightarrow \Delta}$$

$$(\Rightarrow\leftrightarrow)\ \frac{\varphi, \Gamma\Rightarrow \Delta, \psi \qquad \psi, \Gamma \Rightarrow \Delta, \varphi}{\Gamma\Rightarrow \Delta, \varphi \leftrightarrow \psi} \qquad (\Rightarrow\forall)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/a]}{\Gamma\Rightarrow \Delta, \forall x\varphi}$$

$$(\exists\Rightarrow)\ \frac{\varphi[x/a], \Gamma\Rightarrow \Delta}{\exists x\varphi, \Gamma\Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{\Gamma\Rightarrow \Delta, \varphi[x/b]}{\Gamma\Rightarrow \Delta, \exists x\varphi}$$

where a is a fresh parameter (eigenvariable), not present in Γ, Δ and φ.

Figure 1: Rules for pure FOL

presented in Fig. 1 is essentially the calculus G1 of Troelstra and Schwichtenberg [40] for pure (i.e with variables as the only terms) classical FOL. The set of rules displayed in Fig. 2 characterises identity and definite descriptions. $c \approx t$ in rules for $\imath$ means that either $c = t$ or $t = c$ (hence the schemata cover four rules in fact, assuming that c is in its three occurrences always either the left or the right argument of identity). What is specific in this set of rules is the separation of rules for different kinds of identities: atomic, d-identities and mixed identities. The additional rules $(Str \Rightarrow)$ and $(\Rightarrow Str)$ express the characteristic strictness conditions corresponding to denotation principles for predicates stated in the previous section. Note that all rules satisfy the generalised subformula property to the effect that in the premisses we have always the formulae and subformulae (in the sense specified in section 2) of formulae occurring in the conclusion, and possibly identities of the form $b = b$ or $c = d$, where d occurs in the conclusion and c may be a new parameter.

The definition of proofs is standard; they are built as trees with the root labelled with a proven sequent, leaves with axioms and remaining nodes regulated by the application of rules. The height of a proof of a given sequent is the maximal length of branches of its proof. We keep also standard terminology with respect to components of rules; in particular displayed formulae in the schemata of rules are active (principal in the conclusion and side-formulae in the premisses) and remaining ones are parametric and together make a context.

In GRDD there are only three minor modifications of LKID2: (1) the axiom (AX) has added context, (2) two axioms $(\top)$ and $(\bot)$ are added, (3) all context-free (multiplicative) multi-premiss rules are replaced with context-sharing (additive) variants. All these changes are introduced only for simplifying the proof of the interpolation theorem but save all essential features of LKID2, in particular the following:

$$(R)\ \frac{b=b,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\quad(E)\ \frac{b_1=b_2,\Gamma\Rightarrow\Delta}{c=b_1,c=b_2,\Gamma\Rightarrow\Delta}\quad(L)\ \frac{\varphi[x/b],\Gamma\Rightarrow\Delta}{c=b,\varphi[x/c],\Gamma\Rightarrow\Delta}$$

$$(\imath\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\varphi[x/b],c\approx b\qquad c\approx b,\varphi[x/b],\Gamma\Rightarrow\Delta}{c\approx\imath x\varphi,\Gamma\Rightarrow\Delta}$$

$$(\Rightarrow\imath)\ \frac{c\approx a,\Gamma\Rightarrow\Delta,\varphi[x/a]\qquad\varphi[x/a],\Gamma\Rightarrow\Delta,c\approx a}{\Gamma\Rightarrow\Delta,c\approx\imath x\varphi}$$

$$(=\Rightarrow)\ \frac{a_1=d_1,a_2=d_2,a_1=a_2,\Gamma\Rightarrow\Delta}{d_1=d_2,\Gamma\Rightarrow\Delta}$$

$$(\Rightarrow=)\ \frac{\Gamma\Rightarrow\Delta,b_1=d_1\qquad\Gamma\Rightarrow\Delta,b_2=d_2\qquad\Gamma\Rightarrow\Delta,b_1=b_2}{\Gamma\Rightarrow\Delta,d_1=d_2}$$

$$(Str\Rightarrow)\ \frac{a_1=d_1,\dots,a_n=d_n,\varphi[x_1/a_1\dots x_n/a_n],\Gamma\Rightarrow\Delta}{\varphi[x_1/d_1\dots x_n/d_n],\Gamma\Rightarrow\Delta}$$

$$(\Rightarrow Str)\ \frac{\Gamma\Rightarrow\Delta,b_1=d_1\qquad\dots\qquad\Gamma\Rightarrow\Delta,b_n=d_n\qquad\Gamma\Rightarrow\Delta,\varphi[x_1/b_1\dots x_n/b_n]}{\Gamma\Rightarrow\Delta,\varphi[x_1/d_1\dots x_n/d_n]}$$

where: a, a_1, a_2 are different eigenvariables, φ in (L) is a proper atomic formula, $\varphi[x_1/d_1 \dots x_n/d_n]$ in $(Str \Rightarrow)$ and $(\Rightarrow Str)$ is a proper quasi-atomic formula with exactly $n \geq 1$ different definite descriptions (with possibly many occurrences) as arguments and $\varphi[x_1/b_1 \dots x_n/b_n]$ is a proper atomic formula resulting by simultaneous substitution.

Figure 2: Rules for identity and descriptions

1. If $\vdash_k \Gamma \Rightarrow \Delta$, then $\vdash_k \Gamma[a/b] \Rightarrow \Delta[a/b]$, where k is the height of a proof.
2. If $\vdash \Gamma \Rightarrow \Delta$, then $\vdash \Gamma \Rightarrow \Delta$ is cut-free provable.
3. If $\vdash \Gamma \Rightarrow \Delta$ has a cut-free proof, then it satisfies the generalised subformula property.
4. If $\vdash \Gamma \Rightarrow \Delta$ has a cut-free proof, then the only terms occuring in it are those already occuring in Γ, Δ and eigenvariables.

The first three are familiar properties of (1) substitution, (2) cut elimination and (3) the subformula property. The last one follows from the second as can be easily checked by the inspection of rules. Substitution is an auxiliary result needed for proving (2); in particular it helps to change all proofs into regular proofs, in the sense that every parameter which is fresh by side condition on the respective rule must be fresh in the entire proof, not only on the branch where the application of this rule takes place. Thus no eigenvariable is used more than once in the whole regular proof which simplifies the cut elimination proof but also the proof of the interpolation theorem. In what follows we assume that we deal only with such proofs; there is no loss of generality since every proof may be systematically transformed into a regular one by the substitution lemma. (4) is something which may be called the subterm property and distinguishes GRDD from sequent calculi where functional terms are allowed and instantiated in the course of applying $(\forall \Rightarrow)$ and $(\Rightarrow \exists)$. In fact the restriction of terms instantiated in quantifier rules of GRDD is not a real restriction. One may easily prove that the following two rules are derivable:

$$(\forall_d\Rightarrow)\ \frac{\varphi[x/d],\Gamma\Rightarrow\Delta}{\forall x\varphi,\exists xx=d,\Gamma\Rightarrow\Delta}\qquad\qquad(\Rightarrow\exists_d)\ \frac{\Gamma\Rightarrow\Delta,\varphi[x/d]}{\exists xx=d,\Gamma\Rightarrow\Delta,\exists x\varphi}$$

Much more complicated (cf. [20]) is the demonstration that = occurring in different types of identities and characterised by 7 different rules is in fact the identity of the Russellian approach, i.e. with restricted reflexivity but with nonrestricted symmetry, transitivity and Leibniz Law. To this effect we can prove also derivability of the following rules:

$$(RR)\ \frac{d=d,\Gamma\Rightarrow\Delta}{\exists x x=d,\Gamma\Rightarrow\Delta}\quad (S)\ \frac{t_2=t_1,\Gamma\Rightarrow\Delta}{t_1=t_2,\Gamma\Rightarrow\Delta}\quad (T)\ \frac{t_1=t_3,\Gamma\Rightarrow\Delta}{t_1=t_2,t_2=t_3,\Gamma\Rightarrow\Delta}$$

$$(E')\ \frac{t_2=t_3,\Gamma\Rightarrow\Delta}{t_1=t_2,t_1=t_3,\Gamma\Rightarrow\Delta}\quad (LL)\ \frac{\varphi[x/t_2],\Gamma\Rightarrow\Delta}{t_1=t_2,\varphi[x/t_1],\Gamma\Rightarrow\Delta}$$

where t_i are arbitrary terms and φ is an arbitrary formula. Detailed proofs are in [20].

4. Interpolation

Let us recall that a logic L satisfies the *Craig interpolation property* if, for all formulae φ and ψ having common non-logical symbols (predicates and parameters), if $\models\varphi\rightarrow\psi$, then there is an interpolant formula, that is, a formula χ such that $\models\varphi\rightarrow\chi$, $\models\chi\rightarrow\psi$ and χ only contains these non-logical symbols which are common to both φ and ψ. Moreover, if φ and ψ have no non-logical symbols in common, then either $\neg\varphi$ is valid or ψ is valid.

We have mentioned that the most popular method of proving interpolation via sequent calculi is due to Maehara [29] and we follow this route. In fact the Maehara's method justifies a slightly generalised version of the interpolation theorem to the effect that if $\vdash\Gamma\Rightarrow\Delta$, then for any partition $((\Gamma_1,\Delta_1),(\Gamma_2,\Delta_2))$ of Γ,Δ we can construct an interpolant φ (often called a split-interpolant), such that:

1. $\vdash\Gamma_1\Rightarrow\Delta_1,\varphi$
2. $\vdash\varphi,\Gamma_2\Rightarrow\Delta_2$
3. $L(\varphi)\subseteq L(\Gamma_1\sqcup\Delta_1)\cap L(\Gamma_2\sqcup\Delta_2)$

where $\sqcup$ denote a union of multisets which is like ordinary set union $\cup$ but with additional counting of the number of occurrences of the same formula. $L(\Gamma)$ is the set of non-logical symbols (parameters and predicate symbols) in Γ. Conditions 1, 2 can be called provability criteria, and condition 3, a language condition for our interpolant. In what follows the situation where the principal formula of the considered rule belongs to $((\Gamma_1,\Delta_1)$ will be called the left division, and where it belongs to $(\Gamma_2,\Delta_2))$ – the right division.

It is obvious that Maehara's theorem implies Craig's theorem. Since by taking a partition $\Gamma_1:=[\varphi]$, $\Delta_2:=[\psi]$, $\Gamma_2=\Delta_1=\varnothing$ we derive from Maehara's theorem an interpolant χ, such that $\vdash\varphi\Rightarrow\chi$ and $\vdash\chi\Rightarrow\psi$.

A necessary condition for the successfull application of Maehara's procedure is that the sequent calculus satisfies the subformula property. Since our cut-free GRDD satisfies this condition we are free to examine if it satisfies the Maehara's interpolation theorem.

Theorem 4.1. *Cut-free GRDD satisfies Maehera's interpolation theorem.*

Proof. It goes by induction on the height of proof of the sequent. All the details of the proof concerning the axioms and rules for connectives and quantifiers are like in other presentations of this proof for FOL (see e.g. in [40]), it remains only to show that the specific rules for identity and descriptions from Fig. 2 preserve the feature that the interpolant for the conclusion can be computed on the basis of interpolants for premisses.

Consider the case of (R) and arbitrary partition of the conclusion into (Γ_1,Δ_1) and (Γ_2,Δ_2). By the induction hypothesis (IH) we have:

1. $\vdash b=b,\Gamma_1\Rightarrow\Delta_1,\varphi$

2. $\vdash \varphi, \Gamma_2 \Rightarrow \Delta_2$

3. $L(\varphi) \subseteq L(\{b = b\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

with an interpolant φ. If b is not present in φ, then φ satisfies also conditions for being an interpolant of the conclusion; provability and language conditions are trivially satisfied. Otherwise we must consider if $b \in L(\Gamma_1 \sqcup \Delta_1)$ or not. In the first case φ is intact, in the second case it has the form $\forall x\varphi[b/x]$ which guarantees the satisfaction of the language condition. Provability conditions are satisfied by applications of $(\Rightarrow \forall)$ to $\Gamma_1 \Rightarrow \Delta_1, \varphi$ (correct since b is only in φ) and $(\forall \Rightarrow)$ to $\varphi, \Gamma_2 \Rightarrow \Delta_2$. We could equally well consider an arbitrary partition of the premiss with $b = b$ belonging to (Γ_2, Δ_2) and an interpolant φ with the same result, i.e. the interpolant for the conclusion being either φ or $\exists x\varphi[b/x]$ (if b is in φ but not in Γ_2, Δ_2).

In the case of $(=\Rightarrow)$ and $(Str \Rightarrow)$ the situation is even simpler. For any partition of the conclusion, an interpolant φ is inherited from the respective partition of the premiss. It means that if the principal formula belongs to the left (right) division, then we take the same partition of Γ, Δ with all side formulae belonging to the left (right) division. Note that in this case $a_1, ..., a_n$ are eigenvariables so it is not possible that any of them belongs to φ and we do not need any quantification on φ to satisfy the language condition.

In the case of $(\Rightarrow \imath)$, for arbitrary partition, $c \approx \imath x\varphi$ is either in the left or in the right division. In the first case, by the IH we have:

1. $\vdash c \approx a, \Gamma_1 \Rightarrow \Delta_1, \varphi[x/a], \psi_1$

2. $\vdash \psi_1, \Gamma_2 \Rightarrow \Delta_2$

3. $L(\psi_1) \subseteq L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

4. $\vdash \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, c \approx a, \psi_2$

5. $\vdash \psi_2, \Gamma_2 \Rightarrow \Delta_2$

6. $L(\psi_2) \subseteq L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

with ψ_1, ψ_2 being interpolants of these partitions of the premisses. We obtain the interpolant $\psi_1 \vee \psi_2$ of the conclusion in the left division since it holds by 1, 4 and 2, 5:

$$\cfrac{\cfrac{c \approx a, \Gamma_1 \Rightarrow \Delta_1, \varphi[x/a], \psi_1 \qquad \varphi[x/a], \Gamma_1 \Rightarrow \Delta_1, c \approx a, \psi_2}{\Gamma_1 \Rightarrow \Delta_1, c \approx \imath x\varphi, \psi_1, \psi_2}\,(\Rightarrow \imath)}{\Gamma_1 \Rightarrow \Delta_1, c \approx \imath x\varphi, \psi_1 \vee \psi_2}\,(\Rightarrow \vee)$$

$$(\vee \Rightarrow)\,\frac{\psi_1, \Gamma_2 \Rightarrow \Delta_2 \qquad \psi_2, \Gamma_2 \Rightarrow \Delta_2}{\psi_1 \vee \psi_2, \Gamma_2 \Rightarrow \Delta_2}$$

To simplify proof figures we omit explicit application of structural rules like weakening or contraction; this convention applies also to proofs displayed below. The language condition $L(\psi_1 \vee \psi_2) \subseteq L(\{c \approx \imath x\varphi[x/a]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$ is satisfied by 3, 4 and the fact that a is an eigenvariable.

Let $c \approx \imath x\varphi$ be in the right division of arbitrary partition, then by the IH we have:

1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_1$

2. $\vdash \psi_1, c \approx a, \Gamma_2 \Rightarrow \Delta_2, \varphi[x/a]$

3. $L(\psi_1) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$

4. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi_2$

5. $\vdash \psi_2, \varphi[x/a], \Gamma_2 \Rightarrow \Delta_2, c \approx a$

6. $L(\psi_2) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{c \approx a, \varphi[x/a]\} \sqcup \Gamma_2 \sqcup \Delta_2)$

and the interpolant for the conclusion is $\psi_1 \wedge \psi_2$ by symmetric argument.

The situation with computing an interpolant for $(\imath \Rightarrow)$ is similar, i.e. with interpolants being basically either $\psi_1 \vee \psi_2$ or $\psi_1 \wedge \psi_2$, but with an important proviso. Since b is an arbitrary parameter it is possible, in case of the left division of the principal formula, that b is in $\psi_1 \vee \psi_2$ but not in Γ_1, Δ_1. Then the interpolant for the conclusion must be $\forall x(\psi_1 \vee \psi_2)[b/x]$. Similarly, in case of the right division, if b occurs in $\psi_1 \wedge \psi_2$ but not in Γ_2, Δ_2, then it has the form $\exists x(\psi_1 \wedge \psi_2)[b/x]$. All other details of the reasoning remain the same.

Now consider $(\Rightarrow=)$. In case of the left division, by the IH we have:

1. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_1 = d_1, \varphi_1$
2. $\vdash \varphi_1, \Gamma_2 \Rightarrow \Delta_2$
3. $L(\varphi_1) \subseteq L(\{b_1 = d_1\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$
4. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_2 = d_2, \varphi_2$
5. $\vdash \varphi_2, \Gamma_2 \Rightarrow \Delta_2$
6. $L(\varphi_2) \subseteq L(\{b_2 = d_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$
7. $\vdash \Gamma_1 \Rightarrow \Delta_1, b_1 = b_2, \varphi_3$
8. $\vdash \varphi_3, \Gamma_2 \Rightarrow \Delta_2$
9. $L(\varphi_3) \subseteq L(\{b_1 = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

Either $b_1, b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not. Assume the former, then the interpolant is $\varphi_1 \vee \varphi_2 \vee \varphi_3$. The following proofs justify provability conditions on the basis of 1, 4, 7, and 2, 5, 8:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, b_1 = d_1, \varphi_1 \qquad \Gamma_1 \Rightarrow \Delta_1, b_2 = d_2, \varphi_2 \qquad \Gamma_1 \Rightarrow \Delta_1, b_1 = b_2, \varphi_3}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1, \varphi_2, \varphi_3}\,(\Rightarrow=)}{\Gamma_1 \Rightarrow \Delta_1, d_1 = d_2, \varphi_1 \vee \varphi_2 \vee \varphi_3}\,(\Rightarrow \vee)$$

$$\dfrac{\dfrac{\varphi_1, \Gamma_2 \Rightarrow \Delta_2 \qquad \varphi_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2, \Gamma_2 \Rightarrow \Delta_2}\,(\vee \Rightarrow) \qquad \varphi_3, \Gamma_2 \Rightarrow \Delta_2}{\varphi_1 \vee \varphi_2 \vee \varphi_3, \Gamma_2 \Rightarrow \Delta_2}\,(\vee \Rightarrow)$$

$L(\varphi_1 \vee \varphi_2 \vee \varphi_3) \subseteq L(\{d_1 = d_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$ follows from 3, 6, 9. However, in case some of b_1, b_2 (or both) are not present in Γ_1, Δ_1 but occur in interpolants of premisses, an additional quantification is required. Accordingly interpolants for the conclusion should have the form: $\forall x(\varphi_1 \vee \varphi_2 \vee \varphi_3)[b_1/x]$ or $\forall x(\varphi_1 \vee \varphi_2 \vee \varphi_3)[b_2/x]$ or $\forall xy(\varphi_1 \vee \varphi_2 \vee \varphi_3)[b_1/x, b_2/y]$.

By the symmetric argument in the case of the right division, interpolants computed on the basis of three interpolants of the premisses have one of the following forms: $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ or $\exists x(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[b_1/x]$ or $\exists x(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[b_2/x]$ or $\exists xy(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)[b_1/x, b_2/y]$.

There is nothing essentially new in the procedure of computing interpolants for the conclusion of $(\Rightarrow Str)$. For n different descriptions in the principal formula we have, by the IH, $n+1$ interpolants for arbitrary partitions of the premisses. Therefore, in case of the left division our interpolant obtains one of the form $\psi_1 \vee \ldots \psi_{n+1}, \ldots, \forall x_1 \ldots x_{n+1}(\psi_1 \vee \ldots \psi_{n+1})[b_1/x_1, \ldots, b_{n+1}/x_{n+1}]$ depending on the occurrences of b_i in Γ_1, Δ_1 and interpolants. Dually, in case of the right division, suitable interpolants are conjunctions of $\psi_1, \ldots, \psi_{n+1}$, possibly existentially quantified.

The situation is slightly different in the case of (E) and (L). In both rules there are two principal formulae, thus there are four possible classes of partitions of the conclusion. In the case of (E) they are:

(a) $(c = b_1, c = b_2, \Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)$

(b) $(\Gamma_1, \Delta_1), (c = b_1, c = b_2, \Gamma_2, \Delta_2)$

(c) $(c = b_1, \Gamma_1, \Delta_1), (c = b_2, \Gamma_2, \Delta_2)$

(d) $(c = b_2, \Gamma_1, \Delta_1), (c = b_1, \Gamma_2, \Delta_2)$.

In case of the premiss there are only two possible cases, so by the IH we have:

1.1. $\vdash b_1 = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi$

1.2. $\vdash \varphi, \Gamma_2 \Rightarrow \Delta_2$

1.3. $L(\varphi) \subseteq L(\{b_1 = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

and

2.1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \varphi$

2.2. $\vdash \varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2$

2.3. $L(\varphi) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{b_1 = b_2\} \sqcup \Gamma_2 \sqcup \Delta_2)$

For the case (a) from 1.1 we derive $c = b_1, c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi$ by (E), and 1.3. obviously implies $L(\varphi) \subseteq L(\{c = b_1, c = b_2\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$, hence φ satisfies the conditions for being an interpolant and we are done.

By symmetric argument but on the basis of 2.1., 2.2. and 2.3. we prove that in the case (b) the interpolant is also inherited from the respective partition of the premiss.

Consider the case (c). By the IH, 1.1., 1.2., 1.3 hold and either $b_2 \in L(\Gamma_1 \sqcup \Delta_1)$ or not. Assume the former, then the interpolant is $c = b_2 \to \varphi$ and the provability conditions are satisfied by the following arguments on the basis of 1.1. and 1.2.:

$$\dfrac{\dfrac{b_1 = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}{c = b_1, c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi}\,(E)}{c = b_1, \Gamma_1 \Rightarrow \Delta_1, c = b_2 \to \varphi}\,(\Rightarrow\to)$$

$$(\to\Rightarrow)\,\dfrac{c = b_2, \Gamma_2 \Rightarrow \Delta_2, c = b_2 \qquad \varphi, \Gamma_2 \Rightarrow \Delta_2}{c = b_2 \to \varphi, c = b_2, \Gamma_2 \Rightarrow \Delta_2}$$

Note that $L(c = b_2 \to \varphi) \subseteq L(\{c = b_1\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\{c = b_2\} \sqcup \Gamma_2 \sqcup \Delta_2)$ follows by 1.3. and the fact that c is in both parts and b_2 as well by the assumption. Now assume the latter, i.e. $b_2 \notin L(\Gamma_1 \sqcup \Delta_1)$. In this case if b_2 is not in φ we can keep the interpolant intact, but if b_2 occurs in φ, the interpolant must take the form $\forall x(c = x \to \varphi[b_2/x])$. Provability conditions are satisfied by the application of $(\Rightarrow \forall)$ to the interpolant in the first proof figure (correct since b_2 occurs only in the interpolant by assumption) and by $(\forall \Rightarrow)$ in the second.

Now consider the fourth case, i.e. (d). By the IH, 2.1., 2.2., 2.3. hold and again, either $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or not. In the first case the interpolant is $\varphi \wedge c = b_2$ and in the second $\exists x(\varphi[b_2/x] \wedge c = x)$ (provided that b_2 is also in φ). The respective proof figures are the following:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, \varphi \qquad c = b_2, \Gamma_1 \Rightarrow \Delta_1, c = b_2}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \varphi \wedge c = b_2}\,(\Rightarrow \wedge)}{c = b_2, \Gamma_1 \Rightarrow \Delta_1, \exists x(\varphi[b_2/x] \wedge c = x)}\,(\Rightarrow \exists)$$

$$
\dfrac{\dfrac{\dfrac{\varphi, b_1 = b_2, \Gamma_2 \Rightarrow \Delta_2}{\varphi, c = b_1, c = b_2, \Gamma_2 \Rightarrow \Delta_2}\,(E)}{\varphi \wedge c = b_2, c = b_1, \Gamma_2 \Rightarrow \Delta_2}\,(\wedge \Rightarrow)}{\exists x(\varphi[b_2/x] \wedge c = x), c = b_1, \Gamma_2 \Rightarrow \Delta_2}\,(\exists \Rightarrow)
$$

with the last applications of rules for $\exists$ not required if $b_2 \in L(\Gamma_2 \sqcup \Delta_2)$ or does not occur in φ. In the latter case, i.e. if $b_2 \notin L(\Gamma_2 \sqcup \Delta_2)$ but b_2 occurs in φ, the quantification of b_2 is necessary to satisfy the language condition.

In the case of (L) the situation is similar. Again there are four possible classes of partitions of the conclusion:

(a) $(c = b, \varphi[x/c], \Gamma_1, \Delta_1), (\Gamma_2, \Delta_2)$

(b) $(\Gamma_1, \Delta_1), (c = b, \varphi[x/c], \Gamma_2, \Delta_2)$

(c) $(c = b, \Gamma_1, \Delta_1), (\varphi[x/c], \Gamma_2, \Delta_2)$

(d) $(\varphi[x/c], \Gamma_1, \Delta_1), (c = b, \Gamma_2, \Delta_2)$.

and by the IH we have:

1.1. $\vdash \varphi[x/b], \Gamma_1 \Rightarrow \Delta_1, \psi$

1.2. $\vdash \psi, \Gamma_2 \Rightarrow \Delta_2$

1.3. $L(\psi) \subseteq L(\{\varphi[x/b]\} \sqcup \Gamma_1 \sqcup \Delta_1) \cap L(\Gamma_2 \sqcup \Delta_2)$

and

2.1. $\vdash \Gamma_1 \Rightarrow \Delta_1, \psi$

2.2. $\vdash \psi, \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2$

2.3. $L(\psi) \subseteq L(\Gamma_1 \sqcup \Delta_1) \cap L(\{\varphi[x/b]\} \sqcup \Gamma_2 \sqcup \Delta_2)$

The cases (a) and (b) allow us to inherit the interpolant ψ from the premiss. Cases (c) and (d) require similar proofs like for (E). There are two subcases: either $b \in L(\varphi[x/c] \sqcup \Gamma_2 \sqcup \Delta_2$ or not. In the first subcase the interpolant is $\psi \wedge c = b$ obtained from 2.1., 2.2. in the following way:

$$
\dfrac{\Gamma_1 \Rightarrow \Delta_1, \psi \qquad c = b, \Gamma_1 \Rightarrow \Delta_1, c = b}{c = b, \Gamma_1 \Rightarrow \Delta_1, \psi \wedge c = b}\,(\Rightarrow \wedge)
$$

$$
\dfrac{\dfrac{\psi, \varphi[x/b], \Gamma_2 \Rightarrow \Delta_2}{\psi, c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}\,(L)}{\psi \wedge c = b, \varphi[x/c], \Gamma_2 \Rightarrow \Delta_2}\,(\wedge \Rightarrow)
$$

In the latter subcase we need to quantify on b, so the interpolant is $\exists x(\psi[b/x])$ which is derivable in both proof figures by additional applications of $(\Rightarrow \exists)$ and $(\exists \Rightarrow)$ respectively. In the case (d) we obtain $c = b \to \psi$ or $\forall x(c = x \to \psi[b/x])$ (depending on $b \in L(\varphi[x/c] \sqcup \Gamma_1 \sqcup \Delta_1$ or not) as an interpolant on the basis of 1.1., 1.2. In both cases the language condition is satisfied without further inference steps. $\square$

5. Conclusion

In the introduction we have noticed the importance of RDD for investigation into constructive mathematics. In fact, as was proved in [18], GRDD provides an alternative, fully analytic, characterisation of the classical version of the basic system of definedness logics which are treated as a wider family of systems[1] specialised to deal with partial untyped combinatory and lambda calculi (Feferman [12]). Since this logic is more interesting in the constructive version it is important to emphasize that we are in a position to provide an intuitionistic version of this logic too. It is sufficient to change the underlying system from Fig. 1 into intuitionistic one and in a similar way restrict all rules from Fig. 2 to intuitionistic versions having at most one formula in the succedent. This requires only small modification for $(\imath \Rightarrow)$ which must be split into:

$$(\imath \Rightarrow)\ \frac{\Gamma_1 \Rightarrow \varphi[x/b] \quad b = c, \Gamma_2 \Rightarrow \Delta}{c = \imath x\varphi, \Gamma \Rightarrow \Delta}$$

$$(\imath \Rightarrow)\ \frac{\Gamma_1 \Rightarrow b = c \quad \varphi[x/b], \Gamma_2 \Rightarrow \Delta}{c = \imath x\varphi, \Gamma \Rightarrow \Delta}$$

A careful inspection of the proofs shows that all are intuitionistically valid after making suitable corrections. This remark applies both to proofs of equivalence and cut elimination from [20] and to the proof of the interpolation theorem. Hence we obtain the intuitionistic theory of RDD equivalent to definedness logic and satisfying interpolation property.

Acknowledgments

This research is funded by the European Union (ERC, ExtenDD, project number: 101054714). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

I would like to thank the reviewers for their helpful comments.

Declaration on Generative AI

No AI tool was used during the preparation of this work.

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