Deontic Defeasible Description Logic

Giovanni Casini[1,2], Thomas Meyer[2,1], Rafael Peñaloza[3] and Leendert van der Torre[4]

[1]CNR - ISTI, Pisa, Italy
[2]University of Cape Town and CAIR Cape Town, South Africa
[3]Department of Computer Science, Systems, and Communications, University of Milano-Bicocca, Milan, Italy
[3]Department of Computer Science, University of Luxembourg, Esch-sur-Alzette, Luxembourg

Abstract

We introduce an extension of Description Logics (DL), appropriate for modelling and reasoning about deontic notions. Moving from *defeasible* $\mathcal{ALC}$, an extension of the DL $\mathcal{ALC}$ which allows us to model and reason about defeasible information (rules that admit exceptions), we show how to modify the semantics in order to also represent and manage conditional *obligations* and *permissions*, and we present the corresponding, and easily implementable, decision procedures. Also, we define a semantics combining reasoning about expectations and reasoning about norms, pairing it with corresponding decision procedures.

Keywords

Description Logics, Deontic Reasoning, Defeasible Reasoning

1. Introduction

Deontic logic [1, 2] and Description Logics (DLs) [3] have been developed under very different circumstances. Deontic logic started off as a branch of philosophical logic to formalize relations between deontic concepts like norms, obligations and permissions, whereas DLs were developed as a branch of knowledge representation to reason about taxonomies and ontologies. Nevertheless, there are striking similarities between the two formalisms. For example, both use a *conditional* or *rule-based* representation, and consider various kinds of inferences.

Deontic and description logics have both dealt with defeasible reasoning. The former dates back to the eighties, when *prima facie* norms and moral dilemmas became a topic of study, using techniques from non-monotonic logic [4, 5, 6]. Over the past fifteen years, various defeasible DLs were also introduced to reason about exceptions and conflicts [7, 8, 9, 10, 11]. In the philosophical literature, it is common to restrict the attention to propositional modal logic, whereas in the application of deontic logic to natural language semantics, it is common to use full first-order logic. In the area of legal informatics fragments of first-order logic have been used with good computational properties, such as defeasible logics [12]. Deontic defeasible description logics aim at a trade-off between expressive power and good computational properties, still being able to tackle the main reasoning challenges of deontic logic. Classical DLs have already been extended to formalize constitutive norms [13], but defeasible logics are used regularly to model constitutive norms as well. In our formalism we consider defeasible constitutive and regulative norms, which are both described and reasoned about using defeasible DLs.

In Section 2 we briefly introduce dyadic deontic logic and defeasible DLs. Section 3 presents the language, semantics, and an entailment relation for deontic description logics. Section 4 introduces a framework combining expectations and norms in our reasoning.

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giovanni.casini@isti.cnr.it (G. Casini); tmeyer@cair.org.za (T. Meyer); rafael.penaloza@unimib.it (R. Peñaloza); leon.vandertorre@uni.lu (L. van der Torre)
0000-0002-4267-4447 (G. Casini); orcid.org/0000-0003-2204-6969 (T. Meyer); 0000-0002-2693-5790 (R. Peñaloza); 0000-0003-4330-3717 (L. van der Torre)

2. Background

We first provide a very basic introduction to defeasible deontic logic and defeasible description logics. *Deontic Logic*: Let L be a classical logical language (e.g. propositional or first-order logic) and $\rightarrow_C$, $\rightarrow_O$, and $\rightarrow_P$ be conditional implications. A normative system is a tuple $\{\mathcal{C}, \mathcal{R}, \mathcal{A}\}$ containing a set $\mathcal{C} = \{A_i \rightarrow_C B_i\}$ of *constitutive norms*, with $A_i, B_i \in L$, describing the (legal) notions read as "A_i counts as B_i;" a set $\mathcal{R} = \{A_i \rightarrow_X B_i\}$ of *regulative norms*, with $A_i, B_i \in L$ and $X \in \{O, P\}$, describing the regulative part of the normative system; $A \rightarrow_O B$ is read as "if A then it is obligatory that B;" $A \rightarrow_P B$ is read as "if A then B is permitted;" and $\mathcal{A} \subseteq L$. Reasoning in this setting consists of deriving propositions from $\mathcal{C}$ (*institutional facts*), and from $\mathcal{C}$ and $\mathcal{R}$ together (*obligations* and *permissions*).

In deontic logic, conflicts may arise (i) between constitutive norms, expressing exceptions in the legal ontology; (ii) between regulative norms, expressing moral dilemmas or contrary-to-duty reasoning; or (iii) between constitutive and regulative norms, expressing violations. These are managed using techniques from dyadic deontic logic [14, 4, 15, 16, 17, 18, 19, 12, 20, 21]. We follow a similar approach.

Defeasible Description Logic: We assume basic knowledge of the DL $\mathcal{ALC}$, and refer to [3] for the details. In the last 15 years DLs have been extended to handle defeasible knowledge, especially working with preferential semantics [22, 23, 24, 25, 8, 26, 27, 28, 29, 10, 30, 31]. Here we refer in particular to the framework by Britz et al.. *Defeasible $\mathcal{ALC}$* extends $\mathcal{ALC}$ with *defeasible concept inclusions* of the form $C \sqsubsetsim_E D$ (the suffix 'E' referring to *expectations*), expressing that 'if an individual is a C, it *usually* also is a D.' A *defeasible* KB is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, where $\mathcal{T}$ is a TBox and $\mathcal{D}$ is a finite set of defeasible inclusions called a *DBox*.

The semantics is based on *ranked interpretations*, obtained through a *preference relation* $\prec_E^{\mathcal{R}}$ over the interpretation domain, where $a \prec_E^{\mathcal{R}} b$ ($a, b \in \Delta^{\mathcal{I}}$) means that 'the object a is more typical than the object b.' Given a set X and an order $\prec$, $\min_{\prec}(X)$ denotes the set of the minimal elements of X w.r.t. $\prec$; i.e., $\min_{\prec}(X) = \{x \in X \mid \text{for every } y \in X, y \not\prec x\}$.

Definition 1 (Ranked interpretation). *Given a set X, the binary relation $\prec \subseteq X \times X$ is a* ranked order *if there is some mapping $h : X \rightarrow \mathbb{N}$ (the* height function*) such that i) for every $i \in \mathbb{N}$ and $0 \leq j < i$, if $h(x) = i$ for some $x \in X$, then there is a $y \in X$ for which $h(y) = j$; and ii) for every $x, y \in X$, $x \prec y$ iff $h(x) < h(y)$. A* ranked interpretation *is a triple $R = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle$, where $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ is a DL interpretation, and $\prec_E^{\mathcal{R}}$ is a* ranked order *over $\Delta^{\mathcal{R}}$. We indicate with $h_{\prec_E^{\mathcal{R}}}$ the associated height function.*

The *height function* $h_{\prec_E^{\mathcal{R}}}$ can be extended to concepts by considering the lowest height of its members; that is, $h_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) = \min\{h_{\prec_E^{\mathcal{R}}}(x) \mid x \in \Delta^{\mathcal{R}} \text{ and } x \in C^{\mathcal{R}}\}$.

Definition 2 (Satisfaction). *Let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle$ be a ranked interpretation. $\mathcal{R}$* satisfies *(is* a model of*) $C \sqsubsetsim_E D$ (denoted $\mathcal{R} \Vdash C \sqsubsetsim_E D$) iff $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$. $\mathcal{R}$* satisfies *(is* a model of*) $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ iff $\mathcal{R} \Vdash \alpha$ for all the statements $\alpha \in \mathcal{T} \cup \mathcal{D}$. The KB $\mathcal{K}$ is* rank-satisfiable *iff it has a ranked model. Two KBs $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ and $\mathcal{K}' = \langle \mathcal{T}', \mathcal{D}' \rangle$ are* rank-equivalent *iff they have exactly the same ranked models.*

This semantics can model defeasible information in the form of inclusion axioms allowing for exceptions: $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$ means that all the most typical objects in C are in D, allowing for the existence of more exceptional objects in C which are not in D.

Example 1. *In classical $\mathcal{ALC}$, the TBox $\mathcal{T} = \{PhD \sqsubseteq Ad,\ PhD \sqsubseteq \neg W, Ad \sqsubseteq W\}$ where PhD, Ad, and W refer to the concepts 'PhD student,' 'Adult,' and 'Worker' respectively, implies that PhD students cannot exist. Instead we want to indicate that they are exceptional adults who are not workers. To achieve this, it suffices to change the GCI $Ad \sqsubseteq W$ into the defeasible inclusion $Ad \sqsubsetsim_E W$ (adults are* usually *workers). A ranked interpretation $\mathcal{R}$ in which all typical adults (the objects in $\min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}})$) are workers ($\min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}}) \subseteq W^{\mathcal{R}}$), while PhD students are* atypical *adults ($PhD^{\mathcal{R}} \subseteq (Ad^{\mathcal{R}} \setminus \min_{\prec_E^{\mathcal{R}}}(Ad^{\mathcal{R}}))$) is a ranked model of the KB. We want also to enforce the possibility that some PhD students could be particularly young, or that they are working students, hence, we end up with an ontology $\langle \mathcal{T}, \mathcal{D} \rangle$ where $\mathcal{T}$ is empty and $\mathcal{D} = \{PhD \sqsubsetsim_E Ad,\ PhD \sqsubsetsim_E \neg W, Ad \sqsubsetsim_E W\}$.*

It is possible to define multiple kinds of defeasible entailment for DLs [33, 26, 34, 35, 36, 37]. Here we refer exclusively to Rational Closure (RC), which has been characterised in various inferentially-equivalent ways [38, 39, 40, 41, 10, 31, 32], and we are going to refer to one DL-specific characterisation [42, 32]. We briefly introduce the *big ranked model*, the semantic construction used to characterise the RC of a KB $\mathcal{K}$. We refer the reader to the work of Britz et al. (2021) for the details. The construction is based on the *ranked union*, a variant of the *disjoint union* of DL interpretations. We generalise the original definition by Britz et al. to consider models with possibly many ranked orders defined over the domain.

Definition 3 (Ranked Union). *Let $\mathfrak{R}$ be a set of interpretations $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_1^{\mathcal{R}}, \dots, \prec_n^{\mathcal{R}} \rangle$ ($n \geq 0$), with each $\prec_j^{\mathcal{R}}$ being a ranked order defined over the domain $\Delta^{\mathcal{R}}$.*

The Ranked Union *of $\mathfrak{R}$ is an interpretation $\mathcal{R}^{\mathfrak{R}} := \langle \Delta^{\mathfrak{R}}, \cdot^{\mathfrak{R}}, \prec_1^{\mathfrak{R}}, \dots, \prec_n^{\mathfrak{R}} \rangle$ where:*

- *$\Delta^{\mathfrak{R}} := \coprod_{\mathcal{R} \in \mathfrak{R}} \Delta^{\mathcal{R}}$, i.e., the disjoint union of the domains from $\mathfrak{R}$, where each $\mathcal{R} \in \mathfrak{R}$ has the elements $x, y, \dots$ of its domain renamed as $x_{\mathcal{R}}, y_{\mathcal{R}}, \dots$ so that they are all distinct in $\Delta^{\mathfrak{R}}$;*
- *$x_{\mathcal{R}} \in A^{\mathfrak{R}}$ iff $x \in A^{\mathcal{R}}$;*
- *$(x_{\mathcal{R}}, y_{\mathcal{R}'}) \in r^{\mathfrak{R}}$ iff $\mathcal{R} = \mathcal{R}'$ and $(x, y) \in r^{\mathcal{R}}$;*
- *for all i ($1 \leq i \leq n$), $h_{\prec_i^{\mathfrak{R}}}(x_{\mathcal{R}}) = h_{\prec_i^{\mathcal{R}}}(x)$ (By Definition 1, $x \prec_i^{\mathfrak{R}} y$ iff $h_{\prec_i^{\mathfrak{R}}}(x) < h_{\prec_i^{\mathfrak{R}}}(y)$).*

To define the *big ranked model* for a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, we fix a countably infinite domain Δ and we consider the set $Mod_{\Delta}(\mathcal{K})$ of the ranked models of $\mathcal{K}$ with domain Δ: $Mod_{\Delta}(\mathcal{K}) := \{\mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}} \rangle \mid \mathcal{R} \Vdash \mathcal{K}$ and $\mathcal{R}$ is ranked$\}$. We define our entailment relation, modelling RC for DLs, on the model obtained by the ranked union (with $n = 1$) of the models in $Mod_{\Delta}(\mathcal{K})$.

Definition 4 (Big ranked model; Minimal Entailment). *The* big ranked model $\mathcal{O}$ *of a KB $\mathcal{K}$ is the ranked union of all the models in $Mod_{\Delta}(\mathcal{K})$. An inclusion $\alpha ::= C \sqsubseteq D \mid C \,\tilde{\sqsubset}_E\, D$ is* minimally entailed *by $\mathcal{K}$ ($\mathcal{K} \models_{\min} \alpha$) iff $\mathcal{O} \Vdash \alpha$.*

The intuition underlying RC is to enforce the *presumption of typicality* [44]: we reason assuming that every object behaves in the most typical/expected way according to the information available. Semantically, this can be modelled by considering the ranked models of the KB in which the objects are interpreted as low as possible in the ranks: we assume they are as typical as possible modulo the satisfaction of the KB. Britz et al. (2021) prove that the big ranked model and the associated minimal entailment are a correct semantic representation of RC. They also describe a procedure to decide minimal entailment of defeasible inclusions $C \,\tilde{\sqsubset}_E\, D$ based on three main points:

Exceptionality: A concept D is *exceptional* w.r.t. a KB $\mathcal{K}$ iff *all* models of $\mathcal{K}$ satisfy the inclusion $\top \,\tilde{\sqsubset}_E\, \neg D$; i.e., there is no model $\mathcal{R}$ of $\mathcal{K}$ in which an object in $\min_{\prec_C^{\mathcal{R}}}(\top)$ is in $D^{\mathcal{R}}$. The statement $C \,\tilde{\sqsubset}_E\, D$ is exceptional iff C is exceptional. For example, $Penguin \,\tilde{\sqsubset}_E\, \neg Fly$ represents an exceptional conditional w.r.t. a KB containing also the inclusions $Penguin \sqsubseteq Bird$ and $Bird \,\tilde{\sqsubset}_E\, Fly$.

Ranking: Iterating exceptionality yields a *ranking* of all defeasible inclusions in $\mathcal{D}$, starting from rank 0, and moving to more specific ones at every rank. We could, e.g., have a defeasible inclusion $Bird \,\tilde{\sqsubset}_E\, Fly$ at rank 0, an exceptional inclusion $Penguin \,\tilde{\sqsubset}_E\, \neg Fly$ at rank 1, and a more exceptional inclusion $PenguinWithJetpack \,\tilde{\sqsubset}_E\, Fly$ at rank 2. The ranking process shows some strict information that is potentially 'hidden' in $\mathcal{D}$. E.g., if $\mathcal{T} = \{D \sqsubseteq E\}$, the inclusions in $\mathcal{D} = \{C \,\tilde{\sqsubset}_E\, D, C \,\tilde{\sqsubset}_E\, \neg E\}$ convey strict information, since the KB is satisfied only by ranked models satisfying $C \sqsubseteq \bot$.

Rational Closure: Given the ranking of $\mathcal{D}$, one can decide whether an inclusion $C \,\tilde{\sqsubset}_E\, D$ is minimally entailed by $\mathcal{K}$: informally, we calculate the rank i of C, associate to C only the defeasible inclusions in $\mathcal{D}$ of rank $j \geq i$, and check whether we can derive $C \,\tilde{\sqsubset}_E\, D$.

These three points are implemented by three algorithms:

- `Exceptional`$(\langle \mathcal{T}, \mathcal{D} \rangle)$ returns the *exceptional* defeasible concept inclusions in $\mathcal{D}$ w.r.t. $\langle \mathcal{T}, \mathcal{D} \rangle$.
- `ComputeRanking`$(\langle \mathcal{T}, \mathcal{D} \rangle)$ returns, through some calls to `Exceptional`: (i) a rank-equivalent KB $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$ obtained by moving the strict information possibly present in $\mathcal{D}$ into $\mathcal{T}$; (ii) a sequence $\mathcal{E}$=$(\mathcal{E}_0, \dots, \mathcal{E}_n)$ s.t. $\mathcal{E}_0 = \mathcal{D}^*$ and, for each $i < n$, $\mathcal{E}_{i+1} \subset \mathcal{E}_i$, where $\mathcal{E}_{i+1}$ contains the defeasible inclusions in $\mathcal{E}_i$ that are exceptional w.r.t. $\langle \mathcal{T}^*, \mathcal{E}_i \rangle$.

- `RationalClosure`$(\langle \mathcal{T}, \mathcal{D}\rangle, C \sqsubsetsim_E D)$ returns, based on `ComputeRanking`, `true` if $C \sqsubsetsim_E D$ is minimally entailed by $\langle \mathcal{T}, \mathcal{D}\rangle$, and `false` otherwise.

3. Deontic Reasoning in $\mathcal{ALC}$

We use the framework presented before to introduce deontic notions in $\mathcal{ALC}$, reasoning about conditional obligations $C \sqsubsetsim_{\mathcal{N}} D$ ('if something is a C, then it ought to be a D) and conditional permissions $C \sqsubsetsim_{\mathcal{P}} D$ ('if something is a C, then it is allowed to be a D). In this section we consider knowledge bases $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P}\rangle$, where $\mathcal{T}$ is a classical TBox, and $\mathcal{N}$ (resp., $\mathcal{P}$) is a finite set of conditional obligations (resp., permissions). In this section we do not consider defeasible inclusions $\mathcal{D}$ modeling expectations.

Semantically, we formalise deontic reasoning by adapting ranked interpretations to deontic conditionals through a ranked order $\prec_N^{\mathcal{R}}$ that models the preferences of the agent.

Definition 5 (D-interpretation). *A* D-interpretation *(*deontic interpretation*) is a tuple $\mathcal{R}{:=}\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}}\rangle$, with $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}\rangle$ a DL interpretation, and $\prec_N^{\mathcal{R}}$ a ranked order over $\Delta^{\mathcal{R}}$.*

The relation $x \prec_N^{\mathcal{R}} y$ indicates that the object x is preferred to y. In the deontic logic community it is customary, given a pair $x \prec_N^{\mathcal{R}} y$, to consider y the preferred one; we rely on the inverse reading since it is more common in preferential semantics. Obligations are treated as defeasible inclusions except that the preference relation $\prec_N^{\mathcal{R}}$ ranks the objects in the domain according to their conformance to our normative values, and an object in C *ought to* be in D iff all the preferred objects in C are in D ($\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$). We also consider *permissions*, which are considered dual to obligations [5]: something is permitted if its negation is not mandatory. I.e., $C \sqsubsetsim_{\mathcal{P}} D$ holds iff $C \sqsubsetsim_{\mathcal{N}} \neg D$ does not hold.

Definition 6 (Satisfaction). *Let $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}}\rangle$ be a D-interpretation.*

$$\mathcal{R} \Vdash C \sqsubseteq D \text{ iff } C^{\mathcal{R}} \subseteq D^{\mathcal{R}}; \quad \mathcal{R} \Vdash C \sqsubsetsim_{\mathcal{N}} D \text{ iff } \min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}; \quad \mathcal{R} \Vdash C \sqsubsetsim_{\mathcal{P}} D \text{ iff } \begin{cases} \mathcal{R} \nVdash C \sqsubsetsim_{\mathcal{N}} \neg D; \text{ or} \\ \mathcal{R} \Vdash C \sqsubseteq \bot. \end{cases}$$

$\mathcal{R}$ satisfies *(is* a model of*) $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P}\rangle$ iff $\mathcal{R} \Vdash \alpha$ for every $\alpha \in \mathcal{T} \cup \mathcal{N} \cup \mathcal{P}$. A deontic KB $\mathcal{K}$ is* consistent *iff there is at least a D-interpretation $\mathcal{R}$ that satisfies it.*

Note that $\mathcal{R} \nVdash C \sqsubsetsim_{\mathcal{N}} \neg D$ is semantically equivalent to $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \cap D^{\mathcal{R}} \neq \emptyset$. The interpretation of permissions as negated obligations was already formalised in von Wright's seminal work (1951), and in dyadic deontic logic such a duality has simply been reformulated in the conditional form. It works also in our DL framework: $C \sqsubsetsim_{\mathcal{N}} D$ interpreted as $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$ means that 'knowing *only* that an object is in C, according to our normative system it ought to be also in D'; consequently, $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \cap D^{\mathcal{R}} \neq \emptyset$ means that 'knowing *only* that an object is in C, it is consistent with our normative system that it is also in D', since there is a $\prec_N^{\mathcal{R}}$-preferred object in $C^{\mathcal{R}}$, that is, an object in C abiding to all the rules that our system associates to C, that is also in D. That is, 'knowing *only* that an object is in C, it is permitted for it to be also in D'. The second condition in the satisfaction clause for permissions is only meant to cover the limiting case of an empty premise. Modelling permissions as negated obligations in our framework immediately recalls the definition of RC from [40] based on propositional conditionals ($\alpha \mid\!\sim \beta$) and negated conditionals ($\alpha \mid\!\not\sim \beta$). Our semantic construction is indeed inspired by it, despite a few differences, due to the expressivity of the language and a slightly different notion of consistency.

Example 2. *We still consider the class of the PhD students, but for now we put aside expectations and focus on the normative side. Consider a KB $\mathcal{K} = \langle \emptyset, \mathcal{N}, \mathcal{P}\rangle$ where*

$$\mathcal{N} = \{PhD \sqsubsetsim_{\mathcal{N}} \neg\exists pay.T, PhD \sqcap W \sqsubsetsim_{\mathcal{N}} \exists pay.T, PhD \sqsubsetsim_{\mathcal{N}} \neg PT\}, \quad \mathcal{P} = \{PhD \sqcap W \sqsubsetsim_{\mathcal{P}} PT\},$$

meaning that PhD students do not have to pay Taxes (T), but if they work they should; and they are not allowed to be part-time (PT), except if they work.

We consider the derivation of obligations and permissions. We adapt the *big ranked model*, from Section 2 to the case of deontic reasoning. As a result, we model a kind of deontic entailment based on what we could generally call *presumption of non-exceptionality*: we reason assuming that things are as least exceptional as possible, and apply to them the most general obligations and permissions consistent with the available information. The main difference w.r.t. the entailment of expectations from Section 2 is that we consider two kinds of conditionals, obligations and permissions, in our semantics and decision procedures. Since permissions are negated obligations, we need to reformulate the semantic definition of the minimal model and the corresponding decision procedure accordingly.

Given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a fixed countably infinite domain Δ, we consider the set $Mod_\Delta(\mathcal{K})$ of the D-models of $\mathcal{K}$ with domain Δ: $Mod_\Delta(\mathcal{K}) := \{\mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle \mid \mathcal{R} \Vdash \mathcal{K}$ and $\mathcal{R}$ is ranked$\}$, and, again, we define our entailment relation on the ranked union of the models in $Mod_\Delta(\mathcal{K})$.

Definition 7 (Big D-model; Minimal Entailment). *The* big D-model $\mathcal{O}_\mathcal{N}$ *of a KB* $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ *is the ranked union of all the models in* $Mod_\Delta(\mathcal{K})$. *An inclusion* $\alpha ::= C \sqsubseteq D \mid C \,\underset{\sim}{\sqsubset}_\mathcal{N}\, D \mid C \,\underset{\sim}{\sqsubset}_\mathcal{P}\, D$ *is* minimally entailed *by* $\mathcal{K}$ *(*$\mathcal{K} \models_{\min} \alpha$*) iff* $\mathcal{O}_\mathcal{N} \Vdash \alpha$.

As in defeasible $\mathcal{ALC}$, we can prove that, for any consistent deontic KB $\mathcal{K}$, $\mathcal{O}_\mathcal{N}$ is a *minimal* model of $\mathcal{K}$; i.e., for every concept C, the height of C in $\mathcal{O}_\mathcal{N}$ is minimal w.r.t. all the models of $\mathcal{K}$.

Proposition 1. *Let* $\mathcal{K}$ *be a consistent deontic KB and* $\mathcal{O}_\mathcal{N} := \langle \Delta^{\mathcal{O}_\mathcal{N}}, \cdot^{\mathcal{O}_\mathcal{N}}, \prec_N^{\mathcal{O}} \rangle$ *its Big D-model. Then* $\mathcal{O}_\mathcal{N} \Vdash \mathcal{K}$*, and, for any* $C \in \mathcal{L}$*,* $h_{\mathcal{O}_\mathcal{N}}(C) = \min\{h_\mathcal{R}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.

We now present sound and complete decision procedures for minimal entailment in this deontic framework, obtained by modifying the procedures from Section 2 to handle obligations and permissions. Our procedures have the same features as that of Britz et al.: (i) they reduce decision problems in non-monotonic reasoning to a series of calls to classical $\mathcal{ALC}$ reasoners, making the implementation of such procedures particularly easy; and (ii) the runtime behaviour of the overall decision procedure stays in the same complexity class as classical $\mathcal{ALC}$. To reduce all the decision problems to a series of classical $\mathcal{ALC}$ decision problems, we use two transformations: given any set Φ of conditionals $C \sqsubset D$ ($\sqsubset \in \{\sqsubseteq, \underset{\sim}{\sqsubset}_\mathcal{N}, \underset{\sim}{\sqsubset}_\mathcal{P}\}$), $\overline{\Phi} := \{\neg C \sqcup D \mid C \sqsubset D \in \Phi\}$; and $\Phi^\neg := \{\neg C \mid C \sqsubset D \in \Phi\}$ The main procedures refer to three points that are analogous to the ones presented at the end of Section 2.

Exceptionality: concepts that are exceptional w.r.t. preferences, based on the obligations and permissions from the KB. A concept C is *exceptional* w.r.t. $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ iff *all* models of $\mathcal{K}$ satisfy the inclusion $\top \,\underset{\sim}{\sqsubset}_\mathcal{N}\, \neg C$. That is, there is no model $\mathcal{R}$ of $\mathcal{K}$ in which an object violating no obligations (i.e., an object in $\min_{\prec_\mathcal{N}^\mathcal{R}}(\top)$) is in $C^\mathcal{R}$. A conditional obligation $C \,\underset{\sim}{\sqsubset}_\mathcal{N}\, D$ (resp., permission $C \,\underset{\sim}{\sqsubset}_\mathcal{P}\, D$) is exceptional iff C is exceptional. We can, e.g., have one norm that killing is forbidden ($\top \,\underset{\sim}{\sqsubset}_\mathcal{N}\, \neg Kill$), and another norm that, if you kill, you should do it gently ($Kill \,\underset{\sim}{\sqsubset}_\mathcal{N} GentleKill$) and that if you kill you have the right to defend yourself in court ($Kill \,\underset{\sim}{\sqsubset}_\mathcal{P}\, Defend$). $Kill \,\underset{\sim}{\sqsubset}_\mathcal{N}\, GentleKill$ and $Kill \,\underset{\sim}{\sqsubset}_\mathcal{P}\, Defend$ are exceptional inclusions, since they refer to a situation ($Kill$) incompatible with the preferred objects.

The algorithm `NormExceptional` is used to decide exceptionality. From a TBox $\mathcal{T}$, an NBox Φ, and a PBox Ψ, it outputs two (possibly empty) sets Φ' and Ψ', containing the exceptional conditionals in Φ and Ψ, respectively. The main difference w.r.t. defeasible $\mathcal{ALC}$ is that exceptional permissions $C \,\underset{\sim}{\sqsubset}_\mathcal{P}\, D$ impose that the most preferred objects fall under the concept $\neg C$, adding it to the set $\mathcal{E}$.

Example 3. *Running* `NormExceptional`$(\mathcal{T}, \mathcal{N}, \mathcal{P})$ *on the KB from Example 2 we start with* $\mathcal{T} = \emptyset$, $\mathcal{E} = \emptyset$*, and* $\overline{\Phi} = \{\neg PhD \sqcup \neg\exists pay.T, \neg(PhD \sqcap W) \sqcup (\exists pay.T), \neg PhD \sqcup \neg PT\}$. *Since* $\models \bigsqcap \overline{\Phi} \sqsubseteq \neg(PhD \sqcap W \sqcap PT)$*, we conclude that* $PhD \sqcap W \,\underset{\sim}{\sqsubset}_\mathcal{P}\, PT$ *is an exceptional permission (*$\Psi' = \{PhD \sqcap W \,\underset{\sim}{\sqsubset}_\mathcal{P}\, PT\}$ *and* $\mathcal{E} = \{\neg(PhD \sqcap W)\}$*). We also have* $\not\models \bigsqcap \overline{\Phi} \sqcap \bigsqcap \mathcal{E} \sqsubseteq \neg PhD$*, and* $\models \bigsqcap \overline{\Phi} \sqcap \bigsqcap \mathcal{E} \sqsubseteq \neg(PhD \sqcap W)$*, identifying the obligation* $\Phi' = \{PhD \sqcap W \,\underset{\sim}{\sqsubset}_\mathcal{N}\, \exists pay.T\}$ *as exceptional.*

Ranking: As with defeasible $\mathcal{ALC}$, iteratively applying the notion of exceptionality yields a *ranking* of all the defeasible inclusions. The ranking process may identify some pieces of defeasible information in the NBox and the PBox that correspond to strict GCIs. For example, the statements $C \,\underset{\sim}{\sqsubset}_\mathcal{N}\, D$ in the

Procedure NormExceptional($\mathcal{T}, \Phi, \Psi$)

Input: $\mathcal{T}$, $\Phi \subseteq \mathcal{N}$ and $\Psi \subseteq \mathcal{P}$
Output: $\Phi' \subseteq \Phi$ and $\Psi' \subseteq \Psi$ such that Φ' and Ψ' are sets of exceptional axioms w.r.t. $\langle \mathcal{T}, \Phi, \Psi \rangle$

```
Φ' := ∅; Ψ' := ∅; ℰ = ∅;
repeat
    ℱ := ℰ;
    foreach C ⊑̰_𝒫 D ∈ Ψ do
        if 𝒯 ⊨ ⨅Φ̄ ⊓ ⨅ℰ ⊑ ¬(C ⊓ D) then
            Ψ' := Ψ' ∪ {C ⊑̰_𝒫 D};
            ℰ := ℰ ∪ {¬C};
until ℰ = ℱ;
foreach C ⊑̰_𝒩 D ∈ Φ do
    if 𝒯 ⊨ ⨅Φ̄ ⊓ ⨅ℰ ⊑ ¬C then
        Φ' := Φ' ∪ {C ⊑̰_𝒩 D};
return Φ', Ψ'
```

Procedure ComputeNormRanking($\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$)

Input: $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$
Output: Ontology $\langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$, exceptionality ranking $R_\mathcal{N}$ of $\mathcal{N}^\circ$, exceptionality ranking $R_\mathcal{P}$ of $\mathcal{P}^\circ$

```
𝒯°:=𝒯;
repeat
    i := 0;
    Φ_0 := 𝒩;
    Ψ_0 := 𝒫;
    Φ_1, Ψ_1 := NormExceptional(𝒯*, Φ_0, Ψ_0);
    repeat
        i := i + 1;
        Φ_{i+1}, Ψ_{i+1} := NormExceptional(𝒯*, Φ_i, Ψ_i);
    until Φ_{i+1} = Φ_i and Ψ_{i+1} = Ψ_i;
    𝒩_∞ := Φ_i;
    𝒫_∞ := Ψ_i;
    𝒩° := 𝒩 \ 𝒩_∞;
    𝒫° := 𝒫 \ 𝒫_∞;
    𝒯° := 𝒯° ∪ {E ⊑ F | E ⊑̰_𝒩 F ∈ 𝒩_∞} ∪ {E ⊑ ⊥ | E ⊑̰_𝒫 F ∈ 𝒫_∞};
until 𝒩_∞ = ∅ and 𝒫_∞ = ∅;
R_𝒩 := (Φ_0, ..., Φ_{i−1});
R_𝒫 := (Ψ_0, ..., Ψ_{i−1});
return (𝒦° = ⟨𝒯°, 𝒩°, 𝒫°⟩, R_𝒩, R_𝒫);
```

NBox and $C \precsim_\mathcal{P} \neg D$ in the PBox express that it is impossible to have any object in the concept C. That is, an interpretations is a model of $C \precsim_\mathcal{N} D$, $C \precsim_\mathcal{P} \neg D$ iff it is a model of $C \sqsubseteq \bot$.

`ComputeNormRanking` implements the KB ranking. It calls `NormExceptional` iteratively, until reaching a fixed point, and returns two objects: (1) a KB $\mathcal{K}^\circ = \langle \mathcal{T}^\circ, \mathcal{N}^\circ, \mathcal{P}^\circ \rangle$ that is rank-equivalent to $\mathcal{K}$, obtained by moving all the strict information that was 'hidden' in $\mathcal{N}$ and $\mathcal{P}$ into $\mathcal{T}$; and (2) two sequences: $R_\mathcal{N} = (\Phi_0, \ldots, \Phi_n)$ s.t. $\Phi_0 = \mathcal{N}^\circ$ and, for every $i < n$, $\Phi_{i+1} \subseteq \Phi_i$, since Φ_{i+1} contains the obligations in Φ_i which are exceptional w.r.t. $\langle \mathcal{T}^\circ, \Phi_i, \Psi_i \rangle$; and $R_\mathcal{P} = (\Psi_0, \ldots, \Psi_n)$ s.t. $\Psi_0 = \mathcal{P}^\circ$ and, for every $i < n$, $\Psi_{i+1} \subseteq \Psi_i$, contains the permissions in Ψ_i which are exceptional w.r.t. $\langle \mathcal{T}^\circ, \Phi_i, \Psi_i \rangle$.

Example 4. *We run* `ComputeNormRanking`$(\emptyset, \mathcal{N}, \mathcal{P})$ *on the KB from Example 2; the single run of* `Normexceptional`$(\emptyset, \mathcal{N}, \mathcal{P})$ *seen in Example 3 is sufficient to give us the ranking. We end up with* $\mathcal{T}^\circ = \emptyset$; $R_\mathcal{N} = \{\Phi_0, \Phi_1\}$, *with* $\Phi_0 = \mathcal{N}$ *and* $\Phi_1 = \{PhD \sqcap W \precsim_\mathcal{N} \exists pay.T\}$; *and* $R_\mathcal{P} = \{\Psi_0, \Psi_1\}$, *with* $\Psi_0 = \Psi_1 = \mathcal{P} = \{PhD \sqcap W \precsim_\mathcal{P} PT\}$.

Minimal Entailment. Given the rankings $R_\mathcal{N}$ and $R_\mathcal{P}$, one can decide whether some obligation $C \precsim_\mathcal{N} D$ or permission $C \precsim_\mathcal{P} D$ is minimally entailed by $\mathcal{K}$ using (i) `N-DeonticClosure`$(\mathcal{K}, C \precsim_\mathcal{N} D)$, to decide whether $C \precsim_\mathcal{N} D$ is minimally entailed by $\mathcal{K}$; and (ii) `P-DeonticClosure`$(\mathcal{K}, C \precsim_\mathcal{P} D)$, to decide whether $C \precsim_\mathcal{P} D$ is minimally entailed by $\mathcal{K}$. Informally, `N-DeonticClosure`$(\mathcal{K}, C \precsim_\mathcal{N} D)$

computes the rank i of C, associates to C only the defeasible obligations in rank i, plus the negations of all the antecedents of the permissions that have rank $i+1$, and checks whether it can derive $C \precsim_{\mathcal{N}} D$.

Example 5. *To see whether PhD students with family (F) cannot be part time, we run* `N-DeonticClosure`$(\mathcal{K}, PhD \sqcap F \precsim_{\mathcal{N}} \neg PT)$*, with $\mathcal{K}$ from Example 2. $\mathcal{T}^{\circ}$, $R_{\mathcal{N}}$ and $R_{\mathcal{P}}$ are indicated in Example 4. Since $\mathcal{T}^{\circ} \not\models \bigsqcap \overline{\Phi_0} \sqcap \bigsqcap \Psi_1^{\neg} \sqcap (PhD \sqcap F) \sqsubseteq \bot$, the rank i of $PhD \sqcap F$ is 0. The reader can check that $\mathcal{T}^{\circ} \models \bigsqcap \overline{\Phi_0} \sqcap \bigsqcap \Psi_1^{\neg} \sqcap (PhD \sqcap F) \sqsubseteq \neg PT$. That is, with only the rules in the KB, we must conclude that PhD students with families cannot be part time. This example shows that minimal entailment implements the* presumption of non-exceptionality*: based on our KB, PhD students with families do not have any special rule applying to them, and consequently they are not treated as exceptional, at least until extra rules about them are introduced in the KB.*

The procedure `P-DeonticClosure`$(\mathcal{K}, C \precsim_{\mathcal{P}} D)$ decides conditional permissions using Definition 6. It first calls `N-DeonticClosure`$(\mathcal{K}, C \precsim_{\mathcal{N}} D)$ to check whether $\mathcal{K}$ minimally entails $C \precsim_{\mathcal{N}} \neg D$. If that is not the case, then $\mathcal{K}$ minimally entails $C \precsim_{\mathcal{P}} D$. If, instead, $\mathcal{K}$ minimally entails $C \precsim_{\mathcal{N}} \neg D$, the procedure checks whether $C \sqsubseteq \bot$ is minimally entailed by $\mathcal{K}$, and, if not, concludes that $C \precsim_{\mathcal{P}} D$ is not minimally entailed by $\mathcal{K}$. Let $\vdash_{\mathcal{N}}$ be a derivation relation that refers to these procedures.

Definition 8 (Relation $\vdash_{\mathcal{N}}$). *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ be a deontic KB, $\mathcal{K}^{\circ} = \langle \mathcal{T}^{\circ}, \mathcal{N}^{\circ}, \mathcal{P}^{\circ} \rangle =$* `ComputeNormRanking`$(\mathcal{K})$*, $C, D \in \mathcal{L}$, and $\sqsubset \in \{\sqsubseteq, \precsim_{\mathcal{N}}, \precsim_{\mathcal{P}}\}$. $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubset D$ iff*

- *$\sqsubset = \sqsubseteq$ and $\mathcal{T}^{\circ} \models C \sqsubseteq D$;*
- *$\sqsubset = \precsim_{\mathcal{N}}$ and* `N-DeonticClosure`$(\mathcal{K}, C \precsim_{\mathcal{N}} D) =$`true`*;*
- *$\sqsubset = \precsim_{\mathcal{P}}$ and* `P-DeonticClosure`$(\mathcal{K}, C \precsim_{\mathcal{P}} D) =$`true`*.*

It can be proved that the procedures are sound and complete w.r.t. minimal entailment.

Theorem 1. *For any deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and inclusion $C \sqsubset D$ ($\sqsubset \in \{\sqsubseteq, \precsim_{\mathcal{N}}, \precsim_{\mathcal{P}}\}$), we have $\mathcal{K} \models_{\min} C \sqsubset D$ iff $\mathcal{K} \vdash_{\mathcal{N}} C \sqsubset D$.*

An easy consequence is that to decide whether a deontic KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is consistent, it suffices to check the TBox $\mathcal{T}^{\circ}$: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ is ranked-inconsistent iff $\mathcal{T}^{\circ} \models \top \sqsubseteq \bot$.

The computational costs of the overall decision procedure stay is not higher than for classical $\mathcal{ALC}$.

Proposition 2. *Checking minimal entailment for a Deontic KB is EXPTIME-complete.*

4. Deontic Defeasible DL

Now we move to the task of modelling *deontic defeasible reasoning*; i.e., combining the epistemic and deontic modalities, merging reasoning about expectations and about obligations/permissions in order to determine what agents should do in front of uncertain information. The goal is to derive conditionals of the kind 'presumably, a C ought/is permitted to be a D.' To this end, we introduce ranked models in which the two ranked orders, $\prec_E^{\mathcal{R}}$ modelling expectations and $\prec_N^{\mathcal{R}}$ modelling norms, interact.

Procedure N-DeonticClosure$(\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle, C \precsim_{\mathcal{N}} D)$

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a query $C \precsim_{\mathcal{N}} D$.
Output: `true` if $C \precsim_{\mathcal{N}} D$ is in the Deontic Closure of $\mathcal{K}$, `false` otherwise
$(\mathcal{K}^{\circ} = \langle \mathcal{T}^{\circ}, \mathcal{N}^{\circ}, \mathcal{P}^{\circ} \rangle, R_{\mathcal{N}} = (\Phi_0, \ldots, \Phi_n), R_{\mathcal{P}} = (\Psi_0, \ldots, \Psi_n)) :=$ `ComputeNormRanking`$(\mathcal{K})$;
$i := 0$;
$\Psi_{n+1} := \emptyset$;
while $\mathcal{T}^{\circ} \models \bigsqcap \overline{\Phi_i} \sqcap \bigsqcap \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq \bot$ *and* $i \leq n$ **do**
    $i := i + 1$;
if $i \leq n$ **then**
    return $\mathcal{T}^{\circ} \models \bigsqcap \overline{\Phi_i} \sqcap \bigsqcap \Psi_{i+1}^{\neg} \sqcap C \sqsubseteq D$;
else
    return $\mathcal{T}^{\circ} \models C \sqsubseteq D$;

Procedure P-DeonticClosure($\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle, C \precsim_{\mathcal{P}} D$)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$ and a query $C \precsim_{\mathcal{P}} D$.
Output: `true` if $C \precsim_{\mathcal{P}} D$ is in the Deontic Closure of $\mathcal{K}$, `false` otherwise
if `N-DeonticClosure`($\mathcal{K}, C \precsim_{\mathcal{N}} \neg D$) $=$ `false` **then**
  return `true`;
else if $\mathcal{T}^{\circ} \models C \sqsubseteq \bot$ **then**
  return `true`;
else
  return `false`;

Definition 9 (DD-Interpretation). *A* DD-interpretation *(deontic defeasible interpretation) is a tuple* $\mathcal{R} := \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$*, with* $\langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}} \rangle$ *a DL interpretation, and* $\prec_E^{\mathcal{R}}$ *and* $\prec_N^{\mathcal{R}}$ *ranked orders over* $\Delta^{\mathcal{R}}$*.*

The relations $x \prec_E^{\mathcal{R}} y$, $x \prec_N^{\mathcal{R}} y$ indicate that the object x is more in line with our expectations or deontic preferences than y, respectively. The orders $\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$ are independent from each other, each associated to a different modality and with corresponding height functions $h_{\prec_E^{\mathcal{R}}}$ and $h_{\prec_N^{\mathcal{R}}}$, respectively. Given an object $x \in \Delta^{\mathcal{R}}$, $h_{\prec_E^{\mathcal{R}}}(x) = 0$ indicates that x satisfies all the expectations, and the height is proportional to the exceptionality of its properties w.r.t our expectations. Analogously, $h_{\prec_N^{\mathcal{R}}}(x) = 0$ indicates that x satisfies all our normative constraints, and the height is inversely related to compliance with the norms.

We consider KBs $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ containing classical GCIs $C \sqsubseteq D$, conditional expectations $C \precsim_E D$, conditional obligations $C \precsim_{\mathcal{N}} D$, and conditional permissions $C \precsim_{\mathcal{P}} D$; and introduce two new kinds of defeasible inclusions, that can be derived from a KB: *presumptive obligations* $C \precsim_{d\mathcal{N}} D$ and *presumptive permissions* $C \precsim_{d\mathcal{P}} D$. In fact, we will derive the following new inclusions: $C \sqsubseteq D$ (An individual in C is also in D); $C \precsim_E D$ (*Presumably*, an individual in C is in D); $C \precsim_{\mathcal{N}} D$ (An individual in C *should* be in D); $C \precsim_{\mathcal{P}} D$ (An individual in C *is allowed* to be in D); $C \precsim_{d\mathcal{N}} D$ (*Presumably*, an individual in C *should* be in D); $C \precsim_{d\mathcal{P}} D$ (*Presumably*, an individual in C *is allowed* to be in D). We put aside for one moment the newly introduced presumptive obligations and permissions, since they are only derivable and not in the KB. Regarding the inclusions we have considered until now, the satisfaction conditions are preserved from the previous sections.

Definition 10 (Satisfaction). *Let* $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle$ *be a DD-interpretation. The satisfaction relation* $\Vdash$ *is defined as:* $\mathcal{R} \Vdash C \sqsubseteq D$ *iff* $C^{\mathcal{R}} \subseteq D^{\mathcal{R}}$*;* $\mathcal{R} \Vdash C \precsim_E D$ *iff* $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$*;* $\mathcal{R} \Vdash C \precsim_{\mathcal{N}} D$ *iff* $\min_{\prec_N^{\mathcal{R}}}(C^{\mathcal{R}}) \subseteq D^{\mathcal{R}}$*; and* $\mathcal{R} \Vdash C \precsim_{\mathcal{P}} D$ *iff either* $\mathcal{R} \nVdash C \precsim_{\mathcal{N}} \neg D$ *or* $\mathcal{R} \Vdash C \sqsubseteq \bot$*.*

$\mathcal{R}$ satisfies *(is* a model of*)* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ *iff* $\mathcal{R} \Vdash \alpha$ *for all statements* $\alpha \in \mathcal{T} \cup \mathcal{D} \cup \mathcal{N} \cup \mathcal{P}$*. A KB* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ *is* consistent *iff it is satisfied by a DD-interpretation.*

In this multi-modal context, we aim at models in which everything is considered *as least exceptional as possible*, w.r.t. $\prec_E^{\mathcal{R}}$ and $\prec_N^{\mathcal{R}}$, compatibly with $\mathcal{K}$. Hence, we implement once again the same construction seen above, based on a fixed countably infinite domain and the ranked union (Definition 3). Given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and fixed any countably infinite domain Δ, we consider the set $Mod_{\Delta}(\mathcal{K})$ of the DD-models of $\mathcal{K}$ with domain Δ $Mod_{\Delta}(\mathcal{K}) := \{\mathcal{R} = \langle \Delta, \cdot^{\mathcal{R}}, \prec_E^{\mathcal{R}}, \prec_N^{\mathcal{R}} \rangle \mid \mathcal{R} \Vdash \mathcal{K}$ and $\mathcal{R}$ is a DD-interpretation$\}$. We define our entailment relation on the model obtained by the ranked union of the models in $Mod_{\Delta}(\mathcal{K})$.

Definition 11 (Big DD-model & Minimal Entailment). *The* big DD-model $\mathcal{O}_{\mathcal{DN}}$ *of a KB* $\mathcal{K}$ *is the ranked union of all the models in* $Mod_{\Delta}(\mathcal{K})$*. An inclusion* α *(*$\alpha ::= C \sqsubseteq D \mid C \precsim_E D \mid C \precsim_{\mathcal{N}} D \mid C \precsim_{\mathcal{P}} D$*) is* minimally entailed *by* $\mathcal{K}$ *(*$\mathcal{K} \models_{\min} \alpha$*) iff* $\mathcal{O}_{\mathcal{DN}} \Vdash \alpha$*.*

As before, we can prove that, for any consistent deontic KB $\mathcal{K}$, $\mathcal{O}_{\mathcal{DN}}$ is a *minimal* model of $\mathcal{K}$, meaning that for every concept C, the height of C in $\mathcal{O}_{\mathcal{DN}}$ is minimal w.r.t. all the models of $\mathcal{K}$.

Procedure CombinedRanking($\langle\mathcal{T},\mathcal{D},\mathcal{N},\mathcal{P}\rangle$)

Input: $\langle\mathcal{T},\mathcal{D},\mathcal{N},\mathcal{P}\rangle$
Output: $\langle\mathcal{T}^*,\mathcal{D}^*,\mathcal{N}^*,\mathcal{P}^*\rangle$, and rankings $\mathcal{E}=\{\mathcal{E}_0,\dots,\mathcal{E}_n\}$ for $\mathcal{D}^*$, $R_{\mathcal{N}}=\{\Phi_0,\dots,\Phi_m\}$ for $\mathcal{N}^*$, and $R_{\mathcal{P}}=\{\Psi_0,\dots,\Psi_m\}$ for $\mathcal{P}^*$

```
T* := T; D* := D; N* := N; P* := P;
repeat
    T^in := T*;
    D^in := D*;
    N^in := N*;
    ⟨T*, D*⟩, E = (E_0, ..., E_n)) := ComputeRanking(⟨T*, D*⟩);
    ⟨T*, N*, P*⟩, R_N = (Φ_0, ..., Φ_m), R_P = (Ψ_0, ..., Ψ_m)) := ComputeNormRanking(⟨T*, N*, P*⟩);
until T* = T^in;
return K* = ⟨T*, D*, N*, P*⟩, E, R_N, R_P;
```

Proposition 3. *Let $\mathcal{K}$ be a consistent deontic KB and $\mathcal{O}_{\mathcal{DN}} := \langle\Delta^{\mathcal{O}_{\mathcal{DN}}}, \cdot^{\mathcal{O}_{\mathcal{DN}}}, \prec_E^{\mathcal{O}}, \prec_N^{\mathcal{O}}\rangle$ its Big DD-model. Then $\mathcal{O}_{\mathcal{DN}} \Vdash \mathcal{K}$, and, for any $C \in \mathcal{L}$, $h_{\prec_N^{\mathcal{O}}}(C) = \min\{h_{\prec_N^{\mathcal{R}}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$ and $h_{\prec_E^{\mathcal{O}}}(C) = \min\{h_{\prec_E^{\mathcal{R}}}(C) \mid \mathcal{R} \Vdash \mathcal{K}\}$.*

Returning to the new conditionals introduced in this section, the idea to model 'presumably, if something is C then it ought to be D' is that $C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$ is satisfied by a DD-interpretation iff we consider all the expectations that we associate with C, that are represented by the objects in $\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})$, and check if, according to $\prec_N^{\mathcal{R}}$, we associate to such expectations some obligation that enforces D; i.e., $\mathcal{R} \Vdash C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$ iff $\min_{\prec_N^{\mathcal{R}}}(\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})) \subseteq D^{\mathcal{R}}$. Analogously for $C \,\tilde{\sqsubseteq}_{d\mathcal{P}}\, D$, we consider the expectations associated to C and check whether they are compatible, according to $\prec_N^{\mathcal{R}}$, with D. That is, if C is satisfiable in $\mathcal{R}$, $\mathcal{R} \Vdash C \,\tilde{\sqsubseteq}_{d\mathcal{P}}\, D$ if $\min_{\prec_N^{\mathcal{R}}}(\min_{\prec_E^{\mathcal{R}}}(C^{\mathcal{R}})) \cap D^{\mathcal{R}} \neq \emptyset$, while we maintain that in case $C^{\mathcal{R}}$ is empty, we trivially derive that everything is permitted given C. From this definition it is immediate to see that, also for presumptive permissions and obligations, the complementarity of the two notions persists. That is, $\mathcal{R} \Vdash C \,\tilde{\sqsubseteq}_{d\mathcal{P}}\, D$ iff $\mathcal{R} \not\Vdash C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, \neg D$; or $\mathcal{R} \Vdash C \sqsubseteq \bot$. From the semantic definitions it is easy to check that, also in the case of $\tilde{\sqsubseteq}_{d\mathcal{N}}$ and $\tilde{\sqsubseteq}_{d\mathcal{P}}$, if a concept is defeasibly subsumed by $\bot$, it is also classicaly subsumed by $\bot$.

Proposition 4. *For any concept C and DD-interpretation, the following inclusions are equivalent: $C \sqsubseteq \bot$, $C \,\tilde{\sqsubseteq}_E\, \bot$, $C \,\tilde{\sqsubseteq}_{\mathcal{N}}\, \bot$, $C \,\tilde{\sqsubseteq}_{\mathcal{P}}\, \bot$, $C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, \bot$, $C \,\tilde{\sqsubseteq}_{d\mathcal{P}}\, \bot$.*

The procedures we already have at our disposal are almost everything needed to handle deontic defeasible $\mathcal{ALC}$. We just need to model how the two ranked orders interact, and how to decide $\tilde{\sqsubseteq}_{d\mathcal{N}}$- and $\tilde{\sqsubseteq}_{d\mathcal{P}}$-inclusions. To this end, we first use `CombinedRanking`($\langle\mathcal{T},\mathcal{D},\mathcal{N},\mathcal{P}\rangle$) to model how the two rankings, dealing respectively with $\mathcal{D}$ and $\langle\mathcal{N},\mathcal{P}\rangle$, interact. The two ranks can interact through the equivalence of the inclusions with $\bot$ on the right (Proposition 4); i.e., a KB $\mathcal{K} = \langle\mathcal{T},\mathcal{D},\mathcal{N},\mathcal{P}\rangle$ can be divided into the part dealing with expectations, $\langle\mathcal{T},\mathcal{D}\rangle$, and the part dealing with norms, $\langle\mathcal{T},\mathcal{N},\mathcal{P}\rangle$. Each of them is in principle "independent," and can be ranked by `ComputeRanking` and `ComputeNormRanking`, respectively. Yet, the TBox is present in both procedures, and each of them can identify some GCIs which are implicit in the defeasible information and then added to the TBox thus affecting each ranking procedure. The function of `CombinedRanking` is to iterate the two ranking procedures until a fixed point is found and all the GCIs 'hidden' in the expectations and in the norms have been moved to the TBox $\mathcal{T}^*$, and two final rankings have been defined.

From the output of `CombinedRanking`, we can decide $\tilde{\sqsubseteq}_E$-, $\tilde{\sqsubseteq}_{\mathcal{N}}$-, and $\tilde{\sqsubseteq}_{\mathcal{P}}$-inclusions running `RationalClosure`, `N-DeonticClosure` and `P-DeonticClosure`, respectively, substituting each call of `ComputeRanking` and `ComputeNormRanking` with `CombinedRanking`. We introduce new procedures `N-DD-Closure`($\mathcal{K}, C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$) and `P-DD-Closure`($\mathcal{K}, C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$) to decide $\tilde{\sqsubseteq}_{d\mathcal{N}}$- and $\tilde{\sqsubseteq}_{d\mathcal{P}}$-inclusions.

`N-DD-Closure`($\mathcal{K}, C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$), given the query $C \,\tilde{\sqsubseteq}_{d\mathcal{N}}\, D$, and the TBox $\mathcal{T}^*$ plus the rankings of $\mathcal{D}$, $\mathcal{N}$ and $\mathcal{P}$ obtained by `CombinedRanking`($\mathcal{K}$), (1) identifies the rank i of C w.r.t. the $\prec_E^{\mathcal{O}}$, modelling expectations, and associates to C the concepts $\overline{\mathcal{E}}_i$, representing the expectations of rank i; then (2) identifies the rank l of $C \sqcap \bigsqcap \overline{\mathcal{E}}_i$ w.r.t. the $\prec_N^{\mathcal{O}}$, modelling norms, and associates to C the concepts $\overline{\Phi}_l$ and $\Psi_{l+1}^{\neg}$, representing the norms and permissions at rank l; and (3) decides whether we can conclude that a C, presumably, should be a D by checking whether C, together $\overline{\mathcal{E}}_i$, $\overline{\Phi}_l$, and $\Psi_{l+1}^{\neg}$, is subsumed by D. `P-DD-Closure`$(\mathcal{K}, C \sqsubsim_{d\mathcal{P}} D)$ is modelled relying on its relation with $C \sqsubsim_{d\mathcal{N}} \neg D$.

Example 6. *Let $\mathcal{K} = \langle \emptyset, \mathcal{D}, \mathcal{N}, \emptyset \rangle$ be a KB where $\mathcal{D} = \{PhD \sqsubsim_E Ad,\ PhD \sqsubsim_E \neg W, Ad \sqsubsim_E W\}$. On the normative side we consider only obligations, to show the interaction of expectations and norms while preserving simplicity in the example: $\mathcal{N} = \{\neg W \sqsubsim_{\mathcal{N}} \neg\exists pay.T, W \sqsubsim_{\mathcal{N}} \exists pay.T\}$ (non-workers should not pay taxes, while workers should).* `CombinedRanking`$(\mathcal{K})$ *calls* `ComputeRanking`$(\langle \mathcal{T}, \mathcal{D} \rangle)$*, and obtains $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1\}$, with $\mathcal{E}_0 = \mathcal{D}$, $\mathcal{E}_1 = \{PhD \sqsubsim_E Ad,\ PhD \sqsubsim_E \neg W\}$: PhD students are exceptional cases w.r.t. our expectations (since they are non-working adults). $\mathcal{T}$ remains empty. The procedure calls* `ComputeNormRanking`$(\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle)$*. Since $\mathcal{T}$ and $\mathcal{P}$ are both empty,* `NormExceptionality` *is straightforward. $\overline{\Phi} = \{W \sqcup \neg\exists pay.T, \neg W \sqcup \exists pay.T\}$ and, since $\not\models \bigsqcap \overline{\Phi} \sqsubseteq W$ and $\not\models \bigsqcap \overline{\Phi} \sqsubseteq \neg W$, we do not have exceptional rules: $R_{\mathcal{N}} = \{\mathcal{N}\}$.*

We want to know whether presumably PhD students should not pay taxes ($PhD \sqsubsim_{d\mathcal{N}} \neg\exists pay.T$), and run `N-DD-Closure`$(\mathcal{K}, PhD \sqsubsim_{d\mathcal{N}} \neg\exists pay.T)$*. It identifies the rank of PhD w.r.t. $\langle \mathcal{T}, \mathcal{D} \rangle$ (which is 1) and associates the expectations $\mathcal{E}_1$ to it. It then checks the rank of $PhD \sqcap \bigsqcap \overline{\mathcal{E}_1}$ w.r.t. $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$. It concludes that this rank is 0, and associates Φ_0 to it. It can be checked that $\models PhD \sqcap \bigsqcap \overline{\mathcal{E}_1} \sqcap \bigsqcap \overline{\Phi_0} \sqsubseteq \neg\exists pay.T$, since $PhD \sqsubsim_E \neg W \in \mathcal{E}_1$ and $\neg W \sqsubsim_{\mathcal{N}} \neg\exists pay.T \in \Phi_0$. That is, we conclude $PhD \sqsubsim_{d\mathcal{N}} \neg\exists pay.T$: presumably PhD students should not pay taxes.*

We now check if presumably working PhD students should pay taxes ($PhD \sqcap W \sqsubsim_{d\mathcal{N}} \exists pay.T$). The concept $PhD \sqcap W$ is exceptional also w.r.t. $\mathcal{E}_1$, hence its rank is 2 and is associated to $\mathcal{E}_2 = \emptyset$ (hence $\bigsqcap \overline{\mathcal{E}_2} = \top$ and no need to consider it further). For the deontic ranking, $PhD \sqcap W$ is not exceptional w.r.t. $\langle \mathcal{T}, \mathcal{N}, \mathcal{P} \rangle$, and we associate Φ_0 to it. The reader can check that $\models PhD \sqcap W \sqcap \bigsqcap \overline{\Phi_0} \sqsubseteq \exists pay.T$, since $W \sqsubsim_{\mathcal{N}} \exists pay.T \in \Phi_0$. Thus, $PhD \sqcap W \sqsubsim_{d\mathcal{N}} \exists pay.T$: presumably working PhD students should pay taxes.

The relation $\vdash_{\mathcal{DN}}$ is introduced in order to refer to the derivations obtained using the new procedures.

Definition 12 (Relation $\vdash_{\mathcal{DN}}$). *Let $\mathcal{K}$ be a deontic defeasible KB; $C, D \in \mathcal{L}$; $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle =$* `CombinedRanking`$(\mathcal{K})$*; and $\sqsubset \in \{\sqsubseteq, \sqsubsim_E, \sqsubsim_{\mathcal{N}}, \sqsubsim_{\mathcal{P}}, \sqsubsim_{d\mathcal{N}}, \sqsubsim_{d\mathcal{P}}\}$. $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubset D$ iff*

- *$\sqsubset = \sqsubseteq$ and $\mathcal{T}^* \models C \sqsubseteq D$;*
- *$\sqsubset = \sqsubsim_E$ and* `RationalClosure`$^*(\mathcal{K}, C \sqsubsim_E D)$=`true`*;*
- *$\sqsubset = \sqsubsim_{\mathcal{N}}$ and* `N-DeonticClosure`$^*(\mathcal{K}, C \sqsubsim_{\mathcal{N}} D)$=`true`*;*
- *$\sqsubset = \sqsubsim_{\mathcal{P}}$ and* `P-DeonticClosure`$^*(\mathcal{K}, C \sqsubsim_{\mathcal{P}} D)$=`true`*;*
- *$\sqsubset = \sqsubsim_{d\mathcal{N}}$ and* `N-DD-Closure`$(\mathcal{K}, C \sqsubsim_{d\mathcal{N}} D)$=`true`*;*
- *$\sqsubset = \sqsubsim_{d\mathcal{P}}$ and* `P-DD-Closure`$(\mathcal{K}, C \sqsubsim_{d\mathcal{P}} D)$=`true`.

where `RationalClosure`* *indicates the procedure obtained from* `RationalClosure` *[43] by substituting each call of* `ComputeRanking` *with a call of* `CombinedRanking`*; analogously* `N-DeonticClosure`* *and* `P-DeonticClosure`* *are obtained substituting calls to* `ComputeNormRanking` *with calls to* `CombinedRanking`.

The procedures are sound and complete w.r.t. minimal entailment.

Theorem 2. *For any KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ and inclusion $C \sqsubset D$ ($\sqsubset \in \{\sqsubseteq, \sqsubsim_E, \sqsubsim_{\mathcal{N}}, \sqsubsim_{\mathcal{P}}, \sqsubsim_{d\mathcal{N}}, \sqsubsim_{d\mathcal{P}}\}$), we have $\mathcal{K} \models_{\min} C \sqsubset D$ iff $\mathcal{K} \vdash_{\mathcal{DN}} C \sqsubset D$.*

Hence, checking the consistency of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ corresponds to checking the TBox $\mathcal{T}^*$.

Corollary 1. *$\langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$ is inconsistent iff $\mathcal{T}^* \models \top \sqsubseteq \bot$.*

The computational costs of the overall decision procedure stay in the same complexity class as $\mathcal{ALC}$.

Proposition 5. *Checking minimal entailment for a Deontic Defeasible KB is EXPTIME-complete.*

Procedure N-DD-Closure($\mathcal{K}$, α)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$, a query $\alpha = C \sqsubsetsim_{d\mathcal{N}} D$.
Output: `true` if $\mathcal{K} \models_{\min} C \sqsubsetsim_{d\mathcal{N}} D$, `false` ow.
$\langle \mathcal{T}^*, \mathcal{D}^*, \mathcal{N}^*, \mathcal{P}^* \rangle$, $\mathcal{E} = (\mathcal{E}_0, \dots, \mathcal{E}_n)$, $R_{\mathcal{N}} = (\Phi_0, \dots, \Phi_m)$, $R_{\mathcal{P}} = (\Psi_0, \dots, \Psi_m) :=$ `CombinedRanking`($\mathcal{K}$);
$i := 0$;
$\Psi_{m+1} := \emptyset$;
while $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap C \sqsubseteq \bot$ *and* $i \leq n$ **do**
  $i := i + 1$;
if $i = n + 1$ **then**
  $\mathcal{E}_{n+1} := \emptyset$;
$l := 0$;
while $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap \bigsqcap \overline{\Phi_l} \sqcap \bigsqcap \Psi_{l+1}^{\neg} \sqcap C \sqsubseteq \bot$ *and* $l \leq m$ **do**
  $l := l + 1$;
if $l = m + 1$ **then**
  $\Phi_l := \emptyset$;
  $\Psi_{l+1} := \emptyset$;
return $\mathcal{T}^* \models \bigsqcap \overline{\mathcal{E}_i} \sqcap \bigsqcap \overline{\Phi_l} \sqcap \bigsqcap \Psi_{l+1}^{\neg} \sqcap C \sqsubseteq D$;

5. Conclusions

We introduced Deontic Defeasible Description Logic. The main contributions of the paper are (i) a system for modelling deontic reasoning in DLs; (ii) a second system combining reasoning about expectations and norms in DLs. Of both entailment relations we have given a semantic characterisation and the corresponding decision procedures, which are easy to implement (built over classical DL reasoners). Deciding entailment stays in the same complexity class of classical $\mathcal{ALC}$. Our work has a more general contribution to the development of non-monotonic reasoning in DLs, showing that it is possible to easily add negated inclusions in the KB (here represented by permissions), and combine different modalities.

Many frameworks for non-monotonic reasoning have been proficiently used both for modelling deontic reasoning and defeasible reasoning: beyond the preferential framework used here, that has been extensively used for modeling dyadic deontic logic on one hand and defeasible DLs on the other, there has been relevant work in the framework of *Defeasible Logic* [46, 12], both in the direction of deontic reasoning [47, 48, 49, 50] and of defeasible reasoning for DLs [51, 52], but we are not aware of any work in this framework modeling deontic reasoning specifically for DLs. In Section 4 we combine two forms of non-monotonic reasoning, respectively based on expectations and norms; as far as we know, the only other work on this topic is [53], which is based on sequent calculi and a propositional language.

There are various directions in which the present work can be improved. First of all, we have not considered ABox reasoning. It is quite straightforward to introduce reasoning about individuals in a ranked semantics based on minimality, but, as shown for RC in DLs [8, 42], once we reason about individuals we risk to have a multiplicity of minimal configurations, incompatible with each other, associated to the same ABox, making reasoning more complex and computationally expensive. The same happens in this framework; this is an interesting property to be analysed from the deontic point of view, especially in a multi-agent framework. Our approach to minimal entailment has a known limit regarding the application of defeasible information beyond the roles (e.g., we cannot derive $C \sqsubsetsim_E \exists r.E$

Procedure P-DD-Closure($\mathcal{K}$, α)

Input: $\mathcal{K} = \langle \mathcal{T}, \mathcal{D}, \mathcal{N}, \mathcal{P} \rangle$, a query $\alpha = C \sqsubsetsim_{d\mathcal{P}} D$.
Output: `true` if $\mathcal{K} \models_{\min} C \sqsubsetsim_{d\mathcal{P}} D$, `false` otherwise
if `N-DD-Closure`$(\mathcal{K}, C \sqsubsetsim_{\mathcal{N}} \neg D) =$ `false` **then**
  return `true`;
else if $\mathcal{T}^* \models C \sqsubseteq \bot$ **then**
  return `true`;
else
  return `false`;

from $C \sqsubseteq \exists r.D$ and $D \mathrel{\tilde{\sqsubseteq}_E} E$). The proposal in [30] gives a possible refinement of RC for the DL $\mathcal{EL}_\perp$ to overcome this limit. In addition, RC has some well-known inferential limits [44], which can be overcome extending RC [26, 34, 35, 37], which should also be be considered in our framework. Finally, it would also be worth introducing alternative rule-based approaches to deontic reasoning into the framework of defeasible DLs, such as the Input/Output logic approach [54].

Declaration on Generative AI

For the finalisation of this work, the authors used ChatGPT solely for grammar and spelling checks. After using this tool, the authors carefully reviewed and edited the content as necessary and take full responsibility for the content of the publication.

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