Neighbourhood Description Logics for Multiperspective Reasoning

Tiziano Dalmonte[1], Andrea Mazzullo[1] and Rafael Peñaloza[2]

[1]Free University of Bozen-Bolzano, Bolzano, Italy
[2]University of Milano-Bicocca, Milan, Italy

Abstract

Reasoning in the presence of multiple viewpoints or approximate concepts is a long-standing challenge for knowledge representation formalisms. In this paper, we introduce $\mathcal{ALC}^{\lhd\Box}$, a description logic extending the classical $\mathcal{ALC}$ with two pairs of concept constructors that allow us to talk about objects that "on some/all accounts, are acknowledged to be C" (with $\lhd C$ and $\Box C$, respectively), as well as "on all/some accounts, are compatible with being C" (with the duals $\rhd C$ and $\Diamond C$, respectively). Semantically, a function equips each element of the domain with a family of *neighbourhoods*, subsets of the domain that represent multiple, possibly imprecise, or even conflicting, accounts on that element. We show that $\mathcal{ALC}^{\lhd\Box}$ concept satisfiability with respect to a knowledge base is reducible in polynomial-time to the same problem for $\mathcal{ALC}$, hence obtaining an ExpTime-completeness result. Moreover, we investigate additional conditions on the neighbourhood function, so to capture natural constraints that can be imposed on the accounts of an object: existence, consistency, partial consistency, correctness, and partial correctness. In all these cases, we show that the (tight) ExpTime upper bound is preserved, by providing reductions to $\mathcal{ALC}$, $\mathcal{ALCHI}$, or the two-variable guarded fragment GF^2.

Keywords

Neighbourhood Description Logics, Neighbourhood Semantics, Non-Normal Modal Logics, Multiperspective Reasoning

1. Introduction

Description Logics (DLs) [1] have been proposed as logical formalisms to unambiguously describe, and effectively reason about, the conceptual structures underpinning our representations of application domains. Such representations, however, rarely turn out to be monolithic or perfectly crisp. On one hand, information frequently originates from multiple heterogeneous sources, and domain experts can disagree significantly even on fundamental aspects of their conceptualisations. On the other hand, the criteria to determine whether an object in a given context belongs to a certain class may be inherently vague or imprecise.

The *multiperspective* and the *rough* features of conceptualisations are quite pervasive, and frequently show up combined in real-world applications, such as in the medical domain (diagnostic criteria that are both vague and not universally shared, as e.g., in psychiatry), or in biological settings (e.g., taxonomical choices that lead to different classifications, while also admitting boundary individuals that cannot be sharply categorised even within a given perspective). To provide a formal treatment of such aspects using DL languages, several proposals have been considered in the literature.

Standpoint DLs [2, 3, 4] extend standard concepts and inclusions with modal operators $\Box_{\mathsf{s}}$ and $\Diamond_{\mathsf{s}}$, representing, respectively, that something is *unequivocal* from, or *conceivable* by, the standpoint of s (encoding, for instance, the viewpoint of an expert, or an ontology). Semantically, each standpoint is associated with a set of so-called *precisifications* (closely related to the notion of 'world' in relational semantics), and a formula holds unequivocally, or conceivably, according to a standpoint if it holds in all,

DL 2026: 39th International Workshop on Description Logics, July 17–19, 2026, Lisbon, Portugal
Tiziano.Dalmonte@unibz.it (T. Dalmonte); Andrea.Mazzullo@unibz.it (A. Mazzullo); rafael.penaloza@unimib.it (R. Peñaloza)
https://tizianodal.github.io/ (T. Dalmonte); https://andreamazzullo.github.io/ (A. Mazzullo); https://rpenalozan.github.io (R. Peñaloza)
0000-0002-7153-0506 (T. Dalmonte); 0000-0001-8512-1933 (A. Mazzullo); 0000-0002-2693-5790 (R. Peñaloza)

or some, of the classical interpretations associated with its precisifications. This approach is intended to integrate multi-perspective reasoning capabilities within the DL formalism, and complexity results have been shown for the several standpoint extensions of DLs, ranging from the lightweight $\mathcal{EL}$ [2] to expressive logics underlying OWL 2. Semantically, however, standpoint DLs remain essentially anchored to a possible-world setting: structures consist of multiple interpretations, one for each precisification, and the modalities quantify over such interpretations.

Rough DLs [5, 6] instead focus on representing imprecision, approximation, and vagueness. The core idea in rough DLs is to partition the object domain into equivalence classes of *indiscernible* individuals. A vague concept C, with imprecise boundaries, can be seen as squeezed between two other classes: its *lower approximation* $\underline{C}$, consisting of those individuals that have their entire equivalence class contained in C, meaning that they *surely* belong to C; and its *upper approximation* $\overline{C}$, containing the elements whose equivalence classes overlap with C, i.e., objects that *might* be members of C. Recent extensions to a *multi-granular* setting [7] equip rough DLs with a family of indiscernibility relations to represent alternative criteria or levels of granularity. All these approaches require to extend a single DL interpretation with one or more equivalence relations defined over the same "classical" structure (thereby avoiding the jump to a bi-dimensional modal semantics), and indeed the lower and upper approximations can be captured in a sufficiently expressive DL language by means of universal and existential restrictions over a family of reflexive, symmetric, and transitive roles.

In this paper, we generalise the rough semantics even further, by associating to each domain element d not just a single equivalence class of indiscernible elements, but rather an entire *family of neighbourhoods*, that is, a collection $\mathcal{N}(d)$ of sets of individuals. Each neighbourhood $\alpha \in \mathcal{N}(d)$ now represents a possible *account* of d, a family resemblance that can be used to describe it under a certain perspective. In other words, α consists of those objects that, according to that viewpoint, can be imagined as, thought of being like, or even confused with, the object d. Given that the neighbourhoods are not required to be equivalence classes, and do not even need to contain the d element itself, the family $\mathcal{N}(d)$ can be used to represent a collection of multiple, possibly imprecise, overlapping, or even incoherent, accounts on d.

Syntactically, we extend the standard DL language $\mathcal{ALC}$ with two novel concept constructors, $\triangleleft$ and $\Box$, together with their dual operators, $\triangleright$ and $\Diamond$. On the one hand, the concept $\triangleleft C$, read as "on some accounts, it is acknowledged to be C," represents those objects having *some* description such that *all* elements of that description are in the extension C. Dually, $\triangleright C$, that is, "on all accounts, it is compatible with being C," applies to those elements such that *all* of their descriptions have *some* objects intersecting with C. On the other hand, $\Box C$, which we read "on all accounts, it is acknowledged to be C," holds for an object if *all* of its descriptions are such that *all* their elements are in C. Its dual concept, $\Diamond C$, stands for "on some accounts, it is compatible with being C" and it is interpreted as the subset of the domain the elements of which have *some* description with *some* element in common with the extension of C.

For the language so obtained, $\mathcal{ALC}^{\triangleleft\Box}$, we show that concept satisfiability with respect to a knowledge base is polynomial-time reducible to the same problem in plain $\mathcal{ALC}$, leading to an ExpTime-completeness result. Moreover, we investigate several additional conditions that can be imposed on the neighbourhood functions, in order to capture natural assumptions on the descriptions of objects. First, the *existence* condition, requiring that every family of neighbourhood is non-empty, corresponds to the constraint that every object has at least one description. The *consistency* condition, imposing that the empty set does not belong to the family of neighbourhoods, and the *partial consistency* condition, stating that at least one neighbour is non-empty, respectively rule out the inconsistent description and force at least one account to be consistent. Finally, the *correctness* or the *partial correctness* condition, require, respectively, that all, or some, descriptions of an object contain the object itself, thus imposing that all, or some, accounts accurately apply to the object they aim to describe. In all these cases, the ExpTime upper bound is preserved, as shown by similar reductions to $\mathcal{ALC}$, $\mathcal{ALCHI}$, or the *2-variable guarded fragment* GF^2 satisfiability problems.

2. The Description Logic $\mathcal{ALC}^{\lhd\Box}$

We start by formalising the language that we study, with its syntax and neighbourhood-based semantics.

2.1. Syntax

Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be countably infinite and pairwise disjoint sets of *concept*, *role*, and *individual names*, respectively. An $\mathcal{ALC}^{\lhd\Box}$ *concept* is an expression of the form

$$C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C \mid \lhd C \mid \Box C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$. An $\mathcal{ALC}^{\lhd\Box}$ *concept inclusion* (*CI*) is an expression of the form $C \sqsubseteq D$, while an $\mathcal{ALC}^{\lhd\Box}$ *assertion* takes the form $C(a)$ or $r(a,b)$, with C, D $\mathcal{ALC}^{\lhd\Box}$ concepts, $r \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$. An $\mathcal{ALC}^{\lhd\Box}$ *knowledge base* (*KB*) is a pair $\mathcal{K} = (\mathcal{O}, \mathcal{D})$, where $\mathcal{O}$ is an $\mathcal{ALC}^{\lhd\Box}$ *ontology*, i.e., a finite set of $\mathcal{ALC}^{\lhd\Box}$ CIs, and $\mathcal{D}$ is an $\mathcal{ALC}^{\lhd\Box}$ *dataset*, that is, a finite set of $\mathcal{ALC}^{\lhd\Box}$ assertions.

We use the standard abbreviations $(C \sqcup D) := \neg(\neg C \sqcap \neg D)$; $\forall r.C := \neg\exists r.\neg C$; $\bot := A \sqcap \neg A$, $\top := A \sqcup \neg A$ (for an arbitrarily fixed $A \in \mathsf{N_C}$); and set $\rhd C := \neg \lhd \neg C$, and $\Diamond C := \neg\Box\neg C$. We also write $C \equiv D$ to abbreviate $C \sqsubseteq D$ and $D \sqsubseteq C$. The intuitive reading of $\rhd C$ is "on all accounts, it is compatible with being C," while $\lhd C$ stands for "on some accounts, it is acknowledged to be C." Similarly, $\Box C$ can be read as "on all accounts, it is acknowledged to be C," and $\Diamond C$ expresses "on some accounts, it is compatible with being C."

The standard DL language $\mathcal{ALC}$ is obtained from $\mathcal{ALC}^{\lhd\Box}$ by disallowing both $\lhd$ and $\Box$. Additional classic definitions are assumed for standard extensions of $\mathcal{ALC}$, such as $\mathcal{ALCHI}$ concepts and KBs, involving *inverse roles* $\mathsf{N_R^-} = \{r^- \mid r \in \mathsf{N_R}\}$ and *role inclusions* $s_1 \sqsubseteq s_2$, with $s_1, s_2 \in \mathsf{N_R} \cup \mathsf{N_R^-}$.

2.2. Neighbourhood Semantics

An $\mathcal{ALC}^{\lhd\Box}$ *interpretation*, or simply an *interpretation*, is a tuple $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$, where:

- Δ is a non-empty *domain*;
- $\cdot^{\mathcal{I}}$ is a function s.t. for all $A \in \mathsf{N_C}$, $A^{\mathcal{I}} \subseteq \Delta$; for all $r \in \mathsf{N_R}$, $r^{\mathcal{I}} \subseteq \Delta \times \Delta$; for all $a \in \mathsf{N_I}$, $a^{\mathcal{I}} \in \Delta$;
- $\mathcal{N} \colon \Delta \to 2^{2^{\Delta}}$ is a *neighbourhood function.*

The interpretation function $\cdot^{\mathcal{I}}$ is extended to all concepts by setting

$$(\neg D)^{\mathcal{I}} = \Delta \setminus D^{\mathcal{I}}, \quad (D \sqcap E)^{\mathcal{I}} = D^{\mathcal{I}} \cap E^{\mathcal{I}}, \quad (\exists r.D)^{\mathcal{I}} = \{d \in \Delta \mid \exists e \in D^{\mathcal{I}} \text{ s.t. } (d,e) \in r^{\mathcal{I}}\},$$

and for the new constructors:

$$(\lhd D)^{\mathcal{I}} = \{d \in \Delta \mid \exists \alpha \in \mathcal{N}(d) \text{ s.t. } \forall e \in \alpha \colon e \in D^{\mathcal{I}}\},$$
$$(\Box D)^{\mathcal{I}} = \{d \in \Delta \mid \forall \alpha \in \mathcal{N}(d), \forall e \in \alpha \colon e \in D^{\mathcal{I}}\}.$$

By duality, from the definition above, we also have the following:

$$(\rhd D)^{\mathcal{I}} = \{d \in \Delta \mid \forall \alpha \in \mathcal{N}(d) \colon \exists e \in \alpha \text{ s.t. } e \in D^{\mathcal{I}}\},$$
$$(\Diamond D)^{\mathcal{I}} = \{d \in \Delta \mid \exists \alpha \in \mathcal{N}(d), \exists e \in \alpha \text{ s.t. } e \in D^{\mathcal{I}}\}.$$

We say that an $\mathcal{ALC}^{\lhd\Box}$ concept C is *satisfied in* $\mathcal{I}$ if $C^{\mathcal{I}} \neq \emptyset$, that an $\mathcal{ALC}^{\lhd\Box}$ CI $C \sqsubseteq D$ is *satisfied in* $\mathcal{I}$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and that assertions $C(a)$ and $r(a,b)$ are *satisfied in* $\mathcal{I}$ iff, respectively, $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. We say that an $\mathcal{ALC}^{\lhd\Box}$ interpretation $\mathcal{I}$ *satisfies* the KB $\mathcal{K} = (\mathcal{O}, \mathcal{D})$, or is a *model* of $\mathcal{K}$, written $\mathcal{I} \models \mathcal{K}$, if every CI in $\mathcal{O}$ and every assertion in $\mathcal{D}$ is satisfied in $\mathcal{I}$. Then, we say that C is $\mathcal{ALC}^{\lhd\Box}$ *satisfiable with respect to* $\mathcal{K}$ if there exists an $\mathcal{ALC}^{\lhd\Box}$ model $\mathcal{I}$ of $\mathcal{K}$ such that C is satisfied in $\mathcal{I}$. In the following, we focus on $\mathcal{ALC}^{\lhd\Box}$ *concept satisfiability with respect to a knowledge base* as main reasoning problem.

The following CIs are satisfied in *every* $\mathcal{ALC}^{\lhd\Box}$ interpretation:

$$\top \sqsubseteq \Box\top, \qquad \Box C \sqcap \Box D \equiv \Box(C \sqcap D), \qquad \lhd(C \sqcap D) \sqsubseteq \lhd C \sqcap \lhd D,$$
$$\Box C \sqcap \lhd D \sqsubseteq \lhd(C \sqcap D), \qquad \Box C \sqcap \Diamond D \sqsubseteq \lhd C, \qquad \lhd C \sqcap \rhd D \sqsubseteq \Diamond D.$$

Example 1. *Dinosaur classification is a highly debated topic since the birth of palaeontology [8, 9]. Under the* standard hypothesis*, shared by the analyses of Müller & Garcia [10] and Novas et al. [11],* Dinosauria *includes* Saurischia *(which, under these views, contains* Theropoda *and* Sauropodomorpha*) and* Ornithischia*:*

$$\triangleleft\mathsf{Saurischia} \sqcup \mathsf{Ornithischia} \sqsubseteq \triangleleft\mathsf{Dinosauria}, \qquad \mathsf{Theropoda} \sqcup \mathsf{Sauropodomorpha} \sqsubseteq \triangleleft\mathsf{Saurischia}.$$

A non-standard hypothesis*, supported for instance by Baron et al. [12] and Cau [13], introduces an* Ornithoscelida *class, consisting of* Theropoda *and* Ornithischia *grouped together,*

$$\mathsf{Theropoda} \sqcup \mathsf{Ornithischia} \sqsubseteq \triangleleft\mathsf{Ornithoscelida},$$

while leaving Sauropodomorpha *as a separate branch. In this non-standard setting, while both proposals include* Sauropodomorpha *and* Ornithoscelida *as* Dinosauria*,*

$$\mathsf{Sauropodomorpha} \sqcup \triangleleft\mathsf{Ornithoscelida} \sqsubseteq \triangleleft\mathsf{Dinosauria},$$

Cau suggests that this should also constitute a partition, by excluding Herrerasauridae *from* Dinosauria*, thus yielding the most restrictive proposal on the extension of* Dinosauria *(compared to the non-standard hypothesis by Baron et al., and to the standard perspectives):*

$$\Box\mathsf{Dinosauria} \sqsubseteq \mathsf{Sauropodomorpha} \sqcup \triangleleft\mathsf{Ornithoscelida},$$
$$\mathsf{Herrerasauridae} \sqsubseteq \triangleright\mathsf{Dinosauria} \sqcap \Diamond\neg\mathsf{Dinosauria}.$$

Due to the positioning of Herrerasauridae*, the approach by Baron et al. instead aligns with one of the standard proposals, by Novas et al., in suggesting the broader category of* Dracohors *to be formed by either* Silesauridae*, or by those organisms that, both under the standard and the non-standard approach, are classified by at least one proposal as* Dinosauria*:*

$$\triangleright\mathsf{Dracohors} \sqsubseteq \mathsf{Silesauridae} \sqcup \triangleright\mathsf{Dinosauria}$$

Clearly, the standard and the non-standard approaches are incompatible, since Saurischia *and* Ornithoscelida *should be disjoint according to all proposals, but, as we have seen,* Theropoda *are (non-empty and) classified as* Saurischia *under the standard hypothesis and as* Ornithoscelida *in the non-standard accounts:*

$$\Box(\mathsf{Saurischia} \sqcap \mathsf{Ornithoscelida}) \sqsubseteq \bot, \qquad \mathsf{Theropoda} \sqsubseteq \triangleleft\mathsf{Saurischia} \sqcap \triangleleft\mathsf{Ornithoscelida}.$$

Finally, while Ornithischia *can objectively constitute a clade with organisms that some accounts agree to classify as* Saurischia*, they are considered only by the non-standard accounts to form a clade directly with some member of* Theropoda*:*

$$\mathsf{Ornithischia} \sqsubseteq \exists\mathsf{formsCladeWith}.\triangleleft\mathsf{Saurischia}, \qquad \mathsf{Ornithischia} \sqsubseteq \triangleleft\exists\mathsf{formsCladeWith}.\mathsf{Theropoda}.$$

2.3. Existence, Consistency, and Correctness of Neighbourhoods

Some properties of the neighbourhood function over a given individual $d \in \Delta$ can be expressed through $\mathcal{ALC}^{\triangleleft\Box}$ concepts. We present four important cases.

1. $\Box C \sqcap \triangleright\neg C$, or equivalently $\triangleright\bot$, enforces that $\mathcal{N}(d) = \emptyset$, for any d in the interpretation of the concept; i.e., no description applies to d. Indeed, $\Box C$ requires every neighbourhood $\alpha \in \mathcal{N}(d)$ to be contained in $C^{\mathcal{I}}$, while $\triangleright\neg C$ forces every α to contain an element in $\neg C^{\mathcal{I}}$: combined, they imply that $\mathcal{N}(d) = \emptyset$. Equivalently, $\triangleright\bot$ imposes every neighbourhood to contain an element in $\bot^{\mathcal{I}}$, which is impossible.
2. $\triangleleft C \sqcap \Box\neg C$, or equivalently $\triangleleft\bot$, expresses that there exists $\alpha \in \mathcal{N}(d)$ with $\alpha = \emptyset$, i.e., d has an inconsistent description. Indeed, $\triangleleft C$ requires some α to be a subset of $C^{\mathcal{I}}$, but $\Box\neg C$ forces every neighbourhood to be a subset of $\neg C^{\mathcal{I}}$. Therefore, this α must be empty. The existence of a neighbourhood that is a subset of $\bot^{\mathcal{I}}$ is equivalently guaranteed by $\triangleleft\bot$.

3. $\Box C \sqcap \Box\neg C$, or equivalently $\Box\bot$, holds whenever every neighbourhood $\alpha \in \mathcal{N}(d)$ is empty, i.e., d has only (if any) inconsistent descriptions. Indeed, $\Box C$ and $\Box\neg C$ force any element of a neighbourhood of d inside $C^{\mathcal{I}}$ and its complement, meaning that any neighbourhood of d must be empty. The concept $\Box\bot$ corresponds directly to the same constraint.
4. $\triangleleft\top \sqcap \Box\bot$ expresses that $\mathcal{N}(d) = \{\emptyset\}$. Indeed, $\triangleleft\top$ requires the existence of a neighbourhood that is (trivially) in $\top^{\mathcal{I}}$, while $\Box\bot$ forces every neighbourhood to be a subset of $\bot^{\mathcal{I}}$, i.e., empty. Together, they imply that there is exactly one neighbourhood, which is empty, that is $\mathcal{N}(d) = \{\emptyset\}$.

We now introduce selected families of $\mathcal{ALC}^{\triangleleft\Box}$ interpretations $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$ characterised by properties to be imposed on their neighbourhood function over all elements $d \in \Delta$. We consider:

Existence (ex)	$\mathcal{N}(d) \neq \emptyset$;
Consistency (cons)	for every $\alpha \in \mathcal{N}(d)$, $\alpha \neq \emptyset$;
Partial Consistency (pcons)	there exists $\alpha \in \mathcal{N}(d)$, $\alpha \neq \emptyset$;
Correctness (corr)	for every $\alpha \in \mathcal{N}(d)$, $d \in \alpha$;
Partial Correctness (pcorr)	for some $\alpha \in \mathcal{N}(d)$, $d \in \alpha$.

We can see a non-empty neighbourhood as a *consistent* description, and a neighbourhood that contains d itself as a *correct* description of it. Then, the conditions above have the following intuitive meaning:

(ex)	an object has a description;
(cons)	all descriptions are consistent;
(pcons)	some descriptions are consistent;
(corr)	all descriptions are correct;
(pcorr)	some descriptions are correct.

For $\mathsf{L} \subseteq \{\mathsf{ex}, \mathsf{cons}, \mathsf{pcons}, \mathsf{corr}, \mathsf{pcorr}\}$, we call $\mathcal{ALC}^{\triangleleft\Box}_{\mathsf{L}}$ *interpretation* an $\mathcal{ALC}^{\triangleleft\Box}$ interpretation satisfying all conditions in L, for every $d \in \Delta$.

Observe that the following implications hold among the conditions: (corr) $\Rightarrow$ (cons); (pcorr) $\Rightarrow$ (pcons); (pcons) $\Rightarrow$ (ex); (cons) + (ex) $\Rightarrow$ (pcons); (corr) + (ex) $\Rightarrow$ (pcorr). Moreover, the failure of the Conditions (ex), (cons), and (pcons) corresponds to the satisfiability of the concepts identified above in Points 1, 2 and 3, respectively: satisfiability of $\triangleright\bot$ is equivalent to the failure of (ex); satisfiability of $\triangleleft\bot$ to the failure of (cons); and satisfiability of $\Box\bot$ to the failure of (pcons). In addition, the concept $\triangleleft\top \sqcap \Box\bot$, in Point 4 above, is satisfiable if and only if (ex) holds, but (pcons) fails. The Conditions (ex), (cons), and (pcons) can hence be characterised by the following CIs.

Lemma 1. *For every $\mathcal{ALC}^{\triangleleft\Box}$ interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$, the following hold:*

1. *$\mathcal{I}$ is an $\mathcal{ALC}^{\triangleleft\Box}_{\mathsf{ex}}$ interpretation if and only if $\mathcal{I} \models \top \sqsubseteq \triangleleft\top$;*
2. *$\mathcal{I}$ is an $\mathcal{ALC}^{\triangleleft\Box}_{\mathsf{cons}}$ interpretation if and only if $\mathcal{I} \models \top \sqsubseteq \triangleright\top$;*
3. *$\mathcal{I}$ is an $\mathcal{ALC}^{\triangleleft\Box}_{\mathsf{pcons}}$ interpretation if and only if $\mathcal{I} \models \top \sqsubseteq \Diamond\top$.*

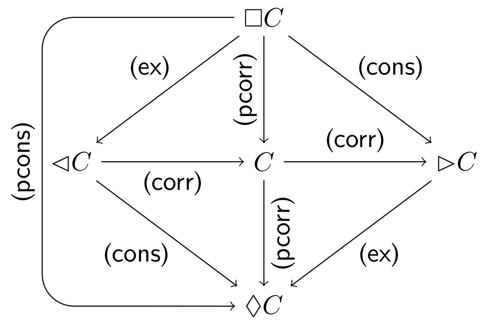
Proof. Point 1. By (ex), we have $\mathcal{N}(d) \neq \emptyset$, meaning that there exists $\alpha \in \mathcal{N}(d)$, for every $d \in \Delta$. Equivalently, for every $d \in \Delta$, there exists $\alpha \in \mathcal{N}(d)$ such that $\alpha \subseteq \Delta = \top^{\mathcal{I}}$, i.e., $\mathcal{I} \models \top \sqsubseteq \triangleleft\top$.

Point 2. By (cons), for every $d \in \Delta$ and every $\alpha \in \mathcal{N}(d)$, we have $\alpha \neq \emptyset$, meaning that there exists $e \in \alpha$, with $e \in \Delta = \top^{\mathcal{I}}$. That is, $\mathcal{I} \models \top \sqsubseteq \triangleright\top$.

Point 3. By (pcons), for every $d \in \Delta$, there exists $\alpha \in \mathcal{N}(d)$ such that $\alpha \neq \emptyset$, i.e., $e \in \alpha$, for some $e \in \Delta = \top^{\mathcal{I}}$. This means that $\mathcal{I} \models \top \sqsubseteq \Diamond\top$. $\square$

Remark 1. *Note that the CIs above are equivalent, by contraposition, to:* (ex) $\triangleright\bot \sqsubseteq \bot$*;* (cons) $\triangleleft\bot \sqsubseteq \bot$*; and* (pcons) $\Box\bot \sqsubseteq \bot$*. Moreover, they are equivalent, respectively, to the schemas (for every concept C): $\Box C \sqsubseteq \triangleleft C$ (by contraposition, $\triangleright C \sqsubseteq \Diamond C$); $\Box C \sqsubseteq \triangleright C$ (by contraposition, $\triangleleft C \sqsubseteq \Diamond C$); and $\Box C \sqsubseteq \Diamond C$.*

On the other hand, correctness and partial correctness cannot be characterised by a single formula.



(ex)	(cons)	(pcons)	(corr)	(pcorr)
$\top \sqsubseteq \lhd\top$	$\top \sqsubseteq \rhd\top$	$\top \sqsubseteq \Diamond\top$	$C \sqsubseteq \rhd C$	$C \sqsubseteq \Diamond C$
$\rhd\bot \sqsubseteq \bot$	$\lhd\bot \sqsubseteq \bot$	$\Box\bot \sqsubseteq \bot$	$\lhd C \sqsubseteq C$	$\Box C \sqsubseteq C$
$\Box C \sqsubseteq \lhd C$	$\Box C \sqsubseteq \rhd C$	$\Box C \sqsubseteq \Diamond C$		
$\rhd C \sqsubseteq \Diamond C$	$\lhd C \sqsubseteq \Diamond C$			

Figure 1: Validities in $\mathcal{ALC}^{\lhd\Box}$ interpretations satisfying conditions among (ex), (cons), (pcons), (corr), (pcorr).

Lemma 2. *There is no $\mathcal{ALC}^{\lhd\Box}$ CI or assertion φ such that $\mathcal{I} \models \varphi$ if and only if $\mathcal{I}$ is an $\mathcal{ALC}^{\lhd\Box}_{\mathsf{corr}}$ interpretation, or $\mathcal{I} \models \varphi$ if and only if $\mathcal{I}$ is an $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$ interpretation.*

Proof. Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$ be an $\mathcal{ALC}^{\lhd\Box}$ interpretation such that $\mathcal{N}(d) = \{\{d\}\}$ for all $d \in \Delta$. Then $\mathcal{I}$ satisfies both correctness and partial correctness. We define the $\mathcal{ALC}^{\lhd\Box}$ interpretation $\mathcal{I}^* = (\Delta^*, \cdot^{\mathcal{I}^*}, \mathcal{N}^*)$ as follows:

- Δ^* is the *disjoint union of two copies* of Δ. We denote d_1, d_2 the two elements of Δ^* corresponding to the element d of Δ.
- $A^{\mathcal{I}^*} = \{d_i \mid d \in A^{\mathcal{I}}\}$, for $i = 1, 2$.
- $r^{\mathcal{I}^*} = \{(d_i, e_1) \mid (d, e) \in r^{\mathcal{I}}\}$, for $i = 1, 2$.
- $a^{\mathcal{I}^*} = d_1$ iff $a^{\mathcal{I}} = d$.
- $\mathcal{N}^*(d_i) = \{\{d_j\}\}$, for $i, j = 1, 2$, $i \neq j$.

It can be seen that $\mathcal{I}^*$ satisfies neither correctness nor partial correctness.

Claim 2.1. *For every $\mathcal{ALC}^{\lhd\Box}$ concept C and every $d \in \Delta$, $d \in C^{\mathcal{I}}$ if and only if $d_1 \in C^{\mathcal{I}^*}$ if and only if $d_2 \in C^{\mathcal{I}^*}$.*

Proof of Claim. By induction on the construction of C. For the base case $C = A$, the claim follows immediately from the definition of $A^{\mathcal{I}^*}$. The cases $C = \neg D$ and $C = D \sqcap E$ are immediate by IH.
[$C = \exists r.D$] By definition, we have $(d, e) \in r^{\mathcal{I}}$ iff $(d_1, e_1) \in r^{\mathcal{I}^*}$ iff $(d_2, e_1) \in r^{\mathcal{I}^*}$. By IH, $e \in D^{\mathcal{I}}$ iff $e_1 \in D^{\mathcal{I}^*}$, hence the claim follows.
[$C = \lhd D$] We have $\mathcal{N}(d) = \{\{d\}\}$, $\mathcal{N}^*(d_1) = \{\{d_2\}\}$ and $\mathcal{N}^*(d_2) = \{\{d_1\}\}$. Moreover, by IH, $d \in D^{\mathcal{I}}$ iff $d_1 \in D^{\mathcal{I}^*}$ iff $d_2 \in D^{\mathcal{I}^*}$. Then we have that there is $\alpha \in \mathcal{N}(d)$ such that for all $e \in \alpha$, $e \in D^{\mathcal{I}}$, iff there is $\alpha \in \mathcal{N}^*(d_1)$ such that for all $e \in \alpha$, $e \in D^{\mathcal{I}^*}$, iff there is $\alpha \in \mathcal{N}^*(d_2)$ such that for all $e \in \alpha$, $e \in D^{\mathcal{I}^*}$. Therefore $d \in (\lhd C)^{\mathcal{I}}$ iff $d_1 \in (\lhd C)^{\mathcal{I}^*}$ iff $d_2 \in (\lhd C)^{\mathcal{I}^*}$.
[$C = \Box D$] This case is analogous to the previous one, considering that, by definition and IH, for every $\alpha \in \mathcal{N}(d)$, every $e \in \alpha$ is such that $e \in D^{\mathcal{I}}$, iff for every $\alpha \in \mathcal{N}^*(d_1)$, every $e \in \alpha$ is such that $e \in D^{\mathcal{I}^*}$ iff for every $\alpha \in \mathcal{N}^*(d_2)$, every $e \in \alpha$ is such that $e \in D^{\mathcal{I}^*}$. ⊣

Claim 2.2. *For every $\mathcal{ALC}^{\lhd\Box}$ CI or assertion φ, $\mathcal{I} \models \varphi$ if and only if $\mathcal{I}^* \models \varphi$.*

Proof of Claim. For CIs, the claim follows immediately from Claim 2.1. Suppose $\varphi = C(a)$, and let $a^{\mathcal{I}} = d$. Then $a^{\mathcal{I}^*} = d_1$. We have $\mathcal{I} \models C(a)$ iff $d \in C^{\mathcal{I}}$ iff, by Claim 2.1, $d_1 \in C^{\mathcal{I}^*}$, iff $\mathcal{I}^* \models C(a)$. ⊣

Hence, no formula can distinguish the two interpretations. This concludes the proof of the lemma. □

While no single formula can characterise the Conditions (corr) and (pcorr), note that the following validities are entailed: (corr) $C \sqsubseteq \rhd C$ and, by contraposition, $\lhd C \sqsubseteq C$; (pcorr) $C \sqsubseteq \Diamond C$ and, by contraposition, $\Box C \sqsubseteq C$. Figure 1 summarises the $\mathcal{ALC}^{\lhd\Box}$ CIs that are valid under the conditions considered.

3. Reasoning in $\mathcal{ALC}^{\triangleleft\Box}$

We now show that $\mathcal{ALC}^{\triangleleft\Box}$ concept satisfiability w.r.t. a KB is ExpTime-complete through a reduction to (classical) $\mathcal{ALC}$ concept satisfiability w.r.t. a KB. Consider a distinguished concept name E, representing that d is a domain element, and two role names r_1, r_2, used to capture, respectively, the relations $\mathcal{N}(d) \ni \alpha$ and $\alpha \ni d$, for a neighbourhood α. We define a translation from $\mathcal{ALC}^{\triangleleft\Box}$ concepts to $\mathcal{ALC}$ concepts:

$$(A)^\dagger = A \sqcap \mathsf{E}, \qquad (\neg C)^\dagger = \neg(C)^\dagger \sqcap \mathsf{E}, \qquad (C \sqcap D)^\dagger = (C)^\dagger \sqcap (D)^\dagger,$$
$$(\exists r.C)^\dagger = \exists r.(C)^\dagger \sqcap \mathsf{E}, \qquad (\triangleleft C)^\dagger = \exists r_1.\forall r_2.(C)^\dagger, \qquad (\Box C)^\dagger = \forall r_1.\forall r_2.(C)^\dagger \sqcap \mathsf{E}.$$

This translation is extended to KBs, by setting $(\mathcal{K})^\dagger = ((\mathcal{O})^\dagger, (\mathcal{D})^\dagger)$, where

$$(\mathcal{O})^\dagger = \{(\alpha)^\dagger \mid \alpha \in \mathcal{O}\} \cup \{\exists r_1.\top \sqsubseteq \mathsf{E}, \top \sqsubseteq \forall r_2.\mathsf{E}\}, \quad (\mathcal{D})^\dagger = \{(\beta)^\dagger \mid \beta \in \mathcal{D}\} \cup \{\mathsf{E}(a) \mid a \in \mathsf{N_I}(\mathcal{D})\},$$
$$(C \sqsubseteq D)^\dagger = (C)^\dagger \sqsubseteq (D)^\dagger, \quad (C(a))^\dagger = (C)^\dagger(a), \quad (r(a,b))^\dagger = r(a,b),$$

with $\mathsf{N_I}(\mathcal{D})$ denoting the set of individual names occurring in $\mathcal{D}$.

Lemma 3. *Given an $\mathcal{ALC}^{\triangleleft\Box}$ concept C and knowledge base $\mathcal{K}$, C is $\mathcal{ALC}^{\triangleleft\Box}$ satisfiable w.r.t. $\mathcal{K}$ if and only if $(C)^\dagger$ is $\mathcal{ALC}$ satisfiable w.r.t. $(\mathcal{K})^\dagger$.*

Proof. ($\Rightarrow$) Let $\mathcal{I} = (\Delta, \cdot^\mathcal{I}, \mathcal{N})$ be an $\mathcal{ALC}^{\triangleleft\Box}$ interpretation such that $\mathcal{I} \models \mathcal{K}$ and $C^\mathcal{I} \neq \emptyset$. We define the $\mathcal{ALC}$ interpretation $\mathcal{I}^* = (\Delta^*, \cdot^{\mathcal{I}^*})$ as follows:

- $\Delta^* = \{(d,0) \mid d \in \Delta\} \cup \{(\alpha,1) \mid \alpha \in \bigcup_{d\in\Delta} \mathcal{N}(d)\}$;
- $\mathsf{E}^{\mathcal{I}^*} = \{(d,0) \mid d \in \Delta\}$;
  $A^{\mathcal{I}^*} = \{(d,0) \mid d \in A^\mathcal{I}\}$, for all $A \in \mathsf{N_C} \setminus \{\mathsf{E}\}$;
- $r_1^{\mathcal{I}^*} = \{((d,0),(\alpha,1)) \mid \alpha \in \mathcal{N}(d)\}$;
  $r_2^{\mathcal{I}^*} = \{((\alpha,1),(d,0)) \mid d \in \alpha\}$;
  $r^{\mathcal{I}^*} = \{((d,0),(e,0)) \mid (d,e) \in r^\mathcal{I}\}$, for all $r \in \mathsf{N_R} \setminus \{r_1, r_2\}$;
- $a^{\mathcal{I}^*} = (a^\mathcal{I}, 0)$, for all $a \in \mathsf{N_I}$.

Observe that, as an immediate consequence of the definition of $\mathcal{I}^*$, we have $\mathcal{I}^* \models \exists r_1.\top \sqsubseteq \mathsf{E}$, $\mathcal{I}^* \models \top \sqsubseteq \forall r_2.\mathsf{E}$ and $\mathcal{I}^* \models \mathsf{E}(a)$, for every $a \in \mathsf{N_I}(\mathcal{D})$.

Claim 3.1. *For every $d \in \Delta$, $d \in C^\mathcal{I}$ if and only if $(d,0) \in ((C)^\dagger)^{\mathcal{I}^*}$.*

Proof of Claim. By induction on the construction of C. The base case of $C = A$ is straightforward, since $d \in A^\mathcal{I}$ iff $(d,0) \in A^{\mathcal{I}^*}$ and $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$ iff $(d,0) \in (A \sqcap \mathsf{E})^{\mathcal{I}^*}$ iff $(d,0) \in ((A)^\dagger)^{\mathcal{I}^*}$. We now show the inductive cases.
[$C = \neg D$] We have $d \in (\neg D)^\mathcal{I}$ iff $d \notin D^\mathcal{I}$ iff, by IH, $(d,0) \notin ((D)^\dagger)^{\mathcal{I}^*}$, and, by definition, $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$, iff $(d,0) \in (\neg(D)^\dagger)^{\mathcal{I}^*}$ and $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$ iff $(d,0) \in (\neg(D)^\dagger \sqcap \mathsf{E})^{\mathcal{I}^*}$ iff $(d,0) \in ((\neg D)^\dagger)^{\mathcal{I}^*}$.
[$C = D \sqcap E$] We have $d \in (D \sqcap E)^\mathcal{I}$ iff $d \in D^\mathcal{I}$ and $d \in E^\mathcal{I}$. By IH, this means $(d,0) \in ((D)^\dagger)^{\mathcal{I}^*}$ and $(d,0) \in ((E)^\dagger)^{\mathcal{I}^*}$. Hence, we have $(d,0) \in ((D)^\dagger \sqcap (E)^\dagger)^{\mathcal{I}^*}$, i.e., $(d,0) \in ((D \sqcap E)^\dagger)^{\mathcal{I}^*}$
[$C = \exists r.D$] We have $d \in (\exists r.D)^\mathcal{I}$ iff there exists $e \in \Delta$ s.t. $(d,e) \in r^\mathcal{I}$ and $e \in D^\mathcal{I}$. By definition and IH, this is equivalent to: $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$ and there exists $(e,0) \in \Delta^{\mathcal{I}^*}$ such that $((d,0),(e,0)) \in r^{\mathcal{I}^*}$ and $(e,0) \in ((D)^\dagger)^{\mathcal{I}^*}$. This means $(d,0) \in (\exists r.(D)^\dagger \sqcap \mathsf{E})^{\mathcal{I}^*}$, that is, $(d,0) \in ((\exists r.D)^\dagger)^{\mathcal{I}^*}$.
[$C = \triangleleft D$] We have $d \in (\triangleleft D)^\mathcal{I}$ iff there is $\alpha \in \mathcal{N}(d)$ such that for all $e \in \alpha$, $e \in D^\mathcal{I}$. By definition and IH, there is $(\alpha,1) \in \Delta^*$ s.t. $((d,0),(\alpha,1)) \in (r_1)^{\mathcal{I}^*}$ and for all $(e,0) \in \Delta^*$, if $((\alpha,1),(e,0)) \in (r_2)^{\mathcal{I}^*}$, then $(e,0) \in ((D)^\dagger)^{\mathcal{I}^*}$. Hence, $(d,0) \in (\exists r_1.\forall r_2.(D)^\dagger)^{\mathcal{I}^*} = ((\triangleleft D)^\dagger)^{\mathcal{I}^*}$.
[$C = \Box D$] We have $d \in (\Box D)^\mathcal{I}$ iff, for every $\alpha \in \mathcal{N}(d)$, every $e \in \alpha$ is such that $e \in D^\mathcal{I}$. By definition and induction hypothesis, this means that for every $(\alpha,1) \in \Delta^*$ such that $((d,0),(\alpha,1)) \in (r_1)^{\mathcal{I}^*}$, and for every $(e,0) \in \Delta^*$ such that $((\alpha,1),(e,0)) \in (r_2)^{\mathcal{I}^*}$, we have $(e,0) \in ((D)^\dagger)^{\mathcal{I}^*}$. Moreover, $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$. Therefore, equivalently, $(d,0) \in (\forall r_1.\forall r_2.(D)^\dagger \sqcap \mathsf{E})^{\mathcal{I}^*} = ((\Box D)^\dagger)^{\mathcal{I}^*}$. $\dashv$

Therefore, $\mathcal{I}^*$ satisfies $(C)^\dagger$.

Claim 3.2. $((C)^\dagger)^{\mathcal{I}^*} \subseteq \mathsf{E}^{\mathcal{I}^*}$.

Proof. The cases of $C = A$, $C = \neg D$, $C = \exists r.D$, $C = \Box D$ are clear by definition of $(C)^\dagger$. The case $C = D \sqcap E$ holds by a straightforward application of IH. The case $C = \lhd D$ can be shown as follows. By definition, $((\lhd D)^\dagger)^{\mathcal{I}^*} = (\exists r_1.\forall r_2.(D)^\dagger)^{\mathcal{I}^*}$. Since $(\exists r_1.\forall r_2.(D)^\dagger)^{\mathcal{I}^*} \subseteq \exists r_1.\top^{\mathcal{I}^*}$, and $\exists r_1.\top^{\mathcal{I}^*} \subseteq \mathsf{E}^{\mathcal{I}^*}$ due to $\exists r_1.\top \sqsubseteq \mathsf{E} \in (\mathcal{O})^\dagger$, we have $((\lhd D)^\dagger)^{\mathcal{I}^*} \subseteq \mathsf{E}^{\mathcal{I}^*}$. $\square$

Claim 3.3. *For every $C \sqsubseteq D \in \mathcal{O}$, $\mathcal{I} \models C \sqsubseteq D$ if and only if $\mathcal{I}^* \models (C \sqsubseteq D)^\dagger$. For every assertion $\delta \in \mathcal{D}$, $\mathcal{I} \models \delta$ if and only if $\mathcal{I}^* \models (\delta)^\dagger$.*

Proof of Claim. Consider a CI $C \sqsubseteq D \in \mathcal{O}$. ($\Rightarrow$) Suppose $\mathcal{I} \models C \sqsubseteq D$, meaning that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, and let $o \in \Delta^*$ be such that $o \in ((C)^\dagger)^{\mathcal{I}^*}$. Since, by Claim 3.2, $((C)^\dagger)^{\mathcal{I}^*} \subseteq \mathsf{E}^{\mathcal{I}^*}$, we have that $o = (d, 0)$, for some $d \in \Delta$. By Claim 3.1, $d \in C^{\mathcal{I}}$, hence $d \in D^{\mathcal{I}}$, thus, again by Claim 3.1, $(d, 0) \in ((D)^\dagger)^{\mathcal{I}^*}$. Therefore, $\mathcal{I}^* \models (C)^\dagger \sqsubseteq (D)^\dagger$, i.e., $\mathcal{I}^* \models (C \sqsubseteq D)^\dagger$. ($\Leftarrow$) Conversely, suppose that $\mathcal{I}^* \models (C)^\dagger \sqsubseteq (D)^\dagger$, and consider $d \in \Delta$ such that $d \in C^{\mathcal{I}}$. By Claim 3.1, we have $(d, 0) \in ((C)^\dagger)^{\mathcal{I}^*}$, and hence $(d, 0) \in ((D)^\dagger)^{\mathcal{I}^*}$. Again by Claim 3.1, this means that $d \in D^{\mathcal{I}}$, thus $\mathcal{I} \models C \sqsubseteq D$.

For an assertion $C(a) \in \mathcal{D}$, we have $\mathcal{I} \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ iff $a^{\mathcal{I}} = d \in \Delta$ and $d \in C^{\mathcal{I}}$ iff, by Claim 3.1 and definition of $\mathcal{I}^*$, $(d, 0) = (a^{\mathcal{I}}, 0) \in ((C)^\dagger)^{\mathcal{I}^*}$ iff $\mathcal{I}^* \models ((C)^\dagger)(a)$.

For an assertion $r(a, b) \in \mathcal{D}$, we have $\mathcal{I} \models r(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ iff, by definition of $\mathcal{I}^*$, $((a^{\mathcal{I}}, 0), (b^{\mathcal{I}}, 0)) = (a^{\mathcal{I}^*}, b^{\mathcal{I}^*}) \in r^{\mathcal{I}^*}$ iff $\mathcal{I}^* \models (r(a, b))^\dagger$. $\dashv$

Therefore we obtain that $\mathcal{I}^* \models (\mathcal{K})^\dagger$, with $((C)^\dagger)^{\mathcal{I}^*} \neq \emptyset$.

($\Leftarrow$) Let $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ be an $\mathcal{ALC}$ interpretation such that $\mathcal{J} \models (\mathcal{K})^\dagger$ and $((C)^\dagger)^{\mathcal{J}} \neq \emptyset$. We define the $\mathcal{ALC}^{\lhd\Box}$ interpretation $\mathcal{J}^* = (\Delta^*, \cdot^{\mathcal{J}^*}, \mathcal{N}^*)$, where:

- $\Delta^* = \mathsf{E}^{\mathcal{J}}$;
- $A^{\mathcal{J}^*} = A^{\mathcal{J}} \cap \mathsf{E}^{\mathcal{J}}$, for all $A \in \mathsf{N_C} \setminus \{\mathsf{E}\}$;
- $r^{\mathcal{J}^*} = r^{\mathcal{J}} \cap (\mathsf{E}^{\mathcal{J}} \times \mathsf{E}^{\mathcal{J}})$, for all $r \in \mathsf{N_R} \setminus \{r_1, r_2\}$;
- $a^{\mathcal{J}^*} = a^{\mathcal{J}}$, if $a \in \mathsf{N_I}(\mathcal{D})$, and arbitrary, otherwise;
- $\mathcal{N}^*(d) = \{(r_2)^{\mathcal{J}}(e) \mid (d, e) \in (r_1)^{\mathcal{J}}\}$, with $(r_2)^{\mathcal{J}}(e) = \{e' \in \Delta \mid (e, e') \in (r_2)^{\mathcal{J}}\}$.

Since $\mathcal{J} \models \exists r_1.\top \sqsubseteq \mathsf{E}$, we have that $(d, e) \in (r_1)^{\mathcal{J}}$ implies $d \in \mathsf{E}^{\mathcal{J}}$. Moreover, since $\mathcal{J} \models \top \sqsubseteq \forall r_2.\mathsf{E}$, for any $e' \in (r_2)^{\mathcal{J}}(e)$, we have $e' \in \mathsf{E}^{\mathcal{J}}$. This means that, for every $d \in \Delta^* = \mathsf{E}^{\mathcal{J}}$, $\mathcal{N}^*(d)$ is a set of subsets of Δ^*, and hence $\mathcal{J}^*$ is a well-defined $\mathcal{ALC}^{\lhd\Box}$ interpretation.

Claim 3.4. *For every $d \in \Delta^*$, $d \in C^{\mathcal{J}^*}$ if and only if $d \in ((C)^\dagger)^{\mathcal{J}}$.*

Proof of Claim. By induction on the construction of C. Similarly to the proof of Claim 3.2, it can be seen that $((C)^\dagger)^{\mathcal{J}} \subseteq \mathsf{E}^{\mathcal{J}} = \Delta^*$, for any $\mathcal{ALC}^{\lhd\Box}$ concept C. For the base case $C = A$, we have $d \in A^{\mathcal{J}^*}$ iff $d \in A^{\mathcal{J}}$ and $d \in \mathsf{E}^{\mathcal{J}}$, by definition of $\mathcal{J}^*$. This means $d \in (A \sqcap \mathsf{E})^{\mathcal{J}}$, i.e., $d \in ((A)^\dagger)^{\mathcal{J}}$. We now show the inductive cases.

$[C = \neg D]$ We have $d \in (\neg D)^{\mathcal{J}^*}$ iff $d \notin D^{\mathcal{J}^*}$. By induction hypothesis, this means $d \notin ((D)^\dagger)^{\mathcal{J}}$. Since $d \in \Delta^*$, this is equivalent to $d \in (\neg(D)^\dagger)^{\mathcal{J}}$ and $d \in \mathsf{E}^{\mathcal{J}}$, i.e. $d \in (\neg(D)^\dagger \sqcap \mathsf{E})^{\mathcal{J}} = ((\neg D)^\dagger)^{\mathcal{J}}$.

$[C = D \sqcap E]$ We have $d \in (D \sqcap E)^{\mathcal{J}^*}$ iff $d \in D^{\mathcal{J}^*}$ and $d \in E^{\mathcal{J}^*}$. By IH this holds iff $d \in ((D)^\dagger)^{\mathcal{J}}$ and $d \in ((E)^\dagger)^{\mathcal{J}}$. The previous step is equivalent to $d \in ((D)^\dagger \sqcap (E)^\dagger)^{\mathcal{J}} = ((D \sqcap E)^\dagger)^{\mathcal{J}}$.

$[C = \exists r.D]$ ($\Rightarrow$) Suppose that $d \in (\exists r.D)^{\mathcal{J}^*}$. This means that there exists $e \in \Delta^*$ such that $(d, e) \in r^{\mathcal{J}^*}$ and $e \in D^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$, we have $(d, e) \in r^{\mathcal{J}^*}$ iff $(d, e) \in r^{\mathcal{J}}$ and $e \in \Delta^*$. Moreover, by induction hypothesis, $e \in D^{\mathcal{J}^*}$ iff $e \in ((D)^\dagger)^{\mathcal{J}}$. Hence, if $d \in (\exists r.D)^{\mathcal{J}^*}$, then there exists $e \in \Delta^* \subseteq \Delta$ with $(d, e) \in r^{\mathcal{J}}$ and $e \in ((D)^\dagger)^{\mathcal{J}}$, i.e. $d \in (\exists r.(D)^\dagger)^{\mathcal{J}}$. Since $d \in \Delta^*$, this means that $d \in (\exists r.(D)^\dagger \sqcap \mathsf{E})^{\mathcal{J}} = ((\exists r.D)^\dagger)^{\mathcal{J}}$. ($\Leftarrow$) Conversely, suppose that $d \in ((\exists r.D)^\dagger)^{\mathcal{J}}$, meaning that there exists $e \in \Delta$ with $(d, e) \in r^{\mathcal{J}}$ and $e \in ((D)^\dagger)^{\mathcal{J}}$. As already observed, we have that $((D)^\dagger)^{\mathcal{J}} \subseteq \mathsf{E}^{\mathcal{J}} = \Delta^*$, then $e \in \Delta^*$. By definition of $\mathcal{J}^*$ and by induction hypothesis, we then obtain that there exists $e \in \Delta^*$ with $(d, e) \in r^{\mathcal{J}^*}$ and $e \in D^{\mathcal{J}^*}$, i.e., $d \in (\exists r.D)^{\mathcal{J}^*}$.

$[C = \lhd D]$ ($\Rightarrow$) Suppose $d \in (\lhd D)^{\mathcal{J}^*}$; i.e., there exists $\alpha \in \mathcal{N}^*(d)$ s.t. $\alpha \subseteq D^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$, this means that there exists $e \in \Delta$ s.t. $(d, e) \in (r_1)^{\mathcal{J}}$ and $(r_2)^{\mathcal{J}}(e) \subseteq D^{\mathcal{J}^*}$. Since $\mathcal{J} \models \top \sqsubseteq \forall r_2.\mathsf{E}$,

for any $e' \in (r_2)^{\mathcal{J}}(e)$ we have $e' \in \mathsf{E}^{\mathcal{J}} = \Delta^*$, and, by IH, $e' \in ((D)^\dagger)^{\mathcal{J}}$. Hence $(r_2)^{\mathcal{J}}(e) \subseteq ((D)^\dagger)^{\mathcal{J}}$, and thus $d \in (\exists r_1.\forall r_2.(D)^\dagger)^{\mathcal{J}} = ((\lhd D)^\dagger)^{\mathcal{J}}$. ($\Leftarrow$) Conversely, suppose that $d \in ((\lhd D)^\dagger)^{\mathcal{J}} = (\exists r_1.\forall r_2.(D)^\dagger)^{\mathcal{J}}$, that is, there exists $e \in \Delta$ such that $(d, e) \in (r_1)^{\mathcal{J}}$ and $(r_2)^{\mathcal{J}}(e) \subseteq ((D)^\dagger)^{\mathcal{J}}$. Since, as observed, $((D)^\dagger)^{\mathcal{J}} \subseteq \mathsf{E}^{\mathcal{J}} = \Delta^*$, by induction hypothesis we obtain that $(r_2)^{\mathcal{J}}(e) \subseteq (D)^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$, $(r_2)^{\mathcal{J}}(e) \in \mathcal{N}^*(d)$, therefore $d \in (\lhd D)^{\mathcal{J}^*}$.
$[C = \Box D]$ ($\Rightarrow$) Suppose $d \in (\Box D)^{\mathcal{J}^*}$; i.e. for every $\alpha \in \mathcal{N}^*(d)$, we have $\alpha \subseteq D^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$, this means that for every $e \in \Delta$ s.t. $(d, e) \in (r_1)^{\mathcal{J}}$, we have $(r_2)^{\mathcal{J}}(e) \subseteq D^{\mathcal{J}^*}$. Since $\mathcal{J} \models \top \sqsubseteq \forall r_2.\mathsf{E}$, for any $e' \in (r_2)^{\mathcal{J}}(e)$ we have $e' \in \mathsf{E}^{\mathcal{J}} = \Delta^*$, and, by IH, $e' \in ((D)^\dagger)^{\mathcal{J}}$. Therefore $(r_2)^{\mathcal{J}}(e) \subseteq ((D)^\dagger)^{\mathcal{J}}$. This means that $d \in (\forall r_1.\forall r_2.(D)^\dagger)^{\mathcal{J}}$. As $d \in \Delta^*$, we get $d \in (\forall r_1.\forall r_2.(D)^\dagger \sqcap \mathsf{E})^{\mathcal{J}} = ((\Box D)^\dagger)^{\mathcal{J}}$. ($\Leftarrow$) Conversely, assume $d \in ((\Box D)^\dagger)^{\mathcal{J}} = (\forall r_1.\forall r_2.(D)^\dagger \sqcap \mathsf{E})^{\mathcal{J}}$, that is, for every $e \in \Delta$ such that $(d, e) \in (r_1)^{\mathcal{J}}$, we have $(r_2)^{\mathcal{J}}(e) \subseteq ((D)^\dagger)^{\mathcal{J}}$. Since $((D)^\dagger)^{\mathcal{J}} \subseteq \mathsf{E}^{\mathcal{J}} = \Delta^*$, by induction hypothesis we obtain $(r_2)^{\mathcal{J}}(e) \subseteq D^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$, if $\alpha \in \mathcal{N}^*(d)$, then $\alpha = (r_2)^{\mathcal{J}}(e)$ for a e such that $(d, e) \in (r_1)^{\mathcal{J}}$. As just shown, we then have $\alpha \subseteq D^{\mathcal{J}^*}$, therefore $d \in (\Box D)^{\mathcal{J}^*}$. $\dashv$

Thus, $\mathcal{J}^*$ satisfies C.

Claim 3.5. *For every $C \sqsubseteq D \in \mathcal{O}$, $\mathcal{J} \models (C \sqsubseteq D)^\dagger$ if and only if $\mathcal{J}^* \models C \sqsubseteq D$. Moreover, for every assertion $\delta \in \mathcal{D}$, $\mathcal{J} \models (\delta)^\dagger$ if and only if $\mathcal{J}^* \models \delta$.*

Proof of Claim. Consider a CI $C \sqsubseteq D \in \mathcal{O}$. ($\Rightarrow$) Suppose $\mathcal{J} \models (C \sqsubseteq D)^\dagger$, i.e., $((C)^\dagger)^{\mathcal{J}} \subseteq ((D)^\dagger)^{\mathcal{J}}$, and consider $d \in \Delta^*$ such that $d \in C^{\mathcal{J}^*}$. By Claim 3.4, $d \in ((C)^\dagger)^{\mathcal{J}}$ and thus $d \in ((D)^\dagger)^{\mathcal{J}}$. By Claim 3.4 again, we obtain $d \in D^{\mathcal{J}^*}$. Hence, $\mathcal{J}^* \models C \sqsubseteq D$. ($\Leftarrow$) Conversely, suppose $\mathcal{J}^* \models C \sqsubseteq D$, meaning that $C^{\mathcal{J}^*} \subseteq D^{\mathcal{J}^*}$. For any $d \in \Delta$ such that $d \in ((C)^\dagger)^{\mathcal{J}}$, we have that $d \in \Delta^*$, since by definition $((C)^\dagger)^{\mathcal{J}} \subseteq E^{\mathcal{J}} = \Delta^*$. Hence, by Claim 3.4, we obtain $d \in C^{\mathcal{J}^*}$, which implies $d \in D^{\mathcal{J}^*}$. Again by Claim 3.4, this means $d \in ((D)^\dagger)^{\mathcal{J}}$. Thus, $((C)^\dagger)^{\mathcal{J}} \subseteq ((D)^\dagger)^{\mathcal{J}}$, i.e., $\mathcal{J} \models (C \sqsubseteq D)^\dagger$.

For an assertion $C(a) \in \mathcal{D}$, we have the following. ($\Rightarrow$) Suppose that $\mathcal{J} \models (C(a))^\dagger$, i.e., $\mathcal{J} \models (C)^\dagger(a)$, meaning that $a^{\mathcal{J}} \in ((C)^\dagger)^{\mathcal{J}}$. Since $\mathcal{J} \models \mathsf{E}(a)$, for every $a \in \mathsf{N_I}(\mathcal{D})$, the previous step implies $a^{\mathcal{J}} \in \mathsf{E}^{\mathcal{J}} = \Delta^*$. Hence, by Claim 3.4, we have $a^{\mathcal{J}^*} = a^{\mathcal{J}} \in C^{\mathcal{J}^*}$, that is, $\mathcal{J}^* \models C(a)$. ($\Leftarrow$) Conversely, suppose that $\mathcal{J}^* \models C(a)$, meaning that $a^{\mathcal{J}^*} \in C^{\mathcal{J}^*}$. By definition of $\mathcal{J}^*$ and Claim 3.4, this means that $a^{\mathcal{J}} \in ((C)^\dagger)^{\mathcal{J}}$, i.e., $\mathcal{J} \models (C(a))^\dagger$.

For an assertion $r(a, b) \in \mathcal{D}$, the following holds. ($\Rightarrow$) Suppose that $\mathcal{J} \models (r(a, b))^\dagger$, i.e., we have $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$. Since $\mathcal{J} \models \mathsf{E}(a)$ and $\mathcal{J} \models \mathsf{E}(b)$, we have $a^{\mathcal{J}}, b^{\mathcal{J}} \in \mathsf{E}^{\mathcal{J}} = \Delta^*$. Thus, we also have $(a^{\mathcal{J}}, b^{\mathcal{J}}) = (a^{\mathcal{J}^*}, b^{\mathcal{J}^*}) \in r^{\mathcal{J}^*}$, meaning that $\mathcal{J}^* \models r(a, b)$. ($\Leftarrow$) Conversely, by assuming that $\mathcal{J}^* \models r(a, b)$, i.e., $(a^{\mathcal{J}^*}, b^{\mathcal{J}^*}) \in r^{\mathcal{J}^*}$, we immediately obtain $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$, i.e., $\mathcal{J} \models (r(a, b))^\dagger$. $\dashv$

Therefore, $\mathcal{J}^* \models \mathcal{K}$, with $C^{\mathcal{J}^*} \neq \emptyset$. This completes the proof of the lemma. $\square$

Since the translation is polynomial in the size of the concept and the KB, the previous result entails that $\mathcal{ALC}^{\lhd\Box}$ concept satisfiability w.r.t. a KB is polynomial-time reducible to the same problem in $\mathcal{ALC}$, and hence its complexity is bounded by that of the classical setting.

Theorem 4. *$\mathcal{ALC}^{\lhd\Box}$ concept satisfiability w.r.t. a knowledge base is ExpTime-complete.*

4. Additional Conditions on Neighbourhood Functions

We show that $\mathcal{ALC}^{\lhd\Box}_{\mathsf{L}}$ concept satisfiability w.r.t. a KB remains ExpTime-complete, under any of the conditions L introduced in Section 2.3. For $\mathsf{L} \in \{\mathsf{ex}, \mathsf{cons}, \mathsf{pcons}\}$, concept satisfiability w.r.t. a KB can be reduced to the same problem in $\mathcal{ALC}$, by adding to the KB the $\mathcal{ALC}^{\lhd\Box}$ CI that characterises the corresponding semantic condition (Lemma 1) and then using the translation given in Section 3. For $\mathsf{L} \in \{\mathsf{corr}, \mathsf{pcorr}\}$, since no single characterising $\mathcal{ALC}^{\lhd\Box}$ formula is available (Lemma 2), we need to provide an *ad hoc* reduction to a more expressive language capable of capturing the corresponding condition. In the following, given a KB $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and a CI or role inclusion φ, with an abuse of notation we write $\mathcal{K} \cup \{\varphi\}$ in place of $(\mathcal{O} \cup \{\varphi\}, \mathcal{D})$.

The $\mathcal{ALC}^{\lhd\Box}_{\mathsf{ex}}$, $\mathcal{ALC}^{\lhd\Box}_{\mathsf{cons}}$, and $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcons}}$ Cases

Lemma 5. *Let* $\mathsf{L} \in \{\mathsf{ex}, \mathsf{cons}, \mathsf{pcons}\}$*. Given a concept* C *and knowledge base* $\mathcal{K}$*,* C *is* $\mathcal{ALC}^{\lhd\Box}_{\mathsf{L}}$*-satisfiable w.r.t.* $\mathcal{K}$ *if and only if* $(C)^{\dagger}$ *is* $\mathcal{ALC}$*-satisfiable w.r.t.* $(\mathcal{K})^{\dagger} \cup \{(\varphi_{\mathsf{L}})^{\dagger}\}$*, where* φ_{L} *is:*

- $\top \sqsubseteq \lhd\top$*, for* $\mathsf{L} = \mathsf{ex}$*;*
- $\top \sqsubseteq \rhd\top$*, for* $\mathsf{L} = \mathsf{cons}$*;*
- $\top \sqsubseteq \Diamond\top$*, for* $\mathsf{L} = \mathsf{pcons}$*.*

Proof. By Lemma 1, a concept C is $\mathcal{ALC}^{\lhd\Box}_{\mathsf{L}}$-satisfiable w.r.t. $\mathcal{K}$ iff C is $\mathcal{ALC}^{\lhd\Box}$-satisfiable w.r.t. $\mathcal{K} \cup \varphi_{\mathsf{L}}$. By Lemma 3, C is $\mathcal{ALC}^{\lhd\Box}$-satisfiable w.r.t. $\mathcal{K} \cup \varphi_{\mathsf{L}}$ iff $(C)^{\dagger}$ is $\mathcal{ALC}$-satisfiable w.r.t. $(\mathcal{K})^{\dagger} \cup (\varphi_{\mathsf{L}})^{\dagger}$. $\Box$

Remark 2. *Observe that* $(\varphi_{\mathsf{L}})^{\dagger}$ *can be replaced by the following simpler formulas, which are equivalent w.r.t.* $(\mathcal{K})^{\dagger}$*:*

- $\mathsf{E} \sqsubseteq \exists r_1.\top$*, for* $\mathsf{L} = \mathsf{ex}$*;*
- $\mathsf{E} \sqsubseteq \forall r_1.\exists r_2.\top$*, for* $\mathsf{L} = \mathsf{cons}$*;*
- $\mathsf{E} \sqsubseteq \exists r_1.\exists r_2.\top$*, for* $\mathsf{L} = \mathsf{pcons}$*.*

Indeed, we have the following.

- *For* $\mathsf{L} = \mathsf{ex}$*, we have by definition* $(\top \sqsubseteq \lhd\top)^{\dagger} = (\top)^{\dagger} \sqsubseteq \exists r_1.\forall r_2.(\top)^{\dagger}$*. Given that* $(\top)^{\dagger}$ *is equivalent to* E*, and that* $\forall r_2.\mathsf{E}$ *is equivalent to* $\top$ *(since* $\top \sqsubseteq \forall r_2.\mathsf{E} \in (\mathcal{O})^{\dagger}$*), we have that* $(\top \sqsubseteq \lhd\top)^{\dagger}$ *is equivalent, over models of* $(\mathcal{K})^{\dagger}$*, to* $\mathsf{E} \sqsubseteq \exists r_1.\top$*.*
- *For* $\mathsf{L} = \mathsf{cons}$*, we have that* $\top \sqsubseteq \rhd\top$ *is equivalent to* $\lhd\bot \sqsubseteq \bot$ *(Remark 1). By definition,* $(\lhd\bot \sqsubseteq \bot)^{\dagger} = \exists r_1.\forall r_2.(\bot)^{\dagger} \sqsubseteq (\bot)^{\dagger}$*. Given that* $(\bot)^{\dagger}$ *is equivalent to* $\bot$*, the translation simplifies to* $\exists r_1.\forall r_2.\bot \sqsubseteq \bot$*, which is equivalent by contraposition to* $\top \sqsubseteq \forall r_1.\exists r_2.\top$*. Finally, since* $\exists r_1.\top \sqsubseteq \mathsf{E} \in (\mathcal{O})^{\dagger}$ *ensures that nothing outside the extension of* E *can have* r_1*-successors, the obtained CI is equivalent, over models of* $(\mathcal{K})^{\dagger}$*, to* $\mathsf{E} \sqsubseteq \forall r_1.\exists r_2.\top$*.*
- *For* $\mathsf{L} = \mathsf{pcons}$*, we have that* $\top \sqsubseteq \Diamond\top$ *is equivalent to* $\Box\bot \sqsubseteq \bot$ *(Remark 1). By definition,* $(\Box\bot \sqsubseteq \bot)^{\dagger} = (\forall r_1.\forall r_2.(\bot)^{\dagger} \sqcap \mathsf{E}) \sqsubseteq (\bot)^{\dagger}$*. Given that* $(\bot)^{\dagger}$ *is equivalent to* $\bot$*, we obtain* $\forall r_1.\forall r_2.\bot \sqcap \mathsf{E} \sqsubseteq \bot$*, which is in turn equivalent to* $\mathsf{E} \sqsubseteq \exists r_1.\exists r_2.\top$*.*

The $\mathcal{ALC}^{\lhd\Box}_{\mathsf{corr}}$ Case

Lemma 6. *Given a concept* C *and knowledge base* $\mathcal{K}$*,* C *is* $\mathcal{ALC}^{\lhd\Box}_{\mathsf{corr}}$*-satisfiable w.r.t.* $\mathcal{K}$ *if and only if* $(C)^{\dagger}$ *is* $\mathcal{ALCHI}$*-satisfiable w.r.t.* $(\mathcal{K})^{\dagger} \cup \{r_1 \sqsubseteq r_2^-\}$*.*

Proof. ($\Rightarrow$) Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$ be a $\mathcal{ALC}^{\lhd\Box}_{\mathsf{corr}}$ interpretation satisfying $\mathcal{K}$ and C, and define the $\mathcal{ALCHI}$ interpretation $\mathcal{I}^* = (\Delta^*, \cdot^{\mathcal{I}^*})$ as in the proof of Lemma 3 ($\Rightarrow$). As already shown, $\mathcal{I}^*$ satisfies $(\mathcal{K})^{\dagger}$ and $(C)^{\dagger}$. We now prove that $\mathcal{I}^*$ satisfies $r_1 \sqsubseteq r_2^-$. If $(x, y) \in (r_1)^{\mathcal{I}^*}$, then $x = (d, 0)$ and $y = (\alpha, 1)$ for some $d \in \Delta$, $\alpha \subseteq \Delta$ such that $\alpha \in \mathcal{N}(d)$. By the condition (corr), $d \in \alpha$, hence by definition $((\alpha, 1), (d, 0)) \in (r_2)^{\mathcal{I}^*}$, and thus $(x, y) \in (r_2^-)^{\mathcal{I}^*}$.

($\Leftarrow$) Let $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ be a $\mathcal{ALCHI}$ interpretation satisfying $(\mathcal{K})^{\dagger} \cup \{r_1 \sqsubseteq r_2^-\}$ and $(C)^{\dagger}$. We define the $\mathcal{ALC}^{\lhd\Box}$ interpretation $\mathcal{J}^* = (\Delta^*, \cdot^{\mathcal{J}^*}, \mathcal{N}^*)$ as in Lemma 3 ($\Leftarrow$). As already shown, $\mathcal{J}^*$ satisfies $\mathcal{K}$ and C. We now prove that $\mathcal{J}^*$ satisfies the condition (corr), i.e., for all $\alpha \in \mathcal{N}^*(d)$, $d \in \alpha$, and hence it is a $\mathcal{ALC}^{\lhd\Box}_{\mathsf{corr}}$ interpretation. If $\alpha \in \mathcal{N}^*(d)$, then $\alpha = (r_2)^{\mathcal{J}}(e)$, for some e such that $(d, e) \in (r_1)^{\mathcal{J}}$. Since $\mathcal{J} \models r_1 \sqsubseteq r_2^-$, we have $(e, d) \in (r_2)^{\mathcal{J}}$, hence $d \in (r_2)^{\mathcal{J}}(e) = \alpha$. $\Box$

Observe that the full expressive power of $\mathcal{ALCHI}$ is actually not needed: indeed, we just require the single role inclusion $r_1 \sqsubseteq r_2^-$, for the fresh roles r_1 and r_2; role inclusions and inverses are not appearing anywhere else in the translation of the KB.

The $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$ Case Following [14], we define the *guarded fragment* GF as the smallest set of first-order (FO) formulas, without function symbols, that are constructed as follows:

1. every atomic formula is in GF;
2. GF is closed under Boolean combinations with $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$;
3. if $\psi \in \mathsf{GF}$, γ is an atomic formula such that $\mathsf{fvar}(\psi) \subseteq \mathsf{fvar}(\gamma)$, and $\bar{x}$ is a non-empty tuple of variables with (the underlying set) $\bar{x} \subseteq \mathsf{fvar}(\gamma)$, then $\forall\bar{x}(\gamma \rightarrow \psi) \in \mathsf{GF}$ and $\exists\bar{x}(\gamma \wedge \psi) \in \mathsf{GF}$;
4. if $\psi \in \mathsf{GF}$ with $\mathsf{fvar}(\psi) = \{x\}$, then $\forall x.\psi \in \mathsf{GF}$ and $\exists x.\psi \in \mathsf{GF}$;

where $\mathsf{fvar}(\psi)$ is the set of free variables of a formula ψ. For $k \geq 2$, the *k-variable guarded fragment* GF^k is the subset of GF using at most k variables.

It is known [14, Theorem 4.8] that the formula satisfiability problem for GF^2 (as well as for GF^k, for any fixed k) is ExpTime-complete. Moreover, it can be readily checked that the formula

$$\forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$$

is in GF^2, as it is the standard translation of any $\mathcal{ALC}$ concept and KB (in the following, with an abuse of notation, we identify an $\mathcal{ALC}$ concept or KB with their standard translation).

Lemma 7. *Given an $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$ concept C and knowledge base $\mathcal{K}$, C is $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$-satisfiable w.r.t. $\mathcal{K}$ if and only if the GF^2 formula $\exists x(C)^\dagger \wedge (\mathcal{K})^\dagger \wedge \forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$ is satisfiable.*

Proof. ($\Rightarrow$) Let $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}}, \mathcal{N})$ be an $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$ interpretation satisfying $\mathcal{K}$ and C, and define the $\mathcal{ALC}$ interpretation $\mathcal{I}^* = (\Delta^*, \cdot^{\mathcal{I}^*})$ as in the proof of Lemma 3 ($\Rightarrow$). As already shown, $\mathcal{I}^*$ satisfies $(\mathcal{K})^\dagger$ and $(C)^\dagger$, hence $\mathcal{I}^*$ (viewed as an FO structure) satisfies (the standard translation of) $\exists x.(C)^\dagger$ and $(\mathcal{K})^\dagger$. It remains to show that $\mathcal{I}^* \models \forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$. Let $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$ be arbitrary. We have $d \in \Delta$, and hence, by the condition (pcorr), there exists $\alpha \in \mathcal{N}(d)$ such that $d \in \alpha$. By definition, $((d,0),(\alpha,1)) \in (r_1)^{\mathcal{I}^*}$ and $((\alpha,1),(d,0)) \in (r_2)^{\mathcal{I}^*}$, since $d \in \alpha$. Therefore, $\mathcal{I}^* \models \exists y(r_1(x,y) \wedge r_2(y,x))$. Since $(d,0) \in \mathsf{E}^{\mathcal{I}^*}$ was arbitrary, we conclude that $\mathcal{I}^* \models \forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$.

($\Leftarrow$) Let $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ be an FO structure satisfying (the standard translation of) $\exists x.((C)^\dagger)^{\mathcal{J}}$, $(\mathcal{K})^\dagger$, and $\forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$. We define the $\mathcal{ALC}^{\lhd\Box}$ interpretation $\mathcal{J}^* = (\Delta^*, \cdot^{\mathcal{J}^*}, \mathcal{N}^*)$ as in Lemma 3 ($\Leftarrow$). As already shown, $\mathcal{J}^*$ satisfies $\mathcal{K}$ and C. We now prove that $\mathcal{J}^*$ satisfies the condition (pcorr), i.e., for every $d \in \Delta^*$, there exists $\alpha \in \mathcal{N}^*(d)$ with $d \in \alpha$, and hence $\mathcal{J}^*$ is an $\mathcal{ALC}^{\lhd\Box}_{\mathsf{pcorr}}$ interpretation. Let $d \in \Delta^* = \mathsf{E}^{\mathcal{J}}$. Since $\mathcal{J} \models \forall x\big(\mathsf{E}(x) \rightarrow \exists y(r_1(x,y) \wedge r_2(y,x))\big)$, there exists $e \in \Delta$ such that $(d,e) \in (r_1)^{\mathcal{J}}$ and $(e,d) \in (r_2)^{\mathcal{J}}$. By definition of $\mathcal{N}^*$, we have $(r_2)^{\mathcal{J}}(e) \in \mathcal{N}^*(d)$ and $d \in (r_2)^{\mathcal{J}}(e)$. Thus, there exists $\alpha \in \mathcal{N}^*(d)$ such that $d \in \alpha$. $\Box$

Given that we are able to reduce $\mathcal{ALC}^{\lhd\Box}_{\mathsf{L}}$, for any singleton subset $\mathsf{L} \subseteq \{\mathsf{ex}, \mathsf{cons}, \mathsf{pcons}, \mathsf{corr}, \mathsf{pcorr}\}$, to either $\mathcal{ALC}$, $\mathcal{ALCHI}$, or GF^2 (which are all known to be ExpTime-complete [1, 15, 14]), and given that GF^2 subsumes both $\mathcal{ALC}$ and $\mathcal{ALCHI}$ as a fragment of first-order logic, our polynomial-time translation gives us the following result for any subset of conditions L.

Theorem 8. *For $\mathsf{L} \subseteq \{\mathsf{ex}, \mathsf{cons}, \mathsf{pcons}, \mathsf{corr}, \mathsf{pcorr}\}$, $\mathcal{ALC}^{\lhd\Box}_{\mathsf{L}}$ concept satisfiability w.r.t. a knowledge base is ExpTime-complete.*

5. Discussion

The neighbourhood semantics here presented, as well as the core idea for the reduction to standard relational semantics, have roots in the literature on propositional *non-normal* modal logics [16, 17, 18], where each world is equipped with a set of neighbourhoods to interpret modal operators. This modal semantics also constitutes the backbone of *non-normal modal DLs* [19], extensions of standard DLs in which modalities can be applied both to concepts and formulas, and where the neighbourhood function is defined on *worlds*, associated with classical DL interpretations. In contrast, $\lhd$ and $\Box$ are here interpreted by means of a neighbourhood function on *domain elements*, within a single DL interpretation, and as such can be thought of as neighbourhood counterparts of role restrictions applied on concepts. As future work, it would be interesting to determine how $\triangleleft$ and $\Box$ can be combined with other non-normal modal operators over formulas that are interpreted using possible-world neighbourhood semantics. Such modalities could be used to encode axioms that are conceivable/acknowledged by some/all accounts, thereby separating them from the factual portions of an ontology, or to model other epistemic, deontic or agentic notions that avoid known paradoxes of normal operators [19].

The operators $\triangleleft$, $\Box$ (and their dual $\triangleright$, $\Diamond$) were originally introduced in the context of propositional modal logic by Brown [20] to represent distinct forms of agent *abilities* to perform actions. Brown provides a complete axiomatisation for his logic, together with a collection of valid formulas capturing the interactions between $\triangleleft$ and $\Box$. Several of these formulas can be directly translated into concept inclusions satisfied by all $\mathcal{ALC}^{\triangleleft\Box}$ interpretations, as shown in Section 2.2. More recently, the same modalities have been reconsidered in a multimodal setting to represent *epistemic attitudes* of intensional groups [21]. Furthermore, Lellmann [22] presents a nested *sequent calculus* for Brown's logic, along with a decision procedure based on his calculus that constructs derivations of valid formulas, as well as neighbourhood countermodels for non-valid ones. In future work, we aim to investigate whether Lellmann's calculus can serve as a basis for developing a tableaux algorithm for reasoning in $\mathcal{ALC}^{\triangleleft\Box}$.

In the context of formalisms for multiperspective and approximate reasoning, we intend to provide a detailed comparison between our neighbourhood DLs, which allow a limited form of quantification over unnamed accounts, and logics from the standpoint DL family, which admit explicit naming of standpoints and richer sharpening statements. Moreover, we plan to investigate the connections with (multi-granular) rough DLs, to understand whether additional constraints on the neighbourhood functions (corresponding, for instance, to the symmetry and the transitivity of the indiscernibility relations in multi-granular rough semantics) can be encoded by a suitable reduction extending those presented in this work. Other interesting semantic constraints to be considered are, e.g., the *closure under non-empty unions* and *intersections* of neighbourhoods, as well as the *nesting* condition, imposing a total order (with respect to inclusion) among the accounts. These options have natural connections with *conditional* logics and their sphere models [23, 24], which are in turn related to the preferential models for DLs of *typicality* [25, 26] and *defeasible* reasoning [27, 28, 29].

Finally, we are interested in investigating the complexity of restricted fragments, such as those based on $\mathcal{EL}$, as well as the impact of adding constructs like *nominals* and *definite descriptions* [30, 31]. These features are particularly useful in scenarios where multiple accounts might disagree on the characterisation of individuals whose precise identity remains uncertain. In contrast to the standpoint DLs, where the interaction with nominals typically increases the complexity of reasoning [4], we conjecture that our reductions remain stable in presence of such referring expression features and other standard DL constructors, as long as they are expressible in the guarded fragment with a fixed number of variables.

Acknowledgments

T. Dalmonte acknowledges the financial support through the 'Abstractron' project funded by the Autonome Provinz Bozen - Südtirol (Autonomous Province of Bolzano/Bozen) through the Research Südtirol/Alto Adige 2022 Call.

Declaration on Generative AI

Generative AI tools were employed to assist in the process of refining ideas and to improve the clarity of writing. The authors take full responsibility for the content and originality of the work.

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