Interpolation above S4 (Extended Abstract)

Simon Santschi[1], Niels C. Vooijs[1,*]

[1]Mathematical Institute, University of Bern, Bern, Switzerland

Abstract

We complete Maksimova's classification of the normal extensions of **S4** with interpolation. In particular, we prove Craig interpolation for the six extensions of **S4** for which Craig interpolation was still open. The proof strategy builds upon the ideas of Smoryński, but employs a novel approach using Fine's frame formulas for splitting clusters.

Keywords

Craig interpolation, S4, frame formulas, Kripke semantics, modal logic

By a celebrated result of Maksimova [1], exactly eight super-intuitionistic logics have the Craig interpolation property (CIP). Shortly thereafter, Maksimova showed, using the intimate connection between super-intuitionistic logics and extensions of the modal logic **S4**, that at most 38 normal extensions of **S4** have the CIP [2], and reduced this to 37 in [3]. Moreover, in [4], she proved the CIP for the majority of these logics, more specifically, the ones that are canonical. Subsequently, Boolos [5], Rautenberg [6], and Maksimova [7] proved the CIP for, in total, another four of the logics. Another two are claimed to have the CIP in [8], referring to an earlier announcement of Maksimova; a proof can be found in [9]. For the final six logics, the status of the CIP has remained open. Regarding the deductive interpolation property (DIP), the story is similar, with at most 49 normal extensions of **S4** enjoying the DIP, and the six open cases coinciding exactly with those for the CIP. A complete overview of the situation is presented in [10].

We prove the CIP for the remaining normal extensions of **S4**. In fact, we give a uniform proof of the CIP for the 18 modal companions of the logics **Int** and **KC** that have the CIP. Since for modal logics, the CIP implies the DIP, this completes Maksimova's characterization for both the CIP and the DIP in normal extensions of **S4**. This resolves Problem 14.3 in Chagrov and Zakharyashev's standard reference on modal logic [11], which was also posed recently in [12, Open Problem 9].

The full paper is available on arXiv [13].

Method. Our approach is based on the method that Smoryński [14] used to show the CIP for **GL**. This approach was also used by Boolos [5] to prove the CIP for **Grz** and by Maksimova [7] for two further normal extensions of **S4**. Smoryński's method relies on syntactically constructing finite models. First, it is assumed that an implication does not have an interpolant. Using this, a finite countermodel for the implication is constructed. For logics that impose restrictions on the clusters in their frames, it is usually necessary to tweak the definition of the accessibility relation, so that the constructed model is a model of the logic, cf. [14, 5, 7]. Thus, every implication lacking an interpolant is invalid in the logic, and the CIP is obtained.

The key difference of our approach compared to the previous ones is that we employ a more flexible and semantic approach towards abiding by the restrictions on clusters. We first construct a standard Smoryński model $\mathfrak{M}$, which is only a model for **S4** (or **S4.2**). The cluster restrictions are then expressed using frame formulas, and it follows from the Truth Lemma for Smoryński models that $\mathfrak{M}$ validates (sufficiently large) substitutions of these frame formulas. Using these, we show that every cluster contains a small enough *adequate* subset of points, which together do not rely on other points in the

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*Corresponding author.
simon.santschi@unibe.ch (S. Santschi); niels.vooijs@unibe.ch (N. C. Vooijs)
0009-0003-9364-5149 (S. Santschi); 0000-0002-4515-9694 (N. C. Vooijs)

cluster for witnessing the satisfaction of formulas. This then finally allows us to modify the accessibility relation into one that does not contain clusters that are too large, while preserving the Truth Lemma; an operation which we will call *refining* the clusters.

1. Preliminaries

We assume that the reader is familiar with the elementary notions of relational semantics for normal unimodal logic, see, e.g., [11, 15].

Formulas are built from atomic propositions using constants $\bot$ and $\top$, unary connectives $\neg$ and $\Box$, and binary connectives $\wedge$ and $\vee$. For a formula χ in atomic propositions $p_0, \dots, p_{n-1}$, and formulas $\varphi_0, \dots, \varphi_{n-1}$, we write $\chi[\varphi_0, \dots, \varphi_{n-1}]$ for the formula obtained by substituting each instance of p_i by φ_i, for $i \in \{0, \dots, n-1\}$.

A logic Λ is said to have the *Craig interpolation property (or CIP for short)* if for all formulas φ, ψ with $\varphi \to \psi \in \Lambda$, there exists a formula χ whose atomic propositions appear in both φ and ψ, such that $\varphi \to \chi, \chi \to \psi \in \Lambda$. In this case the formula χ is called an *interpolant* for the implication $\varphi \to \psi$.

Recall that the logic $\mathbf{S4}$ is complete with respect to preorder models, i.e., models $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$, where $\preceq$ is a preorder on X. The logic $\mathbf{S4.2}$ is $\mathbf{S4} \oplus (\Diamond\Box p \to \Box\Diamond p)$ and is complete with respect to preorder models $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$ that are *confluent*, i.e., for all $x, y, z \in X$, if $x \preceq y$ and $x \preceq z$, then there exists $w \in X$ with $y \preceq w$ and $z \preceq w$.

Let $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$ be a preorder model. For a formula φ, we write $[\![\varphi]\!]_{\mathfrak{M}}$ for the set of points in $\mathfrak{M}$ that satisfy φ, i.e. $\{x \in X \mid \mathfrak{M}, x \vDash \varphi\}$. For a logic Λ, we say that $\mathfrak{M}$ is a Λ-*model*, if $[\![\varphi]\!]_{\mathfrak{M}} = X$ for every $\varphi \in \Lambda$, and $\langle X, \preceq\rangle$ is a Λ-*frame* if every model on it is a Λ-model. A function f between preorders $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$ is *monotone* if $x \preceq_X x'$ implies $f(x) \preceq_X f(x')$ for all $x, x' \in X$. It is called a *p-morphism* if it is monotone, and for all $x \in X$ and $y' \in Y$ with $f(x) \preceq_Y y'$, there exists $x' \in X$ with $f(x') = y'$ and $x \preceq_X x'$.

Let $\preceq$ be a preorder on a set X. We write $x \prec y$ if $x \preceq y$ and $y \not\preceq x$. A non-empty subset $C \subseteq X$ is a *cluster* (of $\preceq$) if for all $x, z \in C$, we have $x \preceq z$, and $x \preceq y \preceq z$ implies $y \in C$. A cluster $C \subseteq X$ is called *final* if there does not exist a point $x \in X$ such that $y \prec x$ for some $y \in C$.

Frame formulas. The logics extending $\mathbf{S4}$ which we are interested in, can be axiomatized using Fine's *frame formulas* [16, Section 2]. For a natural number $n \in \mathbb{N}$, write $[n] := \{0, \dots, n-1\}$. Given a finite preorder $\mathfrak{F} = \langle [n], \preceq\rangle$, the frame formula $\chi(\mathfrak{F})$ of $\mathfrak{F}$ is a formula in n atomic propositions, with the defining property that for any formulas $\varphi_0, \dots, \varphi_{n-1}$, model $\mathfrak{N}$ on a preorder $\mathfrak{G}$, and point $x \in [\![\varphi_0]\!]_{\mathfrak{N}}$, we have $x \notin [\![\chi(\mathfrak{F})[\varphi_0, \dots, \varphi_{n-1}]]\!]_{\mathfrak{N}}$ iff there exists a p-morphism f from $\mathfrak{G}^x$ to $\mathfrak{F}$ such that $f^{-1}(i) = [\![\varphi_i]\!]_{\mathfrak{N}^x}$ [16, Lemma 2.1], where $\mathfrak{N}^x$ denotes the submodel of $\mathfrak{N}$ generated by x and $\mathfrak{G}^x$ is the underlying preorder.[1] For example, $\mathbf{S4.2}$ is axiomatised, over $\mathbf{S4}$, by the frame formula $\chi(\mathfrak{B})$, where $\mathfrak{B}$ is the frame depicted in Figure 1 (a).

The frame formulas that axiomatize our logics of interest encode the absence of final or non-final clusters of cardinality at least m, for a constant $m \in \mathbb{N}$ – in our case either 2 or 3. For this reason we consider the following frames. For $n \geq 1$, define,

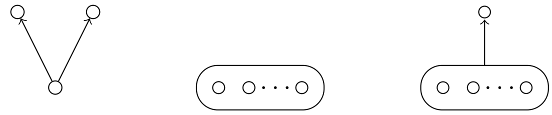
- $\mathfrak{C}_n = \langle [n], \preceq\rangle$, where $i \preceq j$ for any $i, j \in [n]$, and
- $\overline{\mathfrak{C}}_n = \langle [n+1], \preceq\rangle$, where $i \preceq j$ iff either $i, j \in [n]$ or $i = j = n$.

These frames are depicted in Figure 1 (b) and (c). For $m \in \mathbb{N}$, we write

$$\gamma_m = \chi(\mathfrak{C}_{m+1}), \quad \bar{\gamma}_m = \chi(\overline{\mathfrak{C}}_{m+1}), \quad \text{and} \quad \gamma_\omega = \bar{\gamma}_\omega = \top.$$

Then a finite preorder $\mathfrak{G}$ validates γ_m iff $\mathfrak{G}$ contains no final cluster of size $> m$, and it validates $\bar{\gamma}_n$ iff $\mathfrak{G}$ contains no non-final cluster of size $> n$. Moreover, the logic $\mathbf{S4} \oplus \gamma_m \oplus \bar{\gamma}_n$ is complete with respect to finite preorders free of final cluster of size $> m$ and non-final clusters of size $> n$.

[1] Note that we follow the standard convention [17] after Jankov [18], where $\chi(\mathfrak{F})$ is the negation of what Fine [16] defines to be the frame formula.



(a) (b) (c)

Figure 1: The frames $\mathfrak{B}$ (a), $\mathfrak{C}_n$ (b), and $\overline{\mathfrak{C}}_n$ (c)

The logics with the CIP. Maksimova [2] showed that every normal extension of $\mathbf{S4}$ with the CIP (or DIP), is of a certain shape: its intuitionistic fragment Λ has the CIP, and the logic is axiomatized over $\tau(\Lambda)$ by the frame formulas γ_m and $\bar{\gamma}_n$, where $\tau(\Lambda)$ denotes the least modal companion of Λ. More precisely, for a super-intuitionistic logic Λ and $m, n \leq \omega$, define $\Gamma(\Lambda, m, n) \coloneqq \tau(\Lambda) \oplus \gamma_m \oplus \bar{\gamma}_n$ [10, p. 73]. Then, every normal extension of $\mathbf{S4}$ with the DIP (and thus, in particular, those with the CIP) is of the form $\Gamma(\Lambda, m, n)$, for some super-intuitionistic logic Λ with the CIP and $m, n \in \{1, 2, \omega\}$. For example, $\Gamma(\mathbf{Int}, \omega, \omega) = \mathbf{S4}$, $\Gamma(\mathbf{Int}, 1, 1) = \mathbf{Grz}$, $\Gamma(\mathbf{KC}, \omega, \omega) = \mathbf{S4.2}$, and $\Gamma(\mathbf{Cl}, \omega, 0) = \mathbf{S5}$, where $\mathbf{Int}$ denotes intuitionistic propositional logic, $\mathbf{Cl}$ denotes classical propositional logic, and $\mathbf{KC} = \mathbf{Int} + (\neg p \vee \neg\neg p)$. The six normal extensions of $\mathbf{S4}$ for which the CIP remained open, are

$$\Gamma(\Lambda, m, 2) \qquad \text{for } \Lambda \in \{\mathbf{Int}, \mathbf{KC}\} \text{ and } m \in \{1, 2, \omega\},$$

see [10, pp. 256–257]. We show that these six logics have the CIP; in fact, we give a uniform proof of the CIP for the logics $\Gamma(\Lambda, m, n)$ with $\Lambda \in \{\mathbf{Int}, \mathbf{KC}\}$ and $m, n \in \{1, 2, \omega\}$.

2. Smoryński models

In this section, we consider an approach due to Smoryński [14] for syntactically constructing finite models, which we call Smoryński models. A Smoryński model is constructed from two sets Σ_1, Σ_2 of formulas that are 'essentially finite' and satisfy certain closure properties. In the proof of the CIP these two sets will roughly correspond to the subformula-closure of φ and ψ, respectively, for an implication $\varphi \to \psi$ that lacks an interpolant.

We fix, for $i \in \{1, 2\}$, a subformula-closed set of formulas Σ_i, such that for each $\varphi \in \Sigma_i$, both $\neg\varphi$ and $\Box\varphi$ are in Σ_i, and such that Σ_i contains only finitely many formulas up to logical equivalence in $\mathbf{S4}$. Moreover, we fix a normal extension Λ of $\mathbf{S4}$ and write $\Sigma \coloneqq \Sigma_1 \cup \Sigma_2$.

Definition 2.1. Let $T \subseteq \Sigma$ and $T_i = T \cap \Sigma_i$ for $i \in \{1, 2\}$. We call T

- *separable* if there exists a formula ψ with atomic propositions in $\Sigma_1 \cap \Sigma_2$ such that $\bigwedge T_1 \to \psi$, $\bigwedge T_2 \to \neg\psi$ are valid in Λ;
- *inseparable* if T is not separable;
- *maximal* if T is *inseparable* and for each $\varphi \in \Sigma_i$ (for $i \in \{1, 2\}$), $\varphi \in T_i$ or $\neg\varphi \in T_i$.

As the next lemma shows, every inseparable subset can be extended to a maximal subset, and maximal sets are closed under logical equivalence.

Lemma 2.2 ([10, Lemma 5.16])**.**

(i) *Every inseparable subset of Σ can be extended to a maximal subset.*

(ii) *For all maximal $T \subseteq \Sigma$, $i \in \{1, 2\}$, and $\varphi \in \Sigma_i$,*

$$\left(\bigwedge T_i \to \varphi\right) \in \Lambda \iff \varphi \in T_i.$$

In particular, $\bot \notin T_i$, and for all $\varphi, \psi \in \Sigma_i$, if $(\varphi \leftrightarrow \psi) \in \Lambda$, then

$$\varphi \in T_i \iff \psi \in T_i.$$

From these maximal sets – playing a role analogous to ultrafilters in the canonical model – we construct a finite model, which we will call the *Smoryński model*, after Smoryński [14].

Construction 2.3 (Smoryński model)**.** We define the *Smoryński model* $\mathfrak{M}_\Sigma = \langle X_\Sigma, \sqsubseteq, \mathfrak{V}\rangle$ as follows

- X_Σ consists of the maximal subsets of Σ.
- For $T, T' \in X_\Sigma$,
$$T \sqsubseteq T' \iff \forall \Box\varphi \in \Sigma.\ \Box\varphi \in T \implies \Box\varphi \in T'.$$
- For every propositional variable p in Σ,
$$\mathfrak{V}(p) := \{T \in X_\Sigma \mid p \in T\}.$$

Note that clearly $\sqsubseteq$ is a preorder and, by Lemma 2.2 (ii), $\mathfrak{M}_\Sigma$ is finite, since Σ contains only finitely many formulas up to logical equivalence in $\mathbf{S4}$.

Smoryński models satisfy the following Truth Lemma for formulas in Σ.

Lemma 2.4 (Truth Lemma)**.** *For every formula $\varphi \in \Sigma$ and $T \in X_\Sigma$,*

$$\mathfrak{M}_\Sigma, T \models \varphi \iff \varphi \in T.$$

If Λ extends $\mathbf{S4.2}$, then $\mathfrak{M}_\Sigma$ is confluent and hence an $\mathbf{S4.2}$-model [10, p. 146].

To prove the CIP for the logics $\Gamma(\Lambda, m, n)$ with $n, m \in \{1, 2, \omega\}$ and $\Lambda \in \{\mathbf{Int}, \mathbf{KC}\}$ we will assume that an implication $\varphi \to \psi$ does not have an interpolant. Then we construct suitable sets Σ_1 and Σ_2 such that $\varphi \in \Sigma_1$ and $\psi \in \Sigma_2$ and consider the Smoryński model $\mathfrak{M}_\Sigma$. The lack of an interpolant for $\varphi \to \psi$ yields that $\{\varphi, \neg\psi\}$ is inseparable and, by Lemma 2.2 (i), it extends to a maximal subset T. Finally, by the Truth Lemma, the point T in $\mathfrak{M}_\Sigma$ does not satisfy $\varphi \to \psi$.

However, the model $\mathfrak{M}_\Sigma$ might not yet be a model for $\Gamma(\Lambda, m, n)$ if $m < \omega$ or $n < \omega$, since in general it could contain clusters that are too large. In the next section, we show how to salvage this problem.

3. Refining Clusters

In this section we establish a condition that allows us to decrease the size of cluster in models, while preserving the satisfaction of formulas. This will allow us to refine the suitably chosen Smoryński model to a model of our given logic, while maintaining the Truth Lemma for a subset of formulas.

Definition 3.1. Let Σ be a set of formulas, $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$ a preorder model, and $C \subseteq X$ a cluster in $\mathfrak{M}$. A doubleton $\{x, y\} \subseteq C$ is called Σ*-adequate* if for all $\Box\varphi \in \Sigma$ with $x \models \neg\Box\varphi \wedge \varphi$ and $y \models \neg\Box\varphi \wedge \varphi$, there exists $z \in X$ with $x \prec z$ such that $z \models \neg\varphi$.

The notion of a Σ-adequate doubleton in a cluster will be crucial in the proof of the main lemma. It exactly captures the property needed to reduce the size of a cluster by removing edges while still preserving the validity of the formulas in Σ. The next lemma shows how clusters with an adequate doubleton can be refined. The transformation is depicted in Figure 2.

Lemma 3.2. *Let Σ be a Boolean and subformula closed set of formulas, $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$ a preorder model, and C a cluster in $\mathfrak{M}$. If $\{x_1, x_2\} \subseteq C$ is a Σ-adequate doubleton, then the relation*

$$\preceq' := \preceq \setminus \left\{\langle y, z\rangle \in C^2 \;\middle|\; y \neq z \text{ and } z \notin \{x_1, x_2\}\right\}$$

is a preorder such that, for $\mathfrak{M}' := \langle X, \preceq', \mathfrak{V}\rangle$, we have $[\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}'}$ for each $\varphi \in \Sigma$.

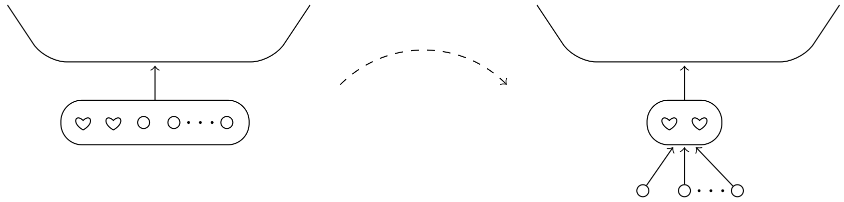


Figure 2: Depiction of the transformation to refine a cluster

Clearly, if $\preceq$ is confluent, then so is $\preceq'$.

A Smoryński model $\mathfrak{M}_\Sigma$ is only guaranteed to satisfy the theorems of the logic that are contained in Σ. In particular, a Smoryński model for $\Gamma(\Lambda, m, n)$ might not satisfy arbitrary substitutions of γ_m and $\bar{\gamma}_n$. However, for our purposes, it suffices that it satisfies 'enough' substitutions, which leads to the following definition of *relative satisfaction.*

Definition 3.3 (Relative satisfaction)**.** Let Σ be a set of formulas, $\mathfrak{M}$ a preorder model, and χ a formula in atomic propositions $p_0, \dots, p_{n-1}$. We say that a cluster C in $\mathfrak{M}$ *satisfies* χ *relative to* Σ if for all $\varphi_0, \dots, \varphi_{n-1} \in \Sigma$, we have $C \subseteq [\![\chi[\varphi_0, \dots, \varphi_{n-1}]]\!]$.

To see how this helps us to refine clusters in a Smoryński model, suppose that a non-final cluster C satisfies $\bar{\gamma}_2 = \chi(\overline{\mathfrak{C}}_3)$ relative to Σ_i, for $i \in \{1, 2\}$, and let $x \in C$. Using the defining property of frame formulas, we can show that there exists $y_i \in C$ such that $\{x, y_i\}$ is a Σ_i-adequate doubleton. If $y_1 = y_2$ then $\{x, y_1\}$ is both Σ_1- and Σ_2-adequate, hence Σ-adequate, and Lemma 3.2 applies. However, even when $y_1 \neq y_2$, a combinatorial argument shows that at least one of the doubletons

$$\{x, y_1\}, \ \{x, y_2\}, \ \text{ and } \ \{y_1, y_2\}$$

must be both Σ_1- and Σ_2-adequate, and thus Σ-adequate. Therefore, Lemma 3.2 applies to the cluster C, and we can refine the cluster, leaving only a 2-cluster and single points.

If C relatively satisfies $\bar{\gamma}_1$, then a simpler argument shows that every singleton in C is Σ-adequate. Applying Lemma 3.2 then leaves no clusters of more than one point. Similar reasoning applies to final clusters satisfing $\bar{\gamma}_1$ and $\bar{\gamma}_2$ relative to Σ_i, for $i \in \{1, 2\}$. Thus, we obtain the following lemma.

Lemma 3.4 (Main Lemma)**.** *For $i \in \{1, 2\}$, let Σ_i be a Boolean and subformula closed set of formulas, $\mathfrak{M} = \langle X, \preceq, \mathfrak{V}\rangle$, and $n \in \{1, 2\}$. Let C be a final cluster in $\mathfrak{M}$ which satisfies γ_n relative to Σ_i, or a non-final cluster in $\mathfrak{M}$ which satisfies $\bar{\gamma}_n$ relative to Σ_i. Then there exists a subpreorder $\preceq'$ of $\preceq$ such that*

(i) *$\preceq$ and $\preceq'$ coincide when restricted to $X^2 \setminus C^2$,*

(ii) *C does not contain a $\preceq'$-cluster of size $> n$, and*

(iii) *$[\![\varphi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}'}$ for all $\varphi \in \Sigma$, where $\mathfrak{M}' := \langle X, \preceq', \mathfrak{V}\rangle$.*

Moreover, if $\mathfrak{M}$ is confluent, then so is $\mathfrak{M}'$.

4. Main result

With the necessary prerequisites established we are able to prove to the main result. Let $\Lambda \in \{\mathbf{Int}, \mathbf{KC}\}$ and $m, n \in \{1, 2, \omega\}$. We claim that the logic $\Gamma = \Gamma(\Lambda, m, n) = \tau(\Lambda) \oplus \gamma_m \oplus \bar{\gamma}_n$ has the CIP. Suppose an implication $\varphi_1 \to \varphi_2$ does not have an interpolant. We prove that $\varphi_1 \to \varphi_2 \notin \Gamma$.

For a set of formulas Σ, let us write $\mathrm{cl}(\Sigma)$ for the Boolean closure of the subformula closure of Σ. Moreover, for a formula χ in atomic propositions $p_0, \dots, p_{n-1}$, define the *χ-closure of* Σ by

$$\mathrm{cl}^\chi(\Sigma) := \mathrm{cl}(\Sigma \cup \{\chi[\varphi_0, \dots, \varphi_{n-1}] \mid \varphi_0, \dots, \varphi_{n-1} \in \Sigma\}).$$

Table 1
The super-intuitionistic logics with the Craig interpolation property.

Logic	Axiomatization
$\mathbf{Int}$	Intuitionistic logic
$\mathbf{KC}$	$\mathbf{Int} + \neg p \vee \neg\neg p$
$\mathbf{LP}_2$	$\mathbf{Int} + p \vee (p \to (q \vee \neg q))$
$\mathbf{LV}$	$\mathbf{LP}_2 + (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$
$\mathbf{LS}$	$\mathbf{LP}_2 + \neg p \vee \neg\neg p$
$\mathbf{LC}$	$\mathbf{Int} + (p \to q) \vee (q \to p)$
$\mathbf{Cl}$	$\mathbf{Int} + p \vee \neg p$
$\mathbf{Fm}$	$\mathbf{Int} + \bot$

Now, for $i \in \{1, 2\}$, we define $\Sigma_i' := \mathrm{cl}(\varphi_i)$, and define Σ_i to be the set obtained by closing $\mathrm{cl}^{\gamma_m}(\Sigma_i') \cup \mathrm{cl}^{\bar{\gamma}_n}(\Sigma_i')$ under boxes and negations.

The sets Σ_1, Σ_2 satisfy all the conditions needed to define the Smoryński model $\mathfrak{M}_\Sigma$, which will be a $\tau(\Lambda)$-model. Moreover, for $i \in \{1, 2\}$, since Σ_i contains the γ_m- and $\bar{\gamma}_n$-closures of Σ_i', Lemmata 2.2 and 2.4 yield that every cluster in $\mathfrak{M}_\Sigma$ satisfies γ_m and $\bar{\gamma}_n$ relative to Σ_i'. Since $\varphi_1 \to \varphi_2$ does not have an interpolant, the set $\{\varphi_1, \neg\varphi_2\}$ is inseparable, and by Lemma 2.2 this set extends to a maximal subset T of Σ. The Truth Lemma yields that $\mathfrak{M}_\Sigma, T$ satisfies φ_1 and $\neg\varphi_2$.

Since the non-final clusters in $\mathfrak{M}_\Sigma$ satisfy $\bar{\gamma}_n$ relative to Σ_i', for $i \in \{1, 2\}$, we can apply the Main Lemma (Lemma 3.4) successively and in each step remove a non-final clusters of size $> n$ which is minimal in the accessibility ordering. Similarly, we apply Lemma 3.4 to the final clusters. Thus, we obtain a finite Γ-model $\mathfrak{M}'$. Moreover, since $\varphi_1 \in \Sigma_1'$ and $\neg\varphi_2 \in \Sigma_2'$, the applications of the Main Lemma preserved their satisfaction. We conclude that $\mathfrak{M}', T \not\models \varphi_1 \to \psi_2$, and hence $\varphi_1 \to \psi_2 \notin \Gamma$.

This establishes our main result.

Theorem 4.1 (Main Theorem)**.** *Let* $\Lambda \in \{\mathbf{Int}, \mathbf{KC}\}$ *and* $m, n \in \{1, 2, \omega\}$*. Then the logic* $\Gamma = \Gamma(\Lambda, m, n)$ *has the Craig interpolation property.*

Remark 4.2. Note that this gives a uniform proof of the CIP for all the modal companions of $\mathbf{Int}$ and $\mathbf{KC}$ with the Craig interpolation property. Therefore, our result subsumes earlier results such as Boolos' [5] proof of the CIP for $\mathbf{Grz} = \Gamma(\mathbf{Int}, 1, 1)$.

Theorem 4.1 completes Maksimova's classification of the normal extensions of $\mathbf{S4}$ with the CIP [10, Theorem 8.45, pp. 256–257]. Moreover, for modal logics, the CIP implies the DIP, and thus, this completes the classification for the DIP as well. Together, this resolves Problem 14.3 of Chagrov and Zakharyaschev [11, p. 469]. We conclude with the complete characterization. For this, recall that the super-intuitionistic logics with the CIP are exactly those listed in Table 1 [1].

Corollary 4.3. *There are exactly 37 normal extensions of* $\mathbf{S4}$ *with the Craig interpolation property and exactly 49 with the deductive interpolation property. In particular:*

(i) *A normal extension of* $\mathbf{S4}$ *has the CIP if and only if it is one of the following logics:*

$$\begin{array}{ll} \mathbf{Fm} \quad (\textit{the inconsistent logic}); & \\ \Gamma(\Lambda, m, n), & \Lambda \in \{\mathbf{Int}, \mathbf{KC}\},\ m, n \in \{1, 2, \omega\}; \\ \Gamma(\Lambda, n, 1),\ \Gamma(\Lambda, 1, n), & \Lambda \in \{\mathbf{LP}_2, \mathbf{LV}, \mathbf{LS}\},\ n \in \{1, 2, \omega\}; \\ \Gamma(\mathbf{Cl}, n, 0), & n \in \{1, 2, \omega\}. \end{array}$$

(ii) *A normal extension of* $\mathbf{S4}$ *has the DIP if and only if it is a logic mentioned in* (i) *or it is one of the following logics:*

$$\Gamma(\Lambda, m, n), \quad \Lambda \in \{\mathbf{LP}_2, \mathbf{LV}, \mathbf{LS}\},\ m, n \in \{2, \omega\}.$$

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Declaration on Generative AI

The authors have not employed any Generative AI tools.

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