Temporal $\mathcal{EL}$ and Equations over Sets of Integers

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Abstract

We study $\mathcal{TEL}^{\bigcirc}$, an extension of the description logic $\mathcal{EL}$ with the temporal operator $\bigcirc^n$ ('in n instants from now'), by encoding its canonical models with sets of integers. We characterise reasoning in $\mathcal{TEL}^{\bigcirc}$ in terms of operations on such sets and arrive at a correspondence between (fragments of) $\mathcal{TEL}^{\bigcirc}$ and (classes of) resolved systems of equations over sets of integers. This generalises the previously established link between $\mathcal{TEL}^{\bigcirc}$ and formal grammars, leading to simpler proofs of known results as well as yielding new results.

Keywords

Temporal description logics, query answering, equations over sets of integers

1. Introduction

Temporal description logics (TDLs) extend classical DLs with means of modelling changes in the application domain; for surveys, see [1, 2, 3, 4]. They are typically interpreted over Cartesian products of standard DL structures and timelines such as $(\mathbb{Z}, <)$ for discrete linear time unbounded in both directions. The two-dimensional nature of temporal interpretations requires a careful choice of the TDL language: *rigid roles*, which do not change over time, are known to lead to undecidability of subsumption in the temporal extension of $\mathcal{ALC}$ with the next-time operator $\bigcirc$ [2, Theorem 4]. We consider $\mathcal{TEL}^{\bigcirc}$, which extends a lightweight DL $\mathcal{EL}$ with temporal operators $\bigcirc^n$ ('in n instants from now'), over the linear timeline $(\mathbb{Z}, <)$; note that n can also be negative, which allows us to look into the past.

Example 1. Suppose that Alice begins a two-year writing course in 2026, denoted $\mathsf{Fresher}(\mathit{Alice}, 2026)$. Graduation inspires her, and inspired writers cannot help but write stories, which typically take three years to develop into complete novels. Finishing a novel proves equally inspiring. This story can be formalised using a *rigid* role $\mathsf{authorOf}$ and $\mathcal{TEL}^{\bigcirc}$ concept inclusions in the following way:

$$\mathsf{Fresher} \sqsubseteq \bigcirc^2(\mathsf{Inspired} \sqcap \mathsf{Writer}), \qquad \mathsf{Inspired} \sqcap \mathsf{Writer} \sqsubseteq \exists \mathsf{authorOf}.\mathsf{Story},$$
$$\mathsf{Writer} \sqsubseteq \bigcirc^1\mathsf{Writer}. \qquad \mathsf{Story} \sqsubseteq \bigcirc^3\mathsf{Novel}, \qquad \exists\mathsf{authorOf}.\mathsf{Novel} \sqsubseteq \mathsf{Inspired}.$$

Figure 1 depicts a selection of facts about Alice that can be inferred from $\mathsf{Fresher}(\mathit{Alice}, 2026)$ using the above concept inclusions. It follows, in particular, that Alice is inspired in 2028, and again in 2031.

Gutiérrez-Basulto et al. [5] studied $\mathcal{TEL}^{\bigcirc}$ and introduced a semantic condition, called *ultimate periodicity* of TBoxes, that guarantees decidability, for example, of subsumption: $\mathcal{T}$ is ultimately periodic if sets $Z_{B,A} = \{n \in \mathbb{Z} \mid B \sqsubseteq_{\mathcal{T}} \bigcirc^n A\}$ are semilinear, for all concept names A and B. Some syntactic fragments of $\mathcal{TEL}^{\bigcirc}$ that ensure ultimate periodicity were also identified, notably those based on certain acyclicity properties, either on the DL or on the temporal side, and those that disallow rigid roles. But the real expressive power of full $\mathcal{TEL}^{\bigcirc}$ remained unclear.

Recently, Bourgaux et al. [6] connected $\mathcal{TEL}^{\bigcirc}$ to conjunctive grammars of Okhotin [7, 8] by using an observation that sets $Z_{B,A}$ can be defined by *unary* languages (each $n \in \mathbb{N}$ corresponds to the word c^n). By applying language-theoretic results [8], they showed, rather surprisingly, that $\mathcal{TEL}^{\bigcirc}$ can define

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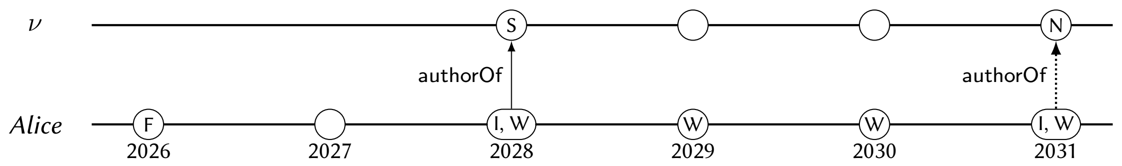

Figure 1: Representation of some inferences in Example 1. Solid lines represent the temporal evolution of the ABox individual *Alice* and the labelled null ν, which stands for the story *Alice* is writing. Concepts are indicated by their initial letters, and dotted arrows represent the relevant relations due to role rigidity.

non-semilinear sets. They further identified the 'linear fragment' of $\mathcal{TEL}^{\bigcirc}$, i.e., without conjunction, that corresponds to context-free grammars (and so, by Parikh [9, Theorem 2], to semilinear sets $Z_{B,A}$).

The unary-language take on sets $Z_{B,A}$, however, has its limitations. Most notably, only non-negative numbers can be encoded as words, and the correspondence to conjunctive grammars is thus limited to the 'future fragment' of $\mathcal{TEL}^{\bigcirc}$ defined over the $(\mathbb{N}, <)$ timeline. Additionally, this view enforces the unary encoding of numbers (which has, however, been common for the TDL complexity results; see, e.g., Chomicki and Imielinski [10], Gutiérrez-Basulto et al. [5], Artale et al. [3]).

In this paper we work directly with sets $Z_{B,A}$ and generalise the established correspondence between future $\mathcal{TEL}^{\bigcirc}$ and conjunctive grammars to full $\mathcal{TEL}^{\bigcirc}$ and *resolved systems of equations* over sets of integers. Introduced and studied by Jeż and Okhotin [11], these equations employ operations of union $\cup$, intersection $\cap$ and set addition, defined by $X + X' = \{n + n' \mid n \in X, n' \in X'\}$, and give us a powerful tool for defining subsets of $\mathbb{Z}$.

Example 2. Consider the following system of equations due to Jeż [12, Theorem 1]:

$$\begin{cases} X_1 &= \{1\} \cup ((X_1 + X_3) \cap (X_2 + X_2)), \\ X_2 &= \{2\} \cup ((X_1 + X_1) \cap (X_2 + X_6)), \\ X_3 &= \{3\} \cup ((X_1 + X_2) \cap (X_6 + X_6)), \\ X_6 &= (X_1 + X_2) \cap (X_3 + X_3), \end{cases}$$

where X_1, X_2, X_3 and X_4 range over sets of integers. It can be seen that the least solution of the system is given by $X_i^* = \{i \cdot 4^n \mid n \in \mathbb{N}\}$, for $i \in \{1, 2, 3, 6\}$, which are *not semilinear sets.* Indeed, for $i \in \{1, 2, 3\}$, we first observe that $i \in X_i^*$; next, as $4^n \in X_1^*$, $3 \cdot 4^n \in X_3^*$ and $2 \cdot 4^n \in X_2^*$, we have $(3 + 1) \cdot 4^n = (2 + 2) \cdot 4^n = 4^{n+1} \in X_1^*$; the other two cases, X_2 and X_3, are similar. For $i = 6$, observe that, as $4 \cdot 4^n \in X_1$, $2 \cdot 4^n \in X_2^*$ and $3 \cdot 4^n \in X_3^*$, we obtain $6 \cdot 4^n \in X_6^*$. These sets can also be seen as representing numbers in base-4 notation, as strings in the alphabet of $\{0, 1, 2, 3\}$, that is, X_1^*, X_2^*, X_3^* and X_6^* consist of numbers of the form $(10^*)_4$, $(20^*)_4$, $(30^*)_4$ and $(120^*)_4$, respectively.

In this setting, the future fragment of $\mathcal{TEL}^{\bigcirc}$ naturally corresponds to equations over subsets of $\mathbb{N}$, and the linear fragment to equations with union and addition only. This new, more unified, connection provides simpler proofs of the known results of Bourgaux et al. [6]. It also extends the known results to the case of the binary encoding of numbers and yields some new results, most notably for the linear fragment $\mathcal{TEL}^{\bigcirc}_{lin}$.

Section 2 introduces the required notions and notation. The correspondence is established in Section 3. Then, in Section 4, we proceed to derive our complexity results.

2. Preliminaries

We begin by formally introducing $\mathcal{TEL}^{\bigcirc}$, a temporal extension of the classical DL $\mathcal{EL}$. Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be countably infinite and pairwise disjoint sets of *concept*, *role* and *individual names*, respectively. The set $\mathsf{N_R}$ is partitioned into two infinite sets, $\mathsf{N_R^{rig}}$ and $\mathsf{N_R^{loc}}$, of *rigid* and *local role names*, respectively.

$\mathcal{TEL}^{\bigcirc}$ *concepts* are defined by the following grammar:

$$C, D \ ::= \ A \ \mid \ C \sqcap D \ \mid \ \exists r.C \ \mid \ \bigcirc^n C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $n \in \mathbb{Z}$. A $\mathcal{TEL}^{\bigcirc}$ *TBox* $\mathcal{T}$ is a finite set of *concept inclusions* (*CIs*) $C \sqsubseteq D$ for $\mathcal{TEL}^{\bigcirc}$ concepts C, D. A *temporal ABox* $\mathcal{A}$ is a finite set of assertions of the form

$$A(a, n) \quad \text{and} \quad r(a, b, n),$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$ and $n \in \mathbb{Z}$. A pair of the form $(\mathcal{T}, \mathcal{A})$ is called a *temporal knowledge base* (*KB*). As usual, we denote by $\mathsf{N_C}(\mathcal{T})$ and $\mathsf{N_I}(\mathcal{A})$ the sets of concept and individual names occurring in $\mathcal{T}$ and $\mathcal{A}$, respectively.

An *interpretation* $\mathfrak{I}$ is a structure $(\Delta^{\mathfrak{I}}, (\mathcal{I}_n)_{n\in\mathbb{Z}})$, where each $\mathcal{I}_n$ is a classical DL interpretation with domain $\Delta^{\mathfrak{I}}$: we have $a^{\mathcal{I}_n} \in \Delta$, for each individual name a; $A^{\mathcal{I}_n} \subseteq \Delta^{\mathfrak{I}}$, for each concept name A; and $r^{\mathcal{I}_n} \subseteq \Delta^{\mathfrak{I}} \times \Delta^{\mathfrak{I}}$, for each role name r. Rigid roles do not change their interpretation in time:

$$r^{\mathcal{I}_n} = r^{\mathcal{I}_0}, \qquad \text{for all } n \in \mathbb{Z} \text{ and } r \in \mathsf{N_R^{rig}}.$$

We usually write $a^{\mathfrak{I},n}$, $A^{\mathfrak{I},n}$ and $r^{\mathfrak{I},n}$ instead of $a^{\mathcal{I}_n}$, $A^{\mathcal{I}_n}$ and $r^{\mathcal{I}_n}$, respectively, and the mapping $\cdot^{\mathfrak{I},n}$ is extended to complex $\mathcal{TEL}^{\bigcirc}$ concepts as follows:

$$(C \sqcap D)^{\mathfrak{I},n} = C^{\mathfrak{I},n} \cap D^{\mathfrak{I},n}, \quad (\exists r.C)^{\mathfrak{I},n} = \{d \mid (d, e) \in r^{\mathfrak{I},n} \text{ for some } e \in C^{\mathfrak{I},n}\}, \quad (\bigcirc^k C)^{\mathfrak{I},n} = C^{\mathfrak{I},n+k}.$$

For ABoxes $\mathcal{A}$, we adopt the *standard name assumption*: $a^{\mathfrak{I},n} = a$ for all $a \in \mathsf{N_I}(\mathcal{A})$ and $n \in \mathbb{Z}$.

An interpretation $\mathfrak{I}$ is said to be a *model* of $C \sqsubseteq D$, written $\mathfrak{I} \models C \sqsubseteq D$, if $C^{\mathfrak{I},n} \subseteq D^{\mathfrak{I},n}$, for all $n \in \mathbb{Z}$, and a *model of a TBox* $\mathcal{T}$, written $\mathfrak{I} \models \mathcal{T}$, if $\mathfrak{I} \models C \sqsubseteq D$ for all $C \sqsubseteq D \in \mathcal{T}$. Thus, TBoxes are interpreted *globally* in the sense that all CIs must be satisfied at every time point. The relation $\models$ is extended to ABoxes by taking $\mathfrak{I} \models A(a, n)$ iff $a \in A^{\mathfrak{I},n}$ and $\mathfrak{I} \models r(a, b, n)$ iff $(a, b) \in r^{\mathfrak{I},n}$. We say that an interpretation $\mathfrak{I}$ is a *model* of $\mathcal{A}$ if $\mathfrak{I} \models \alpha$ for all assertions $\alpha \in \mathcal{A}$; $\mathfrak{I}$ is a *model* of a KB $(\mathcal{T}, \mathcal{A})$, written $\mathfrak{I} \models (\mathcal{T}, \mathcal{A})$, if both $\mathfrak{I} \models \mathcal{T}$ and $\mathfrak{I} \models \mathcal{A}$.

A TBox $\mathcal{T}$ *entails* a CI $C \sqsubseteq D$, written $C \sqsubseteq_{\mathcal{T}} D$, if $\mathfrak{I} \models \mathcal{T}$ implies $\mathfrak{I} \models C \sqsubseteq D$, and a KB $(\mathcal{T}, \mathcal{A})$ entails an ABox assertion α, written $(\mathcal{T}, \mathcal{A}) \models \alpha$, if $\mathfrak{I} \models (\mathcal{T}, \mathcal{A})$ implies $\mathfrak{I} \models \alpha$. Since the language does not contain role inclusions, the entailment problem for role assertions is trivial (boils down to testing ABox membership). So, we concentrate on the following key problem:

Instance checking: $(\mathcal{T}, \mathcal{A}) \models^? A(a, n)$	
Input:	TBox $\mathcal{T}$, ABox $\mathcal{A}$ and assertion $A(a, n)$.
Question:	Does $(\mathcal{T}, \mathcal{A})$ entail $A(a, n)$?

In addition, we study the following variant of the subsumption problem:

Temporal subsumption: $B \sqsubseteq^?_{\mathcal{T}} \bigcirc^n A$	
Input:	TBox $\mathcal{T}$, concept names A, B and $n \in \mathbb{Z}$.
Question:	Does $\mathcal{T}$ entail $B \sqsubseteq \bigcirc^n A$?

Observe that temporal subsumption is a special case of instance checking: indeed, we have $B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$ iff $(\mathcal{T}, \{B(a, 0)\}) \models A(a, n)$.

The *size* of $\mathcal{T}$ and $\mathcal{A}$ is the number of symbols required to write $\mathcal{T}$ and $\mathcal{A}$, respectively. We deal with two ways of representing components of KBs in symbols: numbers $n \in \mathbb{Z}$ in assertions of $\mathcal{A}$ and concepts of the form $\bigcirc^n C$ in $\mathcal{T}$ can be encoded in *unary* or in *binary*.

We present results for both combined and data complexity: for the former, all components are counted as input, while for the latter, the TBox is fixed, that is, for a complexity class $\mathcal{C}$ and a class $\mathcal{X}$ of TBoxes, the problem in $\mathcal{X}$ is *in* $\mathcal{C}$ *in data complexity* if the problem with a fixed $\mathcal{T}$ is in $\mathcal{C}$, for all $\mathcal{T} \in \mathcal{X}$; the problem in $\mathcal{X}$ is $\mathcal{C}$*-hard in data complexity* if the problem with a fixed $\mathcal{T}$ is $\mathcal{C}$-hard for some $\mathcal{T} \in \mathcal{X}$.

As TBox classes $\mathcal{X}$, we consider the following two fragments of $\mathcal{TEL}^{\bigcirc}$:

- in the *future fragment* $\mathcal{TEL}^{\bigcirc}_F$, only $\bigcirc^n$, $n \in \mathbb{N}$, is allowed, and only on the right-hand side of CIs;
- in the *linear fragment* $\mathcal{TEL}^{\bigcirc}_{lin}$, the use of conjunction ($\sqcap$) is not allowed.

Note that *rigid concepts*, which do not change their interpretation in time, can be expressed in both $\mathcal{TEL}^{\bigcirc}$ and $\mathcal{TEL}^{\bigcirc}_{lin}$, but not in $\mathcal{TEL}^{\bigcirc}_F$.

3. Foundations of Reasoning in $\mathcal{TEL}^{\bigcirc}$

In this section, we lay the groundwork for the development of decision procedures for instance checking and temporal subsumption in fragments of $\mathcal{TEL}^{\bigcirc}$. In the sequel, we assume that $\mathcal{TEL}^{\bigcirc}$ TBoxes are in *normal form*, that is, they consist of CIs of the form

$$B \sqcap B' \sqsubseteq A, \qquad B \sqsubseteq \bigcirc^n A, \qquad \exists r.B \sqsubseteq A, \qquad A \sqsubseteq \exists r.B,$$

where $A, B, B' \in \mathsf{N_C}$. It is routine to show that every $\mathcal{TEL}^{\bigcirc}$ TBox can be transformed into the normal form in polynomial time by introducing fresh concept names; see, e.g., [13; 14, Lemma 2]. Moreover, the normal form of a $\mathcal{TEL}^{\bigcirc}$, $\mathcal{TEL}^{\bigcirc}_F$ or $\mathcal{TEL}^{\bigcirc}_{lin}$ TBox belongs to the same fragment. In particular, note that $\bigcirc^n$ is restricted to the right-hand side of CIs: this is by definition for the future fragment $\mathcal{TEL}^{\bigcirc}_F$; in other fragments, this is done without loss of generality as $\bigcirc^n A \sqsubseteq B$ is equivalent to $A \sqsubseteq \bigcirc^{-n} B$.

3.1. Canonical Models

We begin by introducing the notion of the *canonical model* of a $\mathcal{TEL}^{\bigcirc}$ KB, that is, a model that witnesses exactly the ABox assertions entailed by the KB. Given a $\mathcal{TEL}^{\bigcirc}$ KB $(\mathcal{T}, \mathcal{A})$, we construct the canonical model $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ similarly to the case of the temporal DL-Lite [15, 16]. Let $\mathsf{N_N}$ be a countably infinite set of *labelled nulls* disjoint from $\mathsf{N_I}$, and set $\Delta^{\mathfrak{I}_{\mathcal{T},\mathcal{A}}} = \mathsf{N_I}(\mathcal{A}) \cup \mathsf{N_N}$. Following Artale et al. [15, Theorem 6], we represent $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ as a (potentially infinite) set $\mathcal{J}_{\mathcal{T},\mathcal{A}}$ of *atoms* built from $\mathsf{N_R}$, $\mathsf{N_C}$ and $\mathsf{N_I} \cup \mathsf{N_N}$, with $a \in A^{\mathfrak{I}_{\mathcal{T},\mathcal{A}},n}$ iff $A(a,n) \in \mathcal{J}_{\mathcal{T},\mathcal{A}}$ and $(a,b) \in r^{\mathfrak{I}_{\mathcal{T},\mathcal{A}},n}$ iff $r(a,b,n) \in \mathcal{J}_{\mathcal{T},\mathcal{A}}$. We define $\mathcal{J}_{\mathcal{T},\mathcal{A}} = \bigcup_{i \geq 0} \mathcal{J}_i$, where $\mathcal{J}_0 = \mathcal{A}$ and $\mathcal{J}_{i+1}$ is the smallest set including $\mathcal{J}_i$ and the following atoms:

(m1) $r(a,b,k)$, for all $k \in \mathbb{Z}$, if $r(a,b,n) \in \mathcal{J}_i$ and $r \in \mathsf{N_R^{rig}}$;

(m2) $A(a,n)$, if $B(a,n), B'(a,n) \in \mathcal{J}_i$ and $B \sqcap B' \sqsubseteq A \in \mathcal{T}$;

(m3) $A(a,k+n)$, if $B(a,k) \in \mathcal{J}_i$ and $B \sqsubseteq \bigcirc^n A \in \mathcal{T}$;

(m4) $A(a,n)$, if $r(a,b,n), B(b,n) \in \mathcal{J}_i$ and $\exists r.B \sqsubseteq A \in \mathcal{T}$;

(m5) $r(a,\nu,n)$ and $B(\nu,n)$, for some $\nu \in \mathsf{N_N}$ not occuring in $\mathcal{J}_i$, if $A(a,n) \in \mathcal{J}_i$ and $A \sqsubseteq \exists r.B \in \mathcal{T}$.

The above conditions can also be seen as the rules of the *oblivious chase* procedure; see, e.g., Grahne and Onet [17]. Formally, we have the following properties.

Theorem 1. *For every $\mathcal{TEL}^{\bigcirc}$ TBox $\mathcal{T}$ in normal form, we have the following*:

(*i*) $(\mathcal{T}, \mathcal{A}) \models \alpha$ *iff* $\mathfrak{I}_{\mathcal{T},\mathcal{A}} \models \alpha$, *for any ABox $\mathcal{A}$ and ABox assertion α*;

(*ii*) $B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$ *iff* $\mathfrak{I}_{\mathcal{T},\{B(a,k)\}} \models A(a,k+n)$, *for any $A, B \in \mathsf{N_C}$ and $k, n \in \mathbb{Z}$.*

Proof (Sketch). It is not hard to see that $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ is indeed a model of $(\mathcal{T}, \mathcal{A})$, so the 'only if' direction in both (*i*) and (*ii*) trivially holds. For the converse, observe that $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ is a universal model: there is a homomorphism from $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ to every model of $(\mathcal{T}, \mathcal{A})$. That is, for every $\mathfrak{I}$ with $\mathfrak{I} \models (\mathcal{T}, \mathcal{A})$, there exists a mapping $h\colon \Delta^{\mathfrak{I}_{\mathcal{T},\mathcal{A}}} \to \Delta^{\mathfrak{I}}$ that preserves the interpretation of concept and role names. This mapping can be constructed as a union $h = \bigcup_{i \geq 0} h_i$, where h_i is a homomorphism from $\mathcal{J}_i$ to $\mathfrak{I}$ defined by induction. This gives us (*i*). For (*ii*), observe that the relation $\mathfrak{I}_{\mathcal{T},\{B(a,k)\}} \models A(a,k+n)$ is independent of $k \in \mathbb{Z}$: it holds either for all $k \in \mathbb{Z}$ or for no $k \in \mathbb{Z}$. Next, if $\mathfrak{I} \models \mathcal{T}$ and $\mathfrak{I} \models B(a,k)$, then there is a homomorphism from $\mathfrak{I}_{\mathcal{T},\{B(a,k)\}}$ to $\mathfrak{I}$, which gives us (*ii*). ❏

The procedure for constructing the canonical model deals with infinite sets of atoms and is generally not terminating. So, although Theorem 1 provides a criterion for certain answers, it does not immediately yield a decision algorithm. However, by Theorem 1 (*ii*), all labelled nulls introduced in **(m5)** belong to a small number of 'isomorphism types', as formalised in the proof of the following lemma.

Lemma 2. *For every $\mathcal{TEL}^{\bigcirc}$ TBox $\mathcal{T}$ in normal form, ABox $\mathcal{A}$ and a labelled null ν introduced in* **(m5)** *to satisfy $C \sqsubseteq \exists r.B \in \mathcal{T}$ on an element of $C^{\mathfrak{I}_{\mathcal{T},\mathcal{A}}}$ at $k \in \mathbb{Z}$, we have*

$$\mathfrak{I}_{\mathcal{T},\mathcal{A}} \models A(\nu, k+n) \quad \textit{iff} \quad B \sqsubseteq_{\mathcal{T}} \bigcirc^n A, \qquad \textit{for each } A \in \mathsf{N_C} \textit{ and } n \in \mathbb{Z}.$$

Proof (Sketch). Let $\preceq$ be the least partial order on $\mathsf{N_I}(\mathcal{A}) \cup \mathsf{N_N}$ such that $a \preceq b$ if $\mathfrak{I}_{\mathcal{T},\mathcal{A}} \models r(a,b,n)$ for some $r \in \mathsf{N_R}$ and $n \in \mathbb{Z}$. Denote by $\mathfrak{I}^a_{\mathcal{T},\mathcal{A}}$ the restriction of $\mathfrak{I}_{\mathcal{T},\mathcal{A}}$ to $\{b \in \Delta^{\mathfrak{I}_{\mathcal{T},\mathcal{A}}} \mid a \preceq b\}$. It can be seen that, due to **(m1)**–**(m5)**, $\mathfrak{I}^\nu_{\mathcal{T},\mathcal{A}}$ is isomorphic to $\mathfrak{I}_{\mathcal{T},\{B(\nu,k)\}}$. The claim follows by Theorem 1 (*ii*). ❏

One possible approach to obtaining a decision procedure is to replace models with *quasimodels* [5, Theorem 4], where the 'isomorphism types' of nulls are represented as *traces* $\pi_B \colon \mathbb{Z} \to 2^{\mathsf{N_C}(\mathcal{T})}$ with $\pi_B(n)$ consisting of $A \in \mathsf{N_C}(\mathcal{T})$ such that $B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$. Conditions **(m1)**–**(m5)** are then translated to a suitable formalism for dealing with systems of traces. In the case of $\mathcal{TEL}^{\bigcirc}$, however, standard formalisms, e.g., the monadic second-order theory of integers, MSO$(\mathbb{Z}, <)$, a common tool for decidability proofs in temporal DLs [18], or Datalog$_{1S}$ [10], an extension of Datalog with integer variables and one successor function, cannot be easily applied. Intuitively, this is due to the fact that $B \sqsubseteq_{\mathcal{T}} \bigcirc^n C$ and $C \sqsubseteq_{\mathcal{T}} \bigcirc^k B'$ together imply $B \sqsubseteq_{\mathcal{T}} \bigcirc^{n+k} B'$, for all $n, k \in \mathbb{Z}$, but the operation of addition is not expressible in MSO$(\mathbb{Z}, <)$ or Datalog$_{1S}$. It will then follow from our undecidability results that this lack of expressive power cannot be circumvented. We next show that systems of equations over sets of integers can be used for encoding the properties of canonical models provided that the $+$ operation on sets is available.

3.2. From TBoxes to Equations over Sets of Integers: There…

In this section, we reduce our $\mathcal{TEL}^{\bigcirc}$ reasoning tasks to solving systems of equations over sets of integers in *resolved* form

$$\begin{cases} X_1 = \Phi_1(X_1, \dots, X_p), \\ \qquad \dots \\ X_p = \Phi_p(X_1, \dots, X_p), \end{cases}$$

with $\Phi_1, \dots, \Phi_p$ constructed from $X_1, \dots, X_p$ and constants—finite subsets of $\mathbb{Z}$—using operations of union $\cup$, intersection $\cap$ and *set addition* $+$, which is defined by taking

$$X + X' = \{\, n + n' \mid n \in X,\ n' \in X' \,\}.$$

As with KBs, the *size* of a system is the number of symbols required to write it. A resolved system of equations $\overline{X} = \Phi(\overline{X})$, with $\overline{X} = (X_1, \dots, X_p)$, always has the *least solution*, i.e., a solution $\overline{X}^*$ that is minimal with respect to the component-wise set inclusion. Moreover, this solution can be obtained as the component-wise intersection of all solutions of the system of set inclusions $\overline{X} \supseteq \Phi(\overline{X})$; see, e.g., Tarski [19, Theorem 1]. Crucially for us, such systems define (say, as the first components of their least solutions) exactly the recursively enumerable sets [11, Theorem 5].

Our first result shows that one can translate KBs into resolved systems of equations:

Theorem 3. *Let $\mathcal{T}$ be a $\mathcal{TEL}^{\bigcirc}$ TBox in normal form and $\mathcal{A}$ an ABox. One can construct, in polynomial time (under the same encoding of numbers), a resolved system of equations $\overline{Z} = \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z})$ over variables $Z_{d,\,A}$, for $d \in \mathsf{N_I}(\mathcal{A}) \cup \mathsf{N_C}(\mathcal{T})$ and $A \in \mathsf{N_C}(\mathcal{T})$, whose least solution $\overline{Z}^*$ is given by*

$$\begin{aligned} Z^*_{a,\,A} &= \{n \in \mathbb{Z} \mid (\mathcal{T}, \mathcal{A}) \models A(a,n)\}, && \textit{for all } a \in \mathsf{N_I}(\mathcal{A}) \textit{ and } A \in \mathsf{N_C}(\mathcal{T}), \\ Z^*_{B,\,A} &= \{n \in \mathbb{Z} \mid B \sqsubseteq_{\mathcal{T}} \bigcirc^n A\}, && \textit{for all } A, B \in \mathsf{N_C}(\mathcal{T}). \end{aligned} \tag{1}$$

In the rest of the section, we prove Theorem 3. Fix a $\mathcal{TEL}^{\bigcirc}$ TBox $\mathcal{T}$ in normal form and an ABox $\mathcal{A}$. In addition to variables $Z_{d,A}$, we shall use the following constant sets to account for the differences between local and rigid roles:

$$I_r = \begin{cases} \mathbb{Z}, & \text{if } r \in \mathsf{N}_\mathsf{R}^{\mathsf{rig}}, \\ \{0\}, & \text{otherwise.} \end{cases}$$

The TBox CIs and ABox assertions translate into the following set inclusions, for all $d \in \mathsf{N}_\mathsf{I}(\mathcal{A}) \cup \mathsf{N}_\mathsf{C}(\mathcal{T})$:

(e1) $\{n\} \subseteq Z_{a,A}$, for all $A(a,n) \in \mathcal{A}$ with $A \in \mathsf{N}_\mathsf{C}(\mathcal{T})$;

(e2) $Z_{d,\,B} \cap Z_{d,\,B'} \subseteq Z_{d,\,A}$, for all $B \sqcap B' \sqsubseteq A \in \mathcal{T}$;

(e3) $Z_{d,\,B} + \{n\} \subseteq Z_{d,\,A}$, for all $B \sqsubseteq \bigcirc^n A \in \mathcal{T}$;

(e4) $Z_{b,\,B} \cap (I_r + \{n\}) \subseteq Z_{a,\,A}$, for all $r(a,b,n) \in \mathcal{A}$ and $\exists r.B \sqsubseteq A \in \mathcal{T}$;

(e5) $\{0\} \subseteq Z_{A,\,A}$, for all $A \in \mathsf{N}_\mathsf{C}(\mathcal{T})$;

(e6) $Z_{d,\,A} + (Z_{B,\,B'} \cap I_r) \subseteq Z_{d,\,A'}$, for all $A \sqsubseteq \exists r.B,\ \exists r.B' \sqsubseteq A' \in \mathcal{T}$.

Note that for $r \in \mathsf{N}_\mathsf{R}^{\mathsf{rig}}$, the intersections $Z_{b,B} \cap (\mathbb{Z} + \{n\})$ in **(e4)** and $Z_{B,B'} \cap \mathbb{Z}$ in **(e6)** can simply be replaced with $Z_{b,B}$ and $Z_{B,B'}$, respectively, so we effectively use only finite subsets of $\mathbb{Z}$ as constants in set inclusions **(e1)**–**(e6)**.

It can be seen that **(e2)** and **(e3)** are obvious counterparts of **(m2)** and **(m3)**, respectively. Set inclusions **(e1)** and **(e4)** correspond to the requirement that the construction of the canonical model starts from the ABox $\mathcal{A}$ and **(m1)** to account for rigid roles in the ABox. To understand the idea behind **(e5)** and **(e6)**, let us return to Example 1 (see also Figure 1). Then, by **(e3)**, **(e5)** and **(e6)**, we have

$$Z_{d,\,\mathsf{Fresher}} + \{2\} \subseteq Z_{d,\,\mathsf{InspiredWriter}}, \qquad Z_{\mathsf{Story},\,\mathsf{Story}} + \{3\} \subseteq Z_{\mathsf{Story},\,\mathsf{Novel}}, \qquad \{0\} \subseteq Z_{\mathsf{Story},\,\mathsf{Story}},$$
$$Z_{d,\,\mathsf{InspiredWriter}} + Z_{\mathsf{Story},\,\mathsf{Novel}} \subseteq Z_{d,\,\mathsf{Inspired}}.$$

where $\mathsf{InspiredWriter}$ is a 'shortcut' for $\mathsf{Inspired} \sqcap \mathsf{Writer}$ introduced due to normalisation. The last inclusion combines $\mathsf{InspiredWriter} \sqsubseteq \exists\mathsf{authorOf}.\mathsf{Story}$ and $\exists\mathsf{authorOf}.\mathsf{Novel} \sqsubseteq \mathsf{Inspired}$, or rules **(m5)** and **(m4)**, to allow us to entail $\mathsf{Fresher} \sqsubseteq \bigcirc^{2+3}\mathsf{Inspired}$ from $\mathsf{Fresher} \sqsubseteq \bigcirc^{2}\mathsf{InspiredWriter}$ and $\mathsf{Story} \sqsubseteq \bigcirc^{3}\mathsf{Novel}$. Note that **(e6)** also deals with rigid roles, thus covering **(m1)** outside the ABox. We adopt this specific form of **(e6)** because not only does it simplify the proofs, but also highlights the specific case of 'reasoning through a null', where the operation of addition is essential.

Observe now that the right-hand sides of set inclusions **(e1)**–**(e6)** feature no operations, so we can collect all set inclusions of the form $\Psi_1 \subseteq Z, \dots, \Psi_k \subseteq Z$ with the same right-hand side, Z, into a single set inclusion $\Psi_1 \cup \dots \cup \Psi_k \subseteq Z$. Since we are interested only in the least solution, we can replace this set inclusion with the equation $Z = \Psi_1 \cup \dots \cup \Psi_k$ [19, Theorem 1]. The resulting resolved system of equations is denoted by $\overline{Z} = \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z})$.

Lemma 4. *The least solution $\overline{Z}^*$ of $\overline{Z} = \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z})$ satisfies* (1).

Proof (Sketch). We establish the following statements, for all $n \in \mathbb{Z}$:

$$\begin{array}{llll} n \in Z^*_{a,\,A} & \text{iff} & (\mathcal{T},\mathcal{A}) \models A(a,n), & \text{for all } a \in \mathsf{N}_\mathsf{I}(\mathcal{A}) \text{ and } A \in \mathsf{N}_\mathsf{C}(\mathcal{T}), \\ n \in Z^*_{B,\,A} & \text{iff} & B \sqsubseteq_{\mathcal{T}} \bigcirc^n A, & \text{for all } A, B \in \mathsf{N}_\mathsf{C}(\mathcal{T}). \end{array}$$

($\Rightarrow$) For any model $\mathfrak{I}$ of $(\mathcal{T},\mathcal{A})$, consider the following collection $\overline{Z}^{\mathfrak{I}}$ of sets, for $a \in \mathsf{N}_\mathsf{I}(\mathcal{A})$ and $A, B \in \mathsf{N}_\mathsf{C}(\mathcal{T})$:

$$Z^{\mathfrak{I}}_{a,\,A} = \big\{ n \in \mathbb{Z} \mid \mathfrak{I} \models A(a,n) \big\} \quad \text{and} \quad Z^{\mathfrak{I}}_{B,\,A} = \big\{ n \in \mathbb{Z} \mid \mathfrak{I} \models B \sqsubseteq \bigcirc^n A \big\}.$$

We can show that they satisfy set inclusions **(e1)–(e6)**, that is, they satisfy $\overline{Z}^{\mathfrak{J}} \supseteq \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z}^{\mathfrak{J}})$, and so we have $\overline{Z}^{\mathfrak{J}} \supseteq \overline{Z}^*$. Now, if $n \in Z^*_{a,\,A}$, then $n \in Z^{\mathfrak{J}}_{a,\,A}$, and so, by definition, $\mathfrak{J} \models A(a,n)$, for any model $\mathcal{J}$ of $(\mathcal{T},\mathcal{A})$. Thus, $(\mathcal{T},\mathcal{A}) \models A(a,n)$. The argument for $n \in Z^*_{B,\,A}$ is similar.
($\Leftarrow$) Given the least solution $\overline{Z}^*$ of $\overline{Z} = \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z})$, we construct a model $\mathfrak{I}^* = (\Delta^{\mathfrak{I}^*}, (\mathcal{I}^*_n)_{n\in\mathbb{Z}})$ of $(\mathcal{T},\mathcal{A})$ such that $\mathfrak{I}^* \not\models A(a,n)$ if $n \notin Z^*_{a,\,A}$ and $\mathfrak{I}^* \not\models B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$ if $n \notin Z^*_{B,\,A}$. We take

$$\Delta^{\mathfrak{I}^*} = \mathsf{N_I}(\mathcal{A}) \cup \{\nu^k_B \mid B \in \mathsf{N_C}(\mathcal{T}), k \in \mathbb{Z}\}$$

and interpret concept names in $\mathsf{N_C}(\mathcal{T})$ by reading them off $\overline{Z}^*$, bearing in mind that each ν^k_B represents all the nulls witnessing $C \sqsubseteq \exists r.B \in \mathcal{T}$ at moment k, which means that its trace coincides with the trace of ν^0_B shifted by k steps, and the trace of ν^0_B is given by sets $Z_{B,A}$; concept names outside $\mathsf{N_C}(\mathcal{T})$ are interpreted as in the ABox, while all roles are defined according to the ABox and the 'null generation' relation (taking account of rigidity). It can be shown that $\mathfrak{I}^* \models (\mathcal{T},\mathcal{A})$. ❑

3.3. …and Back Again: From Equations over Sets of Integers to TBoxes

We next provide a converse translation to that in Section 3.2, i.e., from resolved systems of equations to $\mathcal{TEL}^\bigcirc$ TBoxes, demonstrating that $\mathcal{TEL}^\bigcirc$ can encode any resolved system of equations over sets of integers with union, intersection, addition and finite constants, and thus inherits the ability to define any recursively enumerable set [11, Theorem 5].

A resolved system of equations $\overline{X} = \Psi(\overline{X})$ with $\overline{X} = (X_1,\dots,X_p)$ is said to be in *normal form* if each of the equations takes one of the following forms:

$$X_i = \{n\}, \qquad X_i = X_j \cup X_k, \qquad X_i = X_j \cap X_k \qquad \text{or} \qquad X_i = X_j + X_k,$$

where $n \in \mathbb{Z}$. Any system can be transformed into normal form by exhaustively replacing complex subexpressions on the right-hand side of equations with fresh variables.

Theorem 5. *Let $\overline{X} = \Psi(\overline{X})$ be a resolved system of equations in normal form, for $\overline{X} = (X_1,\dots,X_p)$. One can construct, in polynomial time (under the same encoding of numbers), a $\mathcal{TEL}^\bigcirc$ TBox $\mathcal{T}_\Psi$ such that the least solution $\overline{X}^*$ of $\overline{X} = \Psi(\overline{X})$ is given by*

$$X^*_i = \{n \in \mathbb{Z} \mid S \sqsubseteq_{\mathcal{T}_\Psi} \bigcirc^n X_i\}, \quad \textit{for each } 1 \le i \le p. \tag{2}$$

In the rest of the section, we prove Theorem 5. Let TBox $\mathcal{T}_\Psi$ use concept names $S, X_1,\dots,X_p$ and rigid role names $r_1,\dots,r_p$ (no local roles) and consist of the following CIs, for each set variable X_i:

$S \sqsubseteq \bigcirc^n X_i$,	if $X_i = \{n\}$ is in Ψ,
$X_j \sqsubseteq X_i$ and $X_k \sqsubseteq X_i$,	if $X_i = X_j \cup X_k$ is in Ψ,
$X_j \sqcap X_k \sqsubseteq X_i$,	if $X_i = X_j \cap X_k$ is in Ψ,
$X_j \sqsubseteq \exists r_j.S$ and $\exists r_j.X_k \sqsubseteq X_i$,	if $X_i = X_j + X_k$ is in Ψ.

The first three cases are straightforward. We illustrate the last case on the following example.

Example 3. Equations in Example 2 can be transformed into the following system $\overline{X} = \Psi_4(\overline{X})$ in normal form:

$$\begin{cases} X'_1 = \{1\}, & X''_1 = X_{1+3} \cap X_{2+2}, & X_1 = X'_1 \cup X''_1, \\ X'_2 = \{2\}, & X''_2 = X_{1+1} \cap X_{2+6}, & X_2 = X'_2 \cup X''_2, \\ X'_3 = \{3\}, & X''_3 = X_{1+2} \cap X_{6+6}, & X_3 = X'_3 \cup X''_3, \\ & X_6 = X_{1+2} \cap X_{3+3} \quad \text{and} & X_{i+j} = X_i + X_j, \quad \text{for } i,j \in \{1,2,3,6\}. \end{cases}$$

Note that i and j strictly speaking need to range over a 7-element subset of indices only, but the extra equations are harmless. TBox $\mathcal{T}_{\Psi_4}$ for Ψ_4 comprises the following CIs with rigid roles r_1, r_2, r_3, r_6:

$$S \sqsubseteq \bigcirc^1 X'_1, \qquad X'_1 \sqsubseteq X_1, \qquad X_{1+3} \sqcap X_{2+2} \sqsubseteq X''_1, \qquad X''_1 \sqsubseteq X_1, \qquad X_1 \sqsubseteq \exists r_1.S,$$

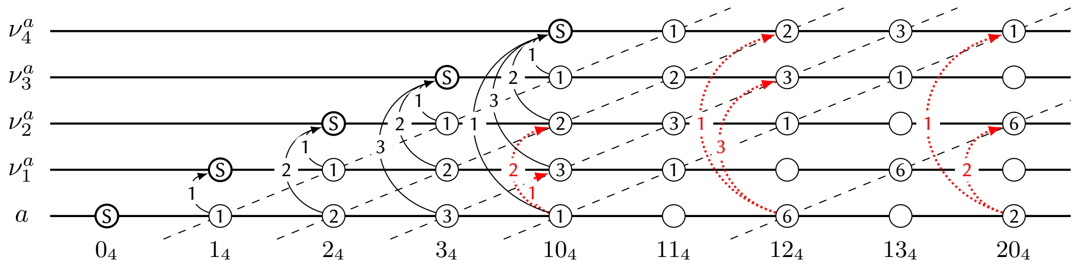

Figure 2: Simplified structure of the canonical model for $(\mathcal{T}_{\Psi_4}, \{S(a,0)\})$ in Example 3: the node content shows which X_i or S the element belongs to, the index j on the arrow specifies the role r_j, while each ν_j^d, for a fixed j, represents a set of identical labelled nulls, one for each solid arrow. The dotted arrows correspond to deriving new X_i using the triples of CIs with $\sqcap$ and appropriate $\exists r_j.X_k$.

$$
\begin{array}{lllll}
S \sqsubseteq \bigcirc^2 X_2', & X_2' \sqsubseteq X_2, & X_{1+1} \sqcap X_{2+6} \sqsubseteq X_2'', & X_2'' \sqsubseteq X_2, & X_2 \sqsubseteq \exists r_2.S,\\
S \sqsubseteq \bigcirc^3 X_3', & X_3' \sqsubseteq X_3, & X_{1+2} \sqcap X_{6+6} \sqsubseteq X_3'', & X_3'' \sqsubseteq X_3, & X_3 \sqsubseteq \exists r_3.S,\\
 & & X_{1+2} \sqcap X_{3+3} \sqsubseteq X_6, & & X_6 \sqsubseteq \exists r_6.S,
\end{array}
$$
$$\exists r_j.X_k \sqsubseteq X_{j+k}, \text{ for } j,k \in \{1,2,3,6\}.$$

Figure 2 gives a simplified depiction of the canonical model $\mathfrak{I}$ for $(\mathcal{T}_{\Psi_4}, \{S(a,0)\})$, where we trivially have $\mathfrak{I} \models X_1(a,1)$, $\mathfrak{I} \models X_2(a,2)$ and $\mathfrak{I} \models X_3(a,3)$ due to $\mathfrak{I} \models S(a,0)$ and the CIs in the first two columns. It then follows that ν_1^a, ν_2^a and ν_3^a are created to witness the CIs in the last column. Now, each of the labelled nulls ν_j^a, for $j \in \{1,2,3\}$, satisfies $S(\nu_j^a, j)$, which allows us to 'restart' counting up to 3 and obtain $X_k(\nu_j^a, j+k)$, for $j,k \in \{1,2,3\}$. Now, the CIs in the last line 'pull' back these shifted counters, and so we obtain, in particular, $X_{1+3}(a,4)$ and $X_{2+2}(a,4)$ (this step is shown by the leftmost pair of dotted arrows in Fig. 2); as a side observation, we could also obtain $X_{3+1}(a,4)$. It then follows that we have $X_1(a,4)$, which triggers the creation of ν_4^a as a witness for $\exists r_1.S$ on a, where we again 'restart' counting along the timeline. We then 'pull' back $X_2(\nu_4^a,6)$ and $X_3(\nu_3^a,6)$ to obtain $X_6(a,6)$. This, in turn, allows us to 'pull' back $X_1(\nu_4^a,8)$ and $X_6(\nu_2^a,8)$ to obtain $X_2(a,8)$; note, however, that both $X_1(\nu_4^a,8)$ and $X_6(\nu_2^a,8)$ require the creation of additional labelled nulls: ν_5^d and ν_6^d for the former and ν_7^d and ν_8^d for the latter, where $d = \nu_4^a$.

The claim of Theorem 5 is established by the following lemma:

Lemma 6. *The least solution $\overline{X}^*$ of $\overline{X} = \Psi(\overline{X})$ satisfies* (2).

Proof. Observe that TBox $\mathcal{T}_\Psi$ is in normal form. Take any ABox $\mathcal{A}$ and consider the resolved system of equations $\overline{Z} = \Phi_{\mathcal{T}_\Psi,\mathcal{A}}(\overline{Z})$ provided for $(\mathcal{T}_\Psi, \mathcal{A})$ by Theorem 3. We focus only on variables Z_{S,X_i}, $1 \leq i \leq p$, and observe that the equations in $\overline{Z} = \Phi_{\mathcal{T}_\Psi,\mathcal{A}}(\overline{Z})$ that contain the Z_{S,X_i} correspond to the equations in Ψ in the following way:

$$
\begin{array}{ll}
Z_{S,X_i} = Z_{S,S} + \{n\}, & \text{for } X_i = \{n\} \text{ in } \Psi,\\
Z_{S,X_i} = (Z_{S,X_j} + \{0\}) \cup (Z_{S,X_k} + \{0\}), & \text{for } X_i = X_j \cup X_k \text{ in } \Psi,\\
Z_{S,X_i} = Z_{S,X_j} \cap Z_{S,X_k}, & \text{for } X_i = X_j \cap X_k \text{ in } \Psi,\\
Z_{S,X_i} = Z_{S,X_j} + Z_{S,X_k}, & \text{for } X_i = X_j + X_k \text{ in } \Psi.
\end{array}
$$

Now, concept S occurs on the right-hand side of CIs in $\mathcal{T}_\Psi$ only in the form of $\exists r_j.S$. Therefore, the only set inclusion from **(e1)–(e6)** applicable to $Z_{S,S}$ is **(e5)**, Thus, the equation for $Z_{S,S}$ in $\overline{Z} = \Phi_{\mathcal{T}_\Psi,\mathcal{A}}(\overline{Z})$ is

$$Z_{S,S} = \{0\}.$$

It is now immediate to see that replacing each $Z_{S,S} + \{n\}$ with $\{n\}$ and each $Z_{S,X_i} + \{0\}$ with Z_{S,X_i} results in an equivalent resolved *sub-system* of equations. It then should be clear that the equation for Z_{S,X_i} in that resulting sub-system coincide with $X_i = \Psi_i(\overline{X})$. The claim is then due to Theorem 3.

❏

Table 1
Complexity of reasoning in $\mathcal{TEL}^{\bigcirc}$ and its fragments. Circuit classes AC ^{0} and ACC ^{0} are DLogTime-uniform.

TBox	problem	unary encoding		binary encoding	
		combined	data	combined	data
$\mathcal{TEL}^{\bigcirc}$	Instance Checking	r.e.-complete			
	Temporal Subsumption				
$\mathcal{TEL}^{\bigcirc}_{F}$	Instance Checking	PTime-complete		ExpTime-complete	
	Temporal Subsumption	PTime-complete	in PTime		
$\mathcal{TEL}^{\bigcirc}_{lin}$	Instance Checking	in NP & PTime-hard	NL-complete	NP-complete	in NP & NL-hard
	Temporal Subsumption		in AC ^{0}		in ACC ^{0} $\setminus$ AC ^{0}

4. Instance Checking and Temporal Subsumption

We consider TBoxes in $\mathcal{TEL}^{\bigcirc}$, $\mathcal{TEL}^{\bigcirc}_{F}$ and $\mathcal{TEL}^{\bigcirc}_{lin}$, together with their corresponding systems of equations, and establish the complexity results summarised in Table 1. Among these, all the bounds under the binary encoding are novel. The bounds on the combined complexity for $\mathcal{TEL}^{\bigcirc}_{lin}$ under the unary encoding are also new, but NL-completeness for data complexity is from Bourgaux et al. [6, Theorem 28]. The remaining results were previously shown by Bourgaux et al. [6]; here, we provide shorter proofs via the correspondence with systems of equations.

For general $\mathcal{TEL}^{\bigcirc}$ TBoxes, recursive enumerability of instance checking and temporal subsumption is straightforward from Theorem 1. For r.e.-hardness, take a resolved system of equations $\overline{X} = \Psi(\overline{X})$ with the least solution $\overline{X}^*$ such that X_1^* is an r.e.-complete set [11, Theorem 5]. By Theorem 5, $n \in X_1^*$ iff $S \sqsubseteq_{\mathcal{T}_\Psi} \bigcirc^n X_1$. Thus, temporal subsumption in $\mathcal{TEL}^{\bigcirc}$ is r.e.-complete for both data and combined complexity. By Theorem 1, temporal subsumption is reducible to instance checking: $B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$ iff $(\mathcal{T}, \{B(a,0)\}) \models A(a,n)$.

4.1. Future-only TBoxes

Knowledge bases with $\mathcal{TEL}^{\bigcirc}_{F}$ TBoxes and ABoxes such that $n \geq 0$, for all $A(a,n) \in \mathcal{A}$ and $r(a,b,n) \in \mathcal{A}$, naturally correspond to systems of equations over sets of *natural numbers*. This connection gives tight complexity bounds for all of our reasoning tasks, save the data complexity of temporal subsumption under the unary encoding of integers, which is thus linked to an open problem for formal languages.

For sets of natural numbers, the definition of systems of equations is the same as for integers apart from that constants are finite subsets of $\mathbb{N}$, not $\mathbb{Z}$. Under the unary encoding, such systems inherit the results on *unary conjunctive grammars*, i.e., formal grammars over the one-letter alphabet $\{c\}$, where a word c^n represents the number $n \in \mathbb{N}$, addition corresponds to string concatenation, union to disjunction and intersection to conjunction [20, 8]. Thus, given a system $\overline{X} = \Phi(\overline{X})$, deciding whether $n \in X_1^*$ is in PTime both when the system is fixed [7] and when given as a part of input [21]; in the latter case, it is actually PTime-complete, as follows from the result of Jones and Laaser [22, Corollary 11] on unary context-free grammars. Under the binary encoding, a trivial exponential-time upper bound is obtained by writing all numbers in unary; the matching lower bound follows from the existence of a system defining an ExpTime-complete set [20, Theorem 3.1].

We now establish the bounds in the second row of Table 1. Let $\mathcal{T}$ be a $\mathcal{TEL}^{\bigcirc}_{F}$ TBox and $\mathcal{A}$ an ABox. Let $n_0 = \min\{n \mid A(a,n) \in \mathcal{A} \text{ or } r(a,b,n) \in \mathcal{A}\}$. One can construct, in polynomial time, an ABox $\mathcal{A}' = \{A(a, n - n_0) \mid A(a,n) \in \mathcal{A}\} \cup \{r(a,b,n-n_0) \mid r(a,b,n) \in \mathcal{A}\}$ such that $(\mathcal{T}, \mathcal{A}) \models A(a,n)$ iff $(\mathcal{T}, \mathcal{A}') \models A(a, n - n_0)$, for all $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$ and $n \in \mathbb{Z}$. Thus, we can assume that $n \geq 0$ in all assertions in $\mathcal{A}$, and so all constants in the system $\overline{Z} = \Phi_{\mathcal{T},\mathcal{A}}(\overline{Z})$ in Theorem 3 are subsets of $\mathbb{N}$. Then all upper bounds follow from the aforementioned results on equations and grammars [21, 7, 20].

For the lower bounds, observe that when $\overline{X} = \Psi(\overline{X})$ is given over sets of natural numbers, $\mathcal{T}_\Psi$ in Theorem 5 is a $\mathcal{TEL}^{\bigcirc}_{F}$ TBox. For ExpTime-hardness under the binary encoding (for both problems and both complexity measures), take $\overline{X} = \Psi(\overline{X})$ to be such that X_1^* is complete for ExpTime [20, Theorem 3.1]. Similarly, the result of Jones and Laaser [22, Corollary 11] translates to PTime-hardness for combined complexity under the unary encoding. Finally, the lower bound for the data complexity of instance checking holds already for the description logic $\mathcal{EL}$, without temporal operators [23, Theorem 4.3].

Note that we have no non-trivial lower bound for the data complexity of temporal subsumption under the unary encoding, as no such bound is known for the systems of equations we rely on. It can only be said that PTime-completeness of this problem is unlikely as there is no PTime-complete unary language unless LogSpace = PTime [24]. For instance checking, we rely on a purely DL result.

Interestingly, the same approach applies to combined complexity: one can obtain PTime-hardness from $\mathcal{EL}$, then establish a polynomial-time reduction of instance checking to temporal subsumption by encoding ABox assertions as TBox CIs, following Bourgaux et al. [6, Theorem 27]. Using the same reduction and the results on unary conjunctive grammars, they also derive the upper bounds under the unary encoding. Moreover, the upper bounds under both unary and binary encodings can also be obtained from the results of Gutiérrez-Basulto et al. [5] on so-called temporally acyclic TBoxes (see the remarks after Theorem 27 in Bourgaux et al. [6]). The lower bound under the binary encoding is new.

We also point out that, since systems of equations over sets of natural numbers can be seen as unary conjunctive grammars, the efficient parsing algorithm of Okhotin and Reitwießner [25] and the parser generator of Okhotin [26] can be employed as tools for instance checking in $\mathcal{TEL}_F^{\bigcirc}$.

4.2. Linear TBoxes

Knowledge bases with $\mathcal{TEL}_{lin}^{\bigcirc}$ TBoxes correspond to resolved systems with union and addition only, although some extra work is required to show this. For $\mathcal{TEL}_{lin}^{\bigcirc}$, the system supplied by Theorem 3 may have intersections of the form $X \cap C$, with C a finite constant set, which are due to local role names in the KB and have to be dealt with.

First, we consider systems with union and addition only. Interpreted over sets of natural numbers (under the unary encoding), such systems are just a syntactic variant of one-letter context-free grammars. The case of integers, and under the binary encoding, is more involved. Recall that for a given ordered alphabet Σ, the *Parikh image* $P\colon \Sigma^* \to \mathbb{N}^{|\Sigma|}$ counts how many times each letter appears in a word; for a formal grammar G, its Parikh image is defined as $P(G) = \{P(w) \mid w \text{ generated by } G\}$ [9]. Every context-free grammar G can be translated in polynomial time to an existential Presburger formula $\varphi_G(x_1, \dots, x_{|\Sigma|})$ defining $P(G)$ [27, Theorem 4]. We remind the reader that existential Presburger formulas are of the form $\exists \overline{y}\, \varphi(\overline{x}, \overline{y})$ with a quantifier-free φ over signature $(\leq, +, 0, 1)$. Validity of existential Presburger sentences can be decided in NP [28, Theorem 6, Corollary 1]. Since every integer can be represented as the difference of two natural numbers, and every natural number is a non-negative integer, the choice of interpreting these formulas over $\mathbb{Z}$ or $\mathbb{N}$ has no effect on complexity.

Lemma 7. *Given a resolved system $\overline{X} = \Phi(\overline{X})$ with union and addition over sets of integers, one can construct, in polynomial time (under the same encoding of numbers), an existential Presburger formula $\psi_\Phi(x)$ that defines the first component of the least solution of $\overline{X} = \Phi(\overline{X})$.*

Proof (Sketch). We assume that $\overline{X} = \Phi(\overline{X})$, for $\overline{X} = (X_1, \dots, X_p)$, is in normal form. Let $\{n_1\}, \dots, \{n_m\} \subseteq \mathbb{Z}$ be all the constant sets in Φ. We construct a context-free grammar G_Φ with nonterminals $X_1, \dots, X_p$, terminals $c_1, \dots, c_m$, start symbol X_1 and the following productions, for each equation $X_i = \Phi_i(\overline{X})$:

$$\begin{aligned}
&X_i \to c_j, && \text{if } X_i = \{n_j\}, \text{ with } 1 \leq j \leq m,\\
&X_i \to X_j \text{ and } X_i \to X_k, && \text{if } X_i = X_j \cup X_k,\\
&X_i \to X_j X_k, && \text{if } X_i = X_j + X_k.
\end{aligned}$$

One can show that $n \in X_1^*$ iff there is $(q_1, \dots, q_m) \in P(G)$ such that $n = \sum_{j=1}^m n_j \cdot q_j$. Following Verma et al. [27, Theorem 4], we construct an existential Presburger formula $\varphi_G(x_1, \dots, x_m)$ for $P(G)$; we then set $\psi_\Phi(x) = \exists x_1 \dots \exists x_m \left[\varphi_G(x_1, \dots, x_m) \wedge (x = \sum_{j=1}^m n_j \cdot x_j)\right]$. Note that each $n_j \cdot x_j$ is expressible in Presburger arithmetic as a sum of n_j copies of x_j if $n_j \geq 0$, or of $-n_j$ copies of $-x_j$, otherwise. To prevent an exponential blow-up for the n_j encoded in binary, we represent each number as the sum $\sum_{k=0}^{s} b_k 2^k$, where $n_j = (b_s \dots b_0)_2$, and encode the exponents 2^k with the help of fresh existentially quantified variables $y_s, \dots, y_0$ satisfying additional conjuncts $y_0 = 1$ and $y_{k+1} = y_k + y_k$, for each $0 \leq k < s$. ❑

With this at hand, we proceed to study the systems that arise from $\mathcal{TEL}^{\bigcirc}_{lin}$ KBs.

Lemma 8. *Given a system $\overline{X} = \Phi(\overline{X})$ over sets of integers using union, addition and intersection with finite constants, and an integer $n \in \mathbb{Z}$, deciding whether $n \in X_1^*$ belongs to* NP. *Moreover, $\overline{X} = \Phi(\overline{X})$ can be effectively transformed into a resolved system using union and addition only, while preserving the least solution; this transformation is feasible in polynomial time with access to an* NP *oracle.*

Proof (Sketch). For systems without intersection, the first claim follows from Lemma 7 and the result of Scarpellini [28, Theorem 6, Corollary 1], while the second is trivial. For the general case, consider a sequence of systems $\overline{X} = \Phi^{(k)}(\overline{X})$ and their least solutions $\overline{X}^{(k)}$, $k \geq 0$, with $\Phi^{(0)}$ obtained from Φ by replacing every equation of the form $X_i = X_j \cap C$ with $X_i = \emptyset$, and $\Phi^{(k+1)}$ by replacing the equation with $X_i = X_j^{(k)} \cap C$ (the intersection is constant). These systems use union and addition only. One can show by induction, using monotonicity of these operations and minimality of solutions, that $\overline{X}^{(k)} \subseteq \overline{X}^{(k+1)} \subseteq \overline{X}^*$, for all $k \geq 0$. Now, if $X_i = X_j \cap C$ is in Φ, then $X_i^* \subseteq C$, and so $\overline{X}^{(m)} = \overline{X}^{(m+1)}$, where m is the sum of the cardinalities of constants C in Φ. By the definition of the least solutions, $\overline{X}^{(m)} = \overline{X}^*$. One can compute each $\Phi^{(k+1)}$ from $\Phi^{(k)}$ by asking the oracle if $c \in X_j^{(k)}$ for every $c \in C$ and $X_i = X_j \cap C$ in Φ. Membership $n \in X_1^*$ is decidable in NP by guessing a complete sequence $\overline{X} = \Phi^{(k)}(\overline{X})$, for $0 \leq k \leq m$, and then using the oracle to confirm that each relevant c is in $X_j^{(k)}$ and to check that the given n is in $X_1^{(m)}$ (any negative oracle's response leads to rejecting the guess). ❑

It follows from Lemmas 7 and 8 and the results of Ginsburg and Spanier [29, Theorem 1.3] that components of the least solutions of resolved systems with union, addition and intersection with finite constants are precisely the semilinear sets, that is, finite unions of sets of the form $\{b_i + kp_i \mid k \in \mathbb{N}\}$. We are now fully equipped to justify the last row in Table 1. All NP upper bounds follow from Theorem 3 and Lemma 8. The lower bounds for combined complexity follow from those for the membership in Parikh images of unary context-free languages [22, Corollary 11; 30, Proposition 5.9]. For the data complexity of instance checking, the lower bounds are trivial from directed graph reachability; the matching upper bound, under the unary encoding, follows by reduction to linear temporal datalog [6, Theorem 28], which relies on the general approach to ultimate periodicity in canonical models [5, Theorem 5].

It remains to consider the data complexity of temporal subsumption. Deciding if $B \sqsubseteq_{\mathcal{T}} \bigcirc^n A$ boils down to checking whether n belongs to one of $\{b_i + kp_i \mid k \in \mathbb{N}\}$. If n is given in unary, then this can trivially be decided in DLogTime-uniform AC^0. If n is given in binary, then the problem is known to be in DLogTime-uniform ACC^0 for natural numbers [31]; it should be clear that the same bound applies to integers. We now reduce the problem MOD_3 of deciding if 3 divides the number of non-zero bits in the input binary sequence, which is known to be outside AC^0 [32, Remark 2; 33, Theorem 2], to temporal subsumption. We use the following folklore idea. Take any $n \in \mathbb{N}$ and let $(b_s \dots b_0)_2$ be its base-2 representation. Consider now $n' = (b_s \dots b_0)_4$, that is, the number with the same digits as n but in base-4 representation. Observe that $4^k \equiv 1 \pmod 3$ for all $k \in \mathbb{N}$. Thus, $n' = \sum_{k=0}^{s} b_k 4^k \equiv \sum_{k=0}^{s} b_k \pmod 3$. It then follows that $n \in \mathrm{MOD}_3$ iff $\sum_{k=0}^{s} b_k \equiv 0 \pmod 3$ iff $n' \equiv 0 \pmod 3$ iff $B \sqsubseteq_{\mathcal{T}_3} \bigcirc^{n'} B$, where $\mathcal{T}_3 = \{B \sqsubseteq \bigcirc^3 B\}$. To complete the reduction, it remains to convert the representation of n' from base-4 to base-2, which is feasible by a constant-depth polynomial-size circuit (effectively replacing each single digit b_k with two digits $0b_k$).

5. Discussion

We connect reasoning in $\mathcal{TEL}^{\bigcirc}$ to finding least solutions of resolved systems of equations over sets of integers. Besides extending the known results from the unary to the binary number encoding, we settle the question of decidability of the reasoning problems in $\mathcal{TEL}^{\bigcirc}_{lin}$ for combined complexity left open by Bourgaux et al. [6]. The key ingredient is Lemma 8, which allows us to eliminate local roles from TBoxes without using conjunction. Although this result might also be proved using formal grammar methods, our proof grew out of the connection to equations over sets of integers. Some complexity bounds for $\mathcal{TEL}^{\bigcirc}_{lin}$ are still not tight, highlighting the subtlety of this seemingly simple logic.

Equations over sets of integers are shown to provide a powerful tool for studying the expressive power of temporal description logics. The correspondence given by Theorems 3 and 5 could likely be extended to other TDLs and to other classes of equations, for example those involving set complement. Beyond point-based time, growing interest in interval-based temporal logics may also motivate the study of equations over sets of rational numbers.

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Declaration on Generative AI

During the preparation of this work, the authors did not employ any Generative AI tools.

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