Coherence Update Semantics for Horn DL-Lite through Stratified Datalog$^\neg$ Rewriting

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Abstract

We investigate a formula-based approach for updating ABoxes in $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{horn}}$, a description logic that enjoys a low computational complexity of reasoning. Specifically, we extend the existing coherence update semantics from $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{core}}$ to deal with conjunctions on the left-hand side of TBox axioms, which introduce ambiguity in the update result. We address this problem by proposing the new prioritized coherence update semantics that employs rankings over conjuncts to select among competing updates while preserving the central properties of minimal change, syntax independence, and uniqueness. We further show that updated ABoxes can be computed via a stratified Datalog$^\neg$ program based on a novel consequence-based calculus. The approach runs in polynomial time, matching the complexity of standard reasoning in $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{horn}}$.

Keywords

DL-Lite, Instance-level Update, Coherence Update Semantics

1. Introduction

The problem of updating description logic ABoxes has been studied in many contexts, such as belief revision [1, 2, 3], ontology repair and inconsistency-tolerant semantics [4, 5, 6, 7, 8], ontology evolution [9, 10, 11, 12, 5, 13, 14], and action formalisms and planning [15, 16, 17, 18, 19]. The main task is to repair/update the ABox while preserving consistency w.r.t. a fixed TBox. That is, given an ABox $\mathcal{A}$ that describes the current state of the world and an update/action that causes new facts about the world to become true and others to be invalidated, we should compute an updated ABox $\mathcal{A}'$ that reflects these changes, but does not contradict the background knowledge in the TBox $\mathcal{T}$. Different update semantics have been proposed, and can be categorized roughly into *model-based* [10, 12, 13] and *formula-based* approaches [9, 11, 5, 14]. In the former, updates operate on the semantic level by considering all possible models of the current ontology $\langle \mathcal{T}, \mathcal{A} \rangle$, updating these models according to the new information, and then trying to describe these models by a new ABox $\mathcal{A}'$. However, model-based updates often do not exist (without changing the ontology language), and are costly in terms of computational complexity.

In contrast, in formula-based approaches, updates operate directly on the syntactic level of the ABox assertions, without considering all possible models explicitly. While such semantics have their own drawbacks, their main advantage is the favourable computational behaviour. In [14], the formula-based *coherence update semantics* for $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{core}}$ was investigated,[1] which is *syntax-independent*: For any two ABoxes $\mathcal{A}_1, \mathcal{A}_2$ that are equivalent w.r.t. $\mathcal{T}$ (i.e., they entail the same assertions), the updated ABoxes $\mathcal{A}'_1, \mathcal{A}'_2$ are also equivalent w.r.t. $\mathcal{T}$. This also ensures that the updated ABox is *unique* up to equivalence, which is a desirable property, e.g. for automated planning [16, 18, 19].

In this paper, we extend the coherence update semantics [14] to $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{horn}}$ (albeit without role disjointness axioms) while preserving its good properties. In contrast to $\textit{DL-Lite}^{(\mathcal{HF})}_{\textit{core}}$, the presence of conjunctions in concept inclusions can cause the existence of multiple possible updated ABoxes. For example, consider the ABox $\mathcal{A} = \{A(a),\ B(a)\}$ and the TBox $\mathcal{T} = \{A \sqcap B \sqsubseteq C\}$. If the goal is to make $C(a)$ false by updating the ABox, there are three possibilities: remove $A(a)$ or $B(a)$ from $\mathcal{A}$, or remove

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[1]Actually $\textit{DL-Lite}_{\mathcal{A}}$, but we do not consider attributes in this paper.

both. The last option violates the principle of *minimal change* that is often adopted [5, 1, 14]. To resolve the choice between the remaining incomparable updated ABoxes $\mathcal{A}' = \{B(a)\}$ and $\mathcal{A}'' = \{A(a)\}$, we introduce a ranking over the conjuncts on the left-hand side of a concept inclusion. For example, for the inclusion $A \sqcap B \sqsubseteq C$, we can declare A to be more important than B, and thus prefer $\mathcal{A}'' = \{A(a)\}$. This allows us to extend the original coherence update semantics [14] without giving up the properties of syntax independence, uniqueness, or minimal change.

While related to semantics of inconsistency-tolerant reasoning, our assumption here is that the original ontology is consistent. Moreover, the focus of update semantics is on computing a single updated ABox that preserves as much information of the original ontology as possible. Although the inconsistency-tolerant IAR/ICAR semantics reduce reasoning to a single ABox, by considering the intersection over all possible repairs, they lose a lot of information [4, 6]. Approaches for computing preferred repairs via weights on the assertions [7] are not feasible in our setting, since for frequent updates we would need an automated way to determine the weights of newly added assertions. Instead, since the TBox does not change, we choose to define rankings on the level of conjunctions, which allows us to compute the update semantics in a uniform way for all possible input ABoxes.

After introducing the new *prioritized coherence update semantics* for *DL-Lite*$_{horn}^{(\mathcal{HF})}$ without role disjointness axioms, we show that we can reduce the problem of computing the updated ABox to the evaluation of a stratified Datalog$^\neg$ program, similarly as in [14]. However, since the rankings are defined on the level of individual concept inclusions, we cannot simply pre-saturate the TBox as commonly done for Datalog rewritings [20, 14], but need to develop an encoding into stratified Datalog$^\neg$ that is more closely aligned with consequence-based approaches for computing TBox entailments [21]. Since no such consequence-based rules exist for *DL-Lite*$_{horn}^{(\mathcal{HF})}$, we start by developing our own calculus, and then use this as the basis to define the translation of our update semantics into stratified Datalog$^\neg$. In contrast to [14], our Datalog$^\neg$ program is recursive, and thus cannot be used directly to obtain an equivalent first-order rewriting. Nevertheless, it allows us to show that the updated ABox can always be computed in polynomial time, similar to standard reasoning in *DL-Lite*$_{horn}^{(\mathcal{HF})}$ [22].

2. Preliminaries

In this section, we introduce *DL-Lite*$_{horn}^{(\mathcal{HF})}$ ontologies and Datalog$^\neg$, as well as additional notations to improve the presentation and conciseness.

The Description Logic DL-Lite$_{\mathbf{horn}}^{(\mathcal{HF})}$. Given disjoint sets $\mathbf{C}, \mathbf{R}, \mathbf{I}$ of *atomic concepts* A, *atomic roles* P, and *individuals* (*constants/objects*) o, *(complex) concepts* and *roles* are formed as follows:

$$\textit{basic role: } R \longrightarrow P \mid P^- \qquad\qquad \textit{role: } S \longrightarrow R \mid \neg R$$
$$\textit{basic concept: } B \longrightarrow \bot \mid A \mid \exists R \qquad\qquad \textit{concept: } M \longrightarrow B \mid M \sqcap M$$

For the *inverse role* P^-, we set $(P^-)^- := P$.

An *ABox* is a finite set of *basic concept assertions* $B(o)$ and *basic role assertions* $R(o, o')$.[2] A *DL-Lite*$_{horn}^{(\mathcal{HF})}$ *TBox* is a finite set of *concept inclusions* (F1), *role inclusions* (F2), *functionality axioms* (F3), *(a)symmetry* axioms (F4), *(ir)reflexivity* axioms (F5), which are of the form

$$M \sqsubseteq B \ \text{(F1)} \qquad R \sqsubseteq S \ \text{(F2)} \qquad \text{funct } R \ \text{(F3)} \qquad \text{(a)sym } P \ \text{(F4)} \qquad \text{(ir)ref } P \ \text{(F5)}$$

where M, B, S, R are as above. For the sake of this paper, we consider only *DL-Lite*$_{horn}^{(\mathcal{HF})}$ axioms of the forms (F1), (F2), and (F3), where (F2) is further restricted to $R \sqsubseteq R$, i.e. we forbid role disjointness, (a)symmetry and (ir)reflexivity axioms. The main reason is that role disjointness axioms $R_1 \sqsubseteq \neg R_2$ can be entailed by concept inclusions like $\exists R_1 \sqcap \exists R_2 \sqsubseteq \bot$, which does not allow to clearly separate reasoning with concept inclusions and reasoning with role inclusions as in *DL-Lite*$_{core}^{(\mathcal{HF})}$. As usual [23, 22],

[2]The reason why we do not just consider atomic assertions will be explained in Section 4.

functional basic roles and their inverses are not allowed to occur on the right-hand side of any inclusions. This has the effect that no new assertions involving functional roles can be derived and lowers the computational complexity. An *ontology* (*knowledge base*) $\mathcal{K}$ is a tuple $\langle \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox. We assume the reader to be familiar with the semantics of *DL-Lite*$^{(\mathcal{HF})}_{horn}$ [22], defined via first-order *interpretations* $\mathcal{I}$ under the unique name assumption, and the entailment relation $\models$. Two ABoxes $\mathcal{A}_1, \mathcal{A}_2$ are *equivalent* w.r.t. a TBox $\mathcal{T}$ if $\langle \mathcal{T}, \mathcal{A}_1 \rangle$ and $\langle \mathcal{T}, \mathcal{A}_2 \rangle$ entail the same assertions.

A *positive concept* D is of the form $A \mid \exists R$ where A is an atomic concept and R is a basic role. Throughout this work, we use B, R, M, S, D, and o to denote basic concepts, basic roles, complex concepts, complex roles, positive concepts, and individuals, respectively. Extending upon the previous notion, a TBox axiom is called *positive* if it is of the form $M \sqsubseteq D$ or $R_1 \sqsubseteq R_2$, where R_1, R_2 are basic roles. Otherwise, it is called *negative*. For a TBox $\mathcal{T}$, we denote by $\mathcal{T}^+$ the set of all positive axioms from $\mathcal{T}$. We also use X to denote either a basic concept B or a basic role R, and $\vec{o}$ to denote a unary or binary vector of individuals, respectively. For an inclusion $\tau = \bigsqcap_{i=1}^{k} X_i \sqsubseteq X$, we define *left*$(\tau)(\vec{o}) := \{X_1(\vec{o}), \ldots, X_k(\vec{o})\}$.

Datalog$^\neg$. A *term* is either a *constant* or a *variable*. An atom has the form $p(\vec{x})$ where p is an n-ary *predicate* and $\vec{x} = (x_1, \ldots, x_n)$ is an n-ary tuple of terms. A fact is an atom without variables. A *Datalog$^\neg$ rule* [24] is a first-order rule of the form $p(\vec{x}) \leftarrow \Phi(\vec{x}, \vec{y})$, where the *head* $p(\vec{x})$ is an atom and the *body* $\Phi(\vec{x}, \vec{y})$ is a conjunction of atoms and *negated* atoms, such that every variable occurring in the head or in a negated body atom also occurs in a positive body atom. Moreover, we write Φ^+/Φ^- to denote the set of positive and negated atoms appearing in Φ, respectively. A finite set of Datalog$^\neg$ rules (*program*) $\mathcal{R}$ is *stratified* if there is a partition $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of the set of predicates $\mathcal{P}$ appearing in $\mathcal{R}$ such that, for each predicate $p_i \in \mathcal{P}_i$ and $p_i(\vec{x}) \leftarrow \Phi(\vec{x}, \vec{y}) \in \mathcal{R}$ ($i \in \{1, \ldots, n\}$), the following holds: (1) If $p_j \in \mathcal{P}_j$ appears in a positive atom in $\Phi(\vec{x}, \vec{y})$, then $j \leq i$; (2) If $p_j \in \mathcal{P}_j$ appears in a negated atom in $\Phi(\vec{x}, \vec{y})$, then $j < i$. Such a partition is called *stratification*.

For a term, atom, or rule α, a *substitution* σ of α is a partial mapping $\alpha\sigma$ from its variables to constants by replacing each occurrence of a variable x in α by $\sigma(x)$. If $\alpha\sigma$ contains no variable, then $\alpha\sigma$ is an *instance* of α. For a set of facts D (*dataset*) and a Datalog program $\mathcal{R}$, the *immediate consequence operator* $T_{\mathcal{R}}(D) = \{p(\vec{x})\sigma \mid p(\vec{x}) \leftarrow \Phi(\vec{x}, \vec{y}) \in \mathcal{R},\ \Phi^+\sigma \subseteq D,\ \Phi^-\sigma \cap D = \emptyset\}$ is the set of facts obtained by applying a rule in the program $\mathcal{R}$ to D. Given a stratification $\lambda = \mathcal{P}_1, \ldots, \mathcal{P}_n$, a Datalog$^\neg$ program $\mathcal{R}$, and a dataset D, we define for each $i \in \{1, \ldots, n\}$ the sequence of datasets: $D_0^\infty = D$, $D_i^1 = D_{i-1}^\infty$, $D_i^{j+1} = D_{i-1}^\infty \cup T_{\mathcal{R}_i}(D_i^j)$, and $D_i^\infty = \cup_{j \geq 1} D_i^j$, where $\mathcal{R}_i = \{p_i(\vec{x}) \leftarrow \Phi(\vec{x}, \vec{y}) \in \mathcal{R},\ p_i \in \mathcal{P}_i\}$. The set D_n^∞ is called the *materialisation* (*evaluation*) of $\mathcal{R}$ over D. For a fact $p(\vec{o})$, we say that $\mathcal{R}$ and D *derive* $p(\vec{o})$ iff $p(\vec{o}) \in D_n^\infty$ and write $\mathcal{R}, D \models p(\vec{o})$. Since the materialisation of $\mathcal{R}$ over D is finite and independent of λ [24], for simplicity we will write $T_{\mathcal{R}}^1(D), \ldots, T_{\mathcal{R}}^m(D)$ instead of $D_0^\infty, \ldots, D_1^\infty, \ldots, D_n^\infty$, pretending that there is a single consequence operator that applies the rules of $\mathcal{R}$ in order of stratification.

3. Consequence-Based Reasoning for DL-Lite$^{(\mathcal{HF})}_{\mathbf{horn}}$

As mentioned in the introduction, since our approach for the prioritised coherence update semantics requires a consequence-based calculus for *DL-Lite*$^{(\mathcal{HF})}_{horn}$ and none is available in the literature, we develop a new calculus for the restricted fragment we consider here.

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a potentially inconsistent *DL-Lite*$^{(\mathcal{HF})}_{horn}$ ontology. We define

$$\mathcal{T}' := \{\exists R \sqsubseteq \bot \mid \mathcal{T} \models \exists R \sqsubseteq \bot\} \cup \{\mathit{funct}\ R \mid \mathcal{T} \models \mathit{funct}\ R\}$$

as the set of all role unsatisfiability and functionality axioms that are entailed by $\mathcal{T}$. Since satisfiability for *DL-Lite*$^{(\mathcal{HF})}_{horn}$ is P-complete [22, Theorem 5.13], $\mathcal{T}'$ can be computed in polynomial time. From this section on, we assume that $\mathcal{T}$ already contains $\mathcal{T}'$.

The completion rules in Figure 1 derive new axioms from $\mathcal{K}$. We use M, N to denote (exponentially many) conjunctions of basic concepts formed from the atomic symbols in $\mathcal{K}$. For conciseness, we write $M \sqsubseteq N$ with $N = \bigsqcap_{i=1}^{k} B_i$ to denote $M \sqsubseteq B_1, \ldots, M \sqsubseteq B_k$.

$$\text{T1}\ \frac{}{X \sqsubseteq X} \qquad \text{T2}\ \frac{R_1 \sqsubseteq R_2 \quad R_2 \sqsubseteq R_3}{R_1 \sqsubseteq R_3} \qquad \text{T3}\ \frac{R_1 \sqsubseteq R_2}{R_1^- \sqsubseteq R_2^-}$$

$$\text{T4}\ \frac{R_1 \sqsubseteq R_2}{\exists R_1 \sqsubseteq \exists R_2} \qquad \text{T5}\ \frac{M \sqsubseteq N \quad N \sqsubseteq B}{M \sqsubseteq B} \qquad \text{T6}\ \frac{}{M \sqcap B \sqsubseteq B} \qquad \text{T7}\ \frac{}{\bot \sqsubseteq D}$$

$$\text{A1}\ \frac{R(o,o') \quad R(o,o'')}{\bot(o)} : \mathit{funct}\, R \in \mathcal{T},\ o'' \neq o' \qquad \text{A2}\ \frac{R_1(o,o')}{R_2(o,o')} : R_1 \sqsubseteq R_2 \in \mathcal{T}^* \qquad \text{A3}\ \frac{R(o,o')}{R^-(o',o)}$$

$$\text{A4}\ \frac{R(o,o')}{\exists R(o)} \qquad \text{A5}\ \frac{B_1(o)\ \ldots\ B_k(o)}{B(o)} : \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$$

Figure 1: Completion rules for *DL-Lite*$_{horn}^{(\mathcal{HF})}$

We categorise the completion rules into T-rules and A-rules, which infer new TBox and ABox axioms, and write $cl(\mathcal{T})$, $cl_{\mathcal{T}}(\mathcal{A})$ to denote the saturations of $\mathcal{T}$ and $\mathcal{A}$ under the T- and A-rules, respectively. Intuitively, rules T1, T6, and T7 derive tautological concept and role inclusions. Then, these tautologies and axioms in $\mathcal{T}$ are extended by Rules T2–T5 at the concept- and role level. Rules T2 and T5 capture the transitive closure for roles and conjunctions of basic concepts, respectively. Rules T3 and T4 deal with inverse roles and existential role restrictions.

Inconsistency of $\mathcal{K}$ can be captured by Rules A1 and A5, which check for violations against functionality constraints and unsatisfiability of conjunctions (when $B = \bot$), respectively. A2 and A5 propagate entailed ABox assertions along the inclusions entailed by the TBox. For this, it suffices to consider axioms in the saturation $\mathcal{T}^*$ of $\mathcal{T}$ under the rules T1, T3, T4, and T7, which we formally demonstrate in the lemma below, whose proof can be found in the appendix.

Let $\langle cl(\mathcal{T}), cl_{\mathcal{T}}(\mathcal{A})\rangle$ be the result of exhaustively applying the completion rules from Figure 1 to $\mathcal{K}$. We directly obtain that $\mathcal{T}^* \subseteq cl(\mathcal{T})$.

Lemma 1. *For a basic role or basic concept X, if $\bigsqcap_{i=1}^{k} X_i \sqsubseteq X \in cl(\mathcal{T})$ and $X_i(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq i \leq k$, then there is $\bigsqcap_{j=1}^{p} X'_j \sqsubseteq X \in \mathcal{T}^*$ such that $X'_j(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq j \leq p$.*

Effectively, this restriction allows us to obtain more elegant results without compromising the correctness and completeness of the complete rules in Figure 1, as seen in later sections. Similar to T3 and T4, A3 and A4 deal with inverse and existentially quantified role assertions.

Lemma 2. *$\mathcal{K}$ is inconsistent iff $\bot(o) \in cl_{\mathcal{T}}(\mathcal{A})$ for some $o \in \mathbf{I}$. If $\mathcal{K}$ is consistent, then for any axiom τ and any assertion α, the following hold:*

1. *$\mathcal{T} \models \tau$ iff $\tau \in cl(\mathcal{T})$.*
2. *$\mathcal{K} \models \alpha$ iff $\alpha \in cl_{\mathcal{T}}(\mathcal{A})$.*

Lemma 2 shows that our completion rules from Figure 1 are sound and complete for restricted *DL-Lite*$_{horn}^{(\mathcal{HF})}$ ontologies, and thus we may write, e.g. $X(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ instead of $\langle \mathcal{T}, \mathcal{A}\rangle \models X(\vec{o})$ in the following. Later, we will use this result to show the correctness of our Datalog$^\neg$ program.

4. Prioritized Coherence Update Semantics for DL-Lite$_{\mathbf{horn}}^{(\mathcal{HF})}$

In the following, we assume that the initial ontology $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$ is consistent, and consider an *(instance-level) update* $\mathcal{U}$, which is a finite set of *insertions* *ins*$(X(\vec{o}))$ and *deletions* *del*$(X(\vec{o}))$, where $X(\vec{o})$ is an assertion [14]. For an update $\mathcal{U}$, the set of assertions occurring in insertions (resp. deletions) in $\mathcal{U}$ is denoted by $\mathcal{A}_\mathcal{U}^{\mathrm{ins}}$ (resp. $\mathcal{A}_\mathcal{U}^{\mathrm{del}}$). As discussed in the introduction, the presence of concept conjunction means that there can be multiple ways of updating the given ABox w.r.t. $\mathcal{U}$. For this reason, we introduce a ranking on the left-hand sides of all concept inclusions in $\mathcal{T}$. For each axiom $\tau = \bigsqcap_{i=1}^k B_i \sqsubseteq B \in \mathcal{T}$, we assume that we are also given a total order $\prec_\tau$ on $\{B_1, \dots, B_k\}$. Unless stated otherwise, we assume w.l.o.g. that $B_1 \prec_\tau B_2 \prec_\tau \dots \prec_\tau B_k$. For a non-empty set $S \subseteq \mathit{left}(\tau)(o)$ for some $o \in \mathbf{I}$, we define $\min_\tau(S) := B_j(o)$, where B_j is the minimal element w.r.t. $\prec_\tau$ for which $B_j(o) \in S$. When it is clear from the context, we may omit τ from the subscript. For convenience, we also extend this notation to role inclusions and axioms from $\mathcal{T}^* \setminus \mathcal{T}$, in which case the left-hand side always contains only one element, and thus $\prec_\tau$ and $\min_\tau$ are uniquely defined.

Definition 1. An ABox $\mathcal{A}'$ *accomplishes the prioritized update* of $\mathcal{K}$ with $\mathcal{U}$ (written $\mathcal{A} \xrightarrow{\prec,\mathcal{T}}_\mathcal{U} \mathcal{A}'$) if $\mathcal{A}'$ is a maximal subset of $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ such that

PU1. $\langle \mathcal{T}, \mathcal{A}' \rangle$ is consistent

PU2. for all $\alpha \in \mathcal{A}_\mathcal{U}^{\mathrm{del}}$, we have $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models \alpha$ and for all $\alpha \in \mathcal{A}_\mathcal{U}^{\mathrm{ins}}$, we have $\langle \mathcal{T}, \mathcal{A}' \rangle \models \alpha$

PU3. for all axioms $\tau = \bigsqcap_{i=1}^k B_i \sqsubseteq B$ from $\mathcal{T}^*$ and constants o from $\mathcal{A}$, if $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models B(o)$ and $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \models \mathit{left}(\tau)(o)$, then $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models \min_\tau \big(\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})\big)$.

Intuitively, PU1 and PU2 require that all requested operations from $\mathcal{U}$ must be respected while maintaining consistency of the updated ABox. Moreover, PU3 ensures that the updated ABox is unique by removing, for each non-entailed $B(o)$ that would be entailed through τ from $\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}}$, the assertion corresponding to the minimal possible element on the left-hand side of τ. Assertions from $cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ are always entailed due to PU2, which is why we have to exclude them before applying $\min_\tau$. Nevertheless, the minimum always exists since $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \models \mathit{left}(\tau)(o)$ would imply $\langle \mathcal{T}, \mathcal{A}' \rangle \models B(o)$ by PU2.

Since the combined ABox $\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}}$ may be inconsistent with $\mathcal{T}$, we consider its consequences only w.r.t. the positive axioms in $\mathcal{T}^+$, to avoid entailing everything through inconsistency. Otherwise, PU3 could result in too many deletions. Although in principle $\mathcal{A}_\mathcal{U}^{\mathrm{ins}}$ may also be inconsistent, we use $cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ instead of $cl_{\mathcal{T}^+}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ since, as we show below, inconsistency of $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$ implies that no updated ABox can exist in the first place.

Theorem 3. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an ontology and $\mathcal{U}$ be an update. An ABox $\mathcal{A}'$ with $\mathcal{A} \xrightarrow{\prec,\mathcal{T}}_\mathcal{U} \mathcal{A}'$ exists iff the following conditions are satisfied:*

1. *$\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{ins} \rangle$ is consistent*
2. *$\mathcal{A}_\mathcal{U}^{del} \cap cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{ins}) = \emptyset$*

If this is the case, then $\mathcal{U}$ is said to be compatible *with $\mathcal{T}$.*

Proof. In the "if" direction, it suffices to show that $\mathcal{A}' = \mathcal{A}_\mathcal{U}^{\mathrm{ins}}$ satisfies PU1–PU3. The first condition implies PU1. It follows from the second condition that, for all $X(\vec{o}) \in \mathcal{A}_\mathcal{U}^{\mathrm{del}}$, we have $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \not\models X(\vec{o})$, i.e. $\mathcal{A}_\mathcal{U}^{\mathrm{ins}}$ satisfies PU2. For PU3, let $\tau \in \mathcal{T}^*$ be an arbitrary axiom $\bigsqcap_{i=1}^k B_i \sqsubseteq B$ s.t. $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \not\models B(o)$ for some object o, and $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \models \{B_1(o), \dots, B_k(o)\}$. Then $\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ cannot be empty, since otherwise $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \models B(o)$ would hold, and thus $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle \not\models \min_\tau \big(\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})\big)$.

In the "only if" direction, since $\langle \mathcal{T}, \mathcal{A}' \rangle$ has a model $\mathcal{I}$ (PU1) and $\mathcal{A}_\mathcal{U}^{\mathrm{ins}} \subseteq cl_\mathcal{T}(\mathcal{A}')$ (PU2), $\mathcal{I}$ is also a model of $\mathcal{A}_\mathcal{U}^{\mathrm{ins}}$. Moreover, the existence of $\alpha \in \mathcal{A}_\mathcal{U}^{\mathrm{del}}$ with $\alpha \in cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}}) \subseteq cl_\mathcal{T}(\mathcal{A}')$ is impossible by PU2. □

Since compatibility does not depend on $\mathcal{A}$, it can be checked statically, i.e. given only the TBox, and then a compatible update can be applied to any ABox. As in [14], we can show that the result of applying a compatible update under our new semantics is always unique.

Theorem 4. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an ontology and $\mathcal{U}$ be an update that is compatible with $\mathcal{T}$. Then there is a unique ABox $\mathcal{A}'$ such that $\mathcal{A} \xrightarrow{\prec,\mathcal{T}}_\mathcal{U} \mathcal{A}'$.*

Proof. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two different ABoxes that accomplish the prioritized update of $\mathcal{K}$ with $\mathcal{U}$. By Definition 1, $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximal subsets of $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$ that satisfy PU1, PU2, and PU3. Thus, we can obtain a contradiction by showing that their union $\mathcal{A}_1 \cup \mathcal{A}_2$ also satisfies these properties. To do this, we first show the following claim.

Claim. $cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2) = cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$.

Proof of claim. The $\supseteq$-direction follows directly from monotonicity of $cl_{\mathcal{T}}$. For the $\subseteq$-direction, consider any $\alpha \in cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2)$. We show that $\alpha \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$ by induction on the number m of rule applications from Figure 1 that derive α from $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$.
Inductive base: $m = 0$ implies that $\alpha \in \mathcal{A}_1 \cup \mathcal{A}_2$, and thus $\alpha \in \mathcal{A}_1 \subseteq cl_{\mathcal{T}}(\mathcal{A}_1)$ or $\alpha \in \mathcal{A}_2 \subseteq cl_{\mathcal{T}}(\mathcal{A}_2)$.
Inductive step: Assume that for any α' derived after $\leq m$ rule applications from $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$, we have $\alpha' \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$. We show the claim for $m + 1$ via case distinction on the last rule from Figure 1 that derives α.

For A1, we have $\alpha = \bot(o), R(o, o'), R(o, o'') \in cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2)$, *funct* $R \in \mathcal{T}$, and $o \neq o''$. However, since R is not allowed on the right-hand side of any axiom in $\mathcal{T}$, $R(o, o')$ and $R(o, o'')$ cannot be derived from any rule, and thus $R(o, o'), R(o, o'') \in \mathcal{A}_1 \cup \mathcal{A}_2$. Since $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$, by the same argument we know that $R(o, o'), R(o, o'') \in \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$. Since $\mathcal{A}, \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}, \mathcal{A}_1, \mathcal{A}_2$ are all consistent w.r.t. $\mathcal{T}$, none of these four ABoxes can contain both $R(o, o')$ and $R(o, o'')$, and thus we can assume w.l.o.g. that $\mathcal{A}_1$ and $\mathcal{A}$ contain $R(o, o')$, but not $R(o, o'')$, and $\mathcal{A}_2$ and $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ contain $R(o, o'')$, but not $R(o, o')$. However, by PU2, $R(o, o'') \in \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ implies that $R(o, o'') \in cl_{\mathcal{T}}(\mathcal{A}_1)$, which together with $R(o, o') \in \mathcal{A}_1$ shows that $\langle \mathcal{T}, \mathcal{A}_1 \rangle$ is inconsistent. Since this contradicts PU1, this case is impossible.

For A2, we know that $\alpha = R_2(o, o')$, $R_1 \sqsubseteq R_2 \in \mathcal{T}^*$, and $R_1(o, o')$ is inferred after $\leq m$ rule applications from $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$. By induction, we obtain that $R_1(o, o') \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$, and thus also $R_2(o, o') \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$ via A2. The arguments for A3 and A4 are analogous.

For A5, we have $\alpha = B(o)$, $\tau = \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$, and $B_1(o), \ldots, B_k(o)$ are inferred after $\leq m$ rule applications from $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$. By induction, we have $\min_\tau \left(\mathit{left}(\tau)(o) \setminus cl_{\mathcal{T}}(\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}) \right) \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$. Moreover, since $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$, we have $B_1(o), \ldots, B_k(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$. By PU3, this implies that $B(o) \in cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2)$.

This concludes the proof of the claim. □

We now show that $\mathcal{A}_1 \cup \mathcal{A}_2$ satisfies Conditions PU1–PU3. For PU1, by consistency of $\langle \mathcal{T}, \mathcal{A}_1 \rangle$ and $\langle \mathcal{T}, \mathcal{A}_2 \rangle$ and Lemma 2, we know that no assertion of the form $\bot(o)$ can be in $cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2) = cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2)$, which shows that $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle$ is consistent. For PU2, by the claim we have $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}} \subseteq cl_{\mathcal{T}}(\mathcal{A}_1) \subseteq cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2)$ and $\mathcal{A}_{\mathcal{U}}^{\mathrm{del}} \cap cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathcal{A}_{\mathcal{U}}^{\mathrm{del}} \cap \left(cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2) \right) = \emptyset$, since $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy PU2. Finally, for PU3, consider $\tau = \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$ with $\langle \mathcal{T}, \mathcal{A}_1 \cup \mathcal{A}_2 \rangle \not\models B(o)$ and $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}} \rangle \models \{B_1(o), \ldots, B_k(o)\}$. Then also $\langle \mathcal{T}, \mathcal{A}_1 \rangle \not\models B(o)$ and $\langle \mathcal{T}, \mathcal{A}_2 \rangle \not\models B(o)$, and thus by PU3 and the claim we have $\min_\tau \left(\mathit{left}(\tau)(o) \setminus cl_{\mathcal{T}}(\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}) \right) \notin cl_{\mathcal{T}}(\mathcal{A}_1) \cup cl_{\mathcal{T}}(\mathcal{A}_2) = cl_{\mathcal{T}}(\mathcal{A}_1 \cup \mathcal{A}_2)$. □

This shows that our new update semantics satisfies *uniqueness*. Moreover, it is *syntax independent* since all properties in Definition 1 are formulated in terms of entailment. Finally, the *minimal change* property is reflected in the fact that we consider the maximal subset of $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$.

5. The Stratified Datalog$^\neg$ Rewriting $\mathcal{R}_\mathcal{T}$

We now present a stratified Datalog$^\neg$ program $\mathcal{R}_\mathcal{T}$ that only depends on $\mathcal{T}$ and can be used to compute the updated ABox $\mathcal{A}'$ for any initial ABox $\mathcal{A}$ and compatible update $\mathcal{U}$. We do not compute precisely the maximal ABox from Definition 1, but a smaller ABox that is equivalent to $\mathcal{A}'$ w.r.t. $\mathcal{T}$, to avoid redundant assertions. For the rewriting, $\mathcal{A}$ and $\mathcal{U}$ are encoded into the dataset

$$D := \mathcal{A} \cup \{Ap\_X(\vec{o}) \mid X(\vec{o}) \in \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}\} \cup \{Am\_X(\vec{o}) \mid X(\vec{o}) \in \mathcal{A}_{\mathcal{U}}^{\mathrm{del}}\},$$

where Ap_X and Am_X are special predicates that encode $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ and $\mathcal{A}_{\mathcal{U}}^{\mathrm{del}}$, respectively. As output, the program $\mathcal{R}_\mathcal{T}$ will compute facts of the forms $ins\_X(\vec{o})$ and $del\_X(\vec{o})$, which encode the actual insertion and deletion operations that have to be executed on $\mathcal{A}$ to obtain the updated ABox. The program consists of three strata.

Stratum 1. In the first layer, we derive facts for the auxiliary predicates $ApCl\_X$ and $AOrApCl\_X$, which represent the assertions in $cl_{\mathcal{T}^+}(\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$ and $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$, respectively. We use $\mathcal{T}^+$ here also for $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$, since at this point $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ could still be inconsistent. First, we initialize the predicates with the assertions in $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ and $\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$, for each *positive* concept or role X that can be formulated over the atomic symbols in $\mathcal{T}$:

$$AOrApCl\_X(\vec{Y}) \leftarrow X(\vec{Y}) \tag{1}$$

$$ApCl\_X(\vec{Y}),\ AOrApCl\_X(\vec{Y}) \leftarrow Ap\_X(\vec{Y}) \tag{2}$$

Here, we summarize two rules into one by writing two head atoms, and $\vec{Y}$ is a vector matching the arity of X. For each basic role R and for each $pred \in \{ApCl, AOrApCl\}$, we implement Rules A3 and A4 from Figure 1:

$$pred\_\exists R(Y_1),\ pred\_R^-(Y_2, Y_1) \leftarrow pred\_R(Y_1, Y_2) \tag{3}$$

For Rules A2 and A5, we consider all $\bigsqcap_{i=1}^{k} X_i \sqsubseteq X \in \mathcal{T}^*$ and all $pred \in \{ApCl, AOrApCl\}$:

$$pred\_X(\vec{Y}) \leftarrow pred\_X_1(\vec{Y}), \ldots, pred\_X_k(\vec{Y}) \tag{4}$$

Since we only consider $\mathcal{T}^+$ here, we do not need to encode Rule A1. However, for $ApCl$, A1 is implicitly implemented in Rule (9) below.

Stratum 2. In the next step, we derive facts of the form $delCl\_X(\vec{o})$, which represent all assertions from $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$ that may have to be deleted to satisfy properties PU1, PU2, and PU3, i.e. for which $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models X(\vec{o})$ should hold. The first rule initializes these facts with $\mathcal{A}_{\mathcal{U}}^{\mathrm{del1}}$ (see PU2):

$$delCl\_X(\vec{Y}) \leftarrow Am\_X(\vec{Y}) \tag{5}$$

Similar to before, $delCl$ also has to be propagated according to A3 and A4; however, the direction is reversed since we are encoding the *non*-entailment of facts. Thus, for A4, we only consider the deletion of role assertions $R(o, o')$ if they are actually present in $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$:

$$delCl\_R^-(Y_2, Y_1) \leftarrow delCl\_R(Y_1, Y_2) \tag{6}$$

$$delCl\_R(Y_1, Y_2) \leftarrow AOrApCl\_R(Y_1, Y_2),\ delCl\_\exists R(Y_1) \tag{7}$$

For functional roles R, we also have to delete existing role assertions if they would contradict new assertions from $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$ (see PU1). Similarly, we need to detect consistency violations that are due to conflicting assertions on functional roles in $\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}}$. Here, the 0-ary predicate $incompatible()$ encodes the fact that $\mathcal{U}$ is not compatible (see Item 1 of Theorem 3):

$$delCl\_R(Y_1, Y_3) \leftarrow Ap\_R(Y_1, Y_2),\ R(Y_1, Y_3),\ Y_2 \neq Y_3 \tag{8}$$

$$incompatible() \leftarrow Ap\_R(Y_1, Y_2),\ Ap\_R(Y_1, Y_3),\ Y_2 \neq Y_3 \tag{9}$$

The following rules deal with the semantics of inclusions for PU3 and A2/A5. For each $\tau = \bigsqcap_{i=1}^{k} X_i \sqsubseteq X \in \mathcal{T}^*$ and each $i \in \{1, \ldots, k\}$, we use $min\_X_i\_in\_\tau$ to check whether $X_i(\vec{o})$ is the least preferred assertion that can be deleted, i.e. is not included in $cl_{\mathcal{T}}(\mathcal{A}_{\mathcal{U}}^{\mathrm{ins}})$ (recall that we assumed $B_1 \prec_\tau \cdots \prec_\tau B_k$):

$$min\_X_i\_in\_\tau(\vec{Y}) \leftarrow ApCl\_X_1(\vec{Y}), \ldots, ApCl\_X_{i-1}(\vec{Y}),\ \neg ApCl\_X_i(\vec{Y}) \tag{10}$$

$$delCl\_X_i(\vec{Y}) \leftarrow delCl\_X(\vec{Y}),\ min\_X_i\_in\_\tau(\vec{Y}),$$

$$AOrApCl\_X_i(\vec{Y}), \ldots, AOrApCl\_X_k(\vec{Y}) \tag{11}$$

In the special case where $X = \bot$, for PU1 we want that $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models \bot(o)$ holds for all $o \in \mathbf{I}$. Thus, we consider $delCl\_\bot(o)$ to be true for all o, and simply omit $delCl\_\bot(\vec{Y})$ from the body of Rule (11). Moreover, we use $incompatible()$ to signal when the set $\min_\tau \left(left(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\text{ins}})\right)$ is empty (see PU3 and Item 2 of Theorem 3):

$$incompatible() \leftarrow delCl\_X(\vec{Y}),\ ApCl\_X_1(\vec{Y}), \ldots, ApCl\_X_k(\vec{Y}) \tag{12}$$

We also need to detect all direct violations of Item 2 of Theorem 3:

$$incompatible() \leftarrow ApCl\_X(\vec{Y}),\ delCl\_X(\vec{Y}) \tag{13}$$

Finally, if we have determined that an assertion from the original ABox $\mathcal{A}$ cannot be entailed by $\langle \mathcal{T}, \mathcal{A}' \rangle$, it needs to be deleted:

$$del\_X(\vec{Y}) \leftarrow X(\vec{Y}),\ delCl\_X(\vec{Y}) \tag{14}$$

Stratum 3. Since we want to keep as many assertions from $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}})$ as possible, we may need to add new assertions into $\mathcal{A}'$ that are not entailed after removing all assertions according to $delCl$, but are not themselves prevented from being entailed. This is implemented by the predicates $insCl\_X$. For example, consider the ABox $\mathcal{A} = \{A(a)\}$, TBox $\mathcal{T} = \{A \sqsubseteq A_1,\ A \sqsubseteq A_2\}$, and update $\mathcal{U} = \{\mathit{del}(A_1(a))\}$. Then $A(a)$ has to be deleted from $\mathcal{A}$, but $A_2(a) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}})$ should be kept, since otherwise the result is not maximal. Hence, we also need to add $A_2(a)$ to the updated ABox $\mathcal{A}'$.

For each $\tau = \bigsqcap_{i=1}^k X_i \sqsubseteq X \in \mathcal{T}^*$, the auxiliary predicate $preInsCl\_\tau$ first determines whether we can add the assertion $X(\vec{o})$ entailed via A2/A5, i.e. it is not required to be deleted.

$$preInsCl\_\tau(\vec{Y}) \leftarrow AOrApCl\_X_1(\vec{Y}), \ldots, AOrApCl\_X_k(\vec{Y}),\ \neg delCl\_X(\vec{Y}) \tag{15}$$

Then, we can trigger the insertion of $X(\vec{o})$ if one of the assertions $X_i(\vec{o})$, $1 \leq i \leq k$, is deleted:

$$insCl\_X(\vec{Y}) \leftarrow delCl\_X_i(\vec{Y}),\ preInsCl\_\tau(\vec{Y}) \tag{16}$$

We also need a similar rule corresponding to A4. For A3, however, it does not make sense to add the entailed assertion $R^-(o', o)$ if $R(o, o')$ is deleted, because these assertions are semantically equivalent.

$$insCl\_\exists R(Y_1) \leftarrow delCl\_R(Y_1, Y_2),\ AOrApCl\_R(Y_1, Y_2),\ \neg delCl\_\exists R(Y_1) \tag{17}$$

Similarly to Rule (14), we can then add all of these assertions (as well as the ones from $\mathcal{A}_\mathcal{U}^{\text{ins}}$) to the updated ABox, if they are not already in $\mathcal{A}$:

$$ins\_X(\vec{Y}) \leftarrow \neg X(\vec{Y}),\ insCl\_X(\vec{Y}) \tag{18}$$

$$ins\_X(\vec{Y}) \leftarrow \neg X(\vec{Y}),\ Ap\_X(\vec{Y}) \tag{19}$$

In this construction, it is important that we also consider assertions over basic concepts, and not just atomic concepts. If in the last example we use the TBox $\mathcal{T} = \{A \sqsubseteq A_1,\ A \sqsubseteq \exists R,\ \exists R \sqsubseteq A_2\}$ instead, then we need to add the assertion $\exists R(a)$ to keep as much information from $cl_\mathcal{T}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}})$ as possible.

5.1. Results

We dedicate this section to showing the theoretical properties of the program $\mathcal{R}_\mathcal{T}$, in particular its correctness w.r.t. Definition 1. All missing proofs can be found in the appendix. We begin by showing that the predicates $ApCl\_X$ and $AOrApCl\_X$ from the first stratum behave as claimed.

Lemma 5. *For every basic assertion $X(\vec{o})$, the following hold:*

1. $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_X(\vec{o})$ *iff* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{ins})$.
2. $\mathcal{R}_\mathcal{T}, D \models ApCl\_X(\vec{o})$ *iff* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A}_\mathcal{U}^{ins})$.

We now focus on the behaviour of $\mathcal{R}_\mathcal{T}, D$ w.r.t. to the negative axioms in $\mathcal{T}$, particularly the case where an update is contradictory, as categorized in Theorem 3.

Lemma 6. $\mathcal{R}_\mathcal{T}, D \models incompatible()$ *iff* $\mathcal{U}$ *is not compatible with* $\mathcal{T}$.

From now on, we assume that $\mathcal{R}_\mathcal{T}, D \not\models incompatible()$, i.e. by Theorem 3 there exists an ABox $\mathcal{A}'$ such that $\mathcal{A} \xrightarrow{\prec,\mathcal{T}}_\mathcal{U} \mathcal{A}'$. In particular, this means that $\mathcal{A}_\mathcal{U}^{\text{ins}}$ is consistent, i.e. $cl_{\mathcal{T}^+}(\mathcal{A}_\mathcal{U}^{\text{ins}}) = cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\text{ins}})$. Thus, we can now also show that the predicates $min\_X_i\_in\_\tau$ behave as expected, i.e. they check whether an assertion $X_i(o)$ is the minimal element to be deleted w.r.t. an axiom τ and $\mathcal{A}_\mathcal{U}^{\text{ins}}$ (see Definition 1).

Lemma 7. *For* $\tau \in \mathcal{T}^*$*, we have* $\mathcal{R}_\mathcal{T}, D \models min\_X_i\_in\_\tau(\vec{o})$ *iff* $X_i(\vec{o}) = \min_\tau \big(left(\tau)(\vec{o}) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{ins})\big)$.

Now, let `del` be the set of all $X(\vec{o})$ such that $\mathcal{R}_\mathcal{T}, D \models del\_X(\vec{o})$ and let `ins` be defined similarly for $ins\_X(\vec{o})$. Then $\mathcal{A}^*$ is the ABox obtained from $\mathcal{A}$ by removing all assertions in `del` and inserting all assertions in `ins`, i.e. $\mathcal{A}^* = (\mathcal{A} \setminus \texttt{del}) \cup \texttt{ins}$. Intuitively, $\mathcal{A}^*$ is the unique updated ABox (up to logical equivalence) according to Theorem 4. To show this, we first prove that `ins` and `del` are disjoint, i.e. we do not add and delete the same fact concurrently.

Lemma 8. $\texttt{ins} \cap \texttt{del} = \emptyset$.

Proof. Assume the contrary that there is $X(\vec{o}) \in \texttt{ins} \cap \texttt{del}$. By Rule (5), we have $\mathcal{R}_\mathcal{T}, D \models delCl\_X(\vec{o})$. Moreover, $X(\vec{o}) \in \texttt{ins}$ implies the application of either Rule (18) or (19).

If Rule (18) is applied, then $\mathcal{R}_\mathcal{T}, D \models insCl\_X(\vec{o})$, which implies the application of either Rule (16) or (17). If Rule (16) is applied, then there is $\tau = \bigsqcap_{i=1}^k X_i \sqsubseteq X \in \mathcal{T}^*$ and $\mathcal{R}_\mathcal{T}, D \models preInsCl\_\tau(\vec{o})$. Thus, by Rule (15) that derives $preInsCl\_\tau(\vec{o})$, we obtain $\mathcal{R}_\mathcal{T}, D \not\models delCl\_X(\vec{o})$, which is a contradiction. If Rule (17) is applied, then $insCl\_X(\vec{o}) = insCl\_\exists R(o)$ for some role R and constant o. Thus, we again have $\mathcal{R}_\mathcal{T}, D \not\models delCl\_\exists R(o)$ by construction of the rule.

If Rule (19) is applied, then $Ap\_X(\vec{o}) \in D$ and thus $\mathcal{R}_\mathcal{T}, D \models ApCl\_X(\vec{o})$ due to Rule (2). However, by Lemma 6 and Rule (13), this violates Item 2 of Theorem 3. □

We can use this to show that $\mathcal{A}^*$ complies with all deletions required by the predicates $delCl\_X$.

Lemma 9. *Let* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{ins})$*. Then,* $\mathcal{R}_\mathcal{T}, D \models delCl\_X(\vec{o})$ *iff* $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X(\vec{o})$.

We use this result to argue that $delCl$ "pinpoints" all assertions violating PU1, PU2, and PU3.

Recall that our semantics is actually defined using the maximal subset of $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}})$ satisfying these three properties (see Definition 1). We will show that this set is exactly $\mathcal{A}' = cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}}) \setminus \texttt{del}$, which, however, is equivalent to the ABox $\mathcal{A}^*$ computed by $\mathcal{R}_\mathcal{T}$ over D.

Lemma 10. $\mathcal{A}^*$ *and* $\mathcal{A}'$ *are equivalent w.r.t.* $\mathcal{T}$.

Proof. In the "only if" direction, it suffices to show that $X(\vec{o}) \in \mathcal{A}^*$ implies $X(\vec{o}) \in \mathcal{A}'$. Recall that $\mathcal{A}^* = (\mathcal{A} \setminus \texttt{del}) \cup \texttt{add}$. We consider the membership of $X(\vec{o})$ in either $\mathcal{A} \setminus \texttt{del}$ or `add`. If $X(\vec{o}) \in \mathcal{A} \setminus \texttt{del}$, then it is also in $cl_\mathcal{T}(\mathcal{A}) \setminus \texttt{del}$. If $X(\vec{o}) \in \texttt{add}$, then we observe that only Rules (19) and (18) derive $ins\_X(\vec{o})$ in $\mathcal{R}_\mathcal{T}$.

For Rule (19), we have $Ap\_X(\vec{o}) \in D$ and thus $X(\vec{o}) \in \mathcal{A}_\mathcal{U}^{\text{ins}}$. If Rule (18) derives $ins\_X(\vec{o})$, then $\mathcal{R}_\mathcal{T}, D \models insCl\_X(\vec{o})$. We first show the auxiliary result below, which relates $insCl\_X(\vec{o})$ to the corresponding assertion in $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\text{ins}})$.

Claim. $\mathcal{R}_\mathcal{T}, D \models insCl\_X(\vec{o})$ *implies* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{ins})$.

Proof of claim. In $\mathcal{R}_\mathcal{T}$, $insCl\_X(\vec{o})$ is derivable from the either Rule (16) or (17).

If Rule (16) is applied, then there is $\tau = \bigsqcap_{i=1}^{k} X_i \sqsubseteq X \in \mathcal{T}$ where $\mathcal{R}_\mathcal{T}, D \models preInsCl\_\tau(\vec{o})$. By Rule (15), we must have $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_X_j(\vec{o})$ for all $1 \leq j \leq k$. Hence, $X_j(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$ for all $1 \leq j \leq k$, and thus we obtain $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$ via Rule A5 from Figure 1.

If Rule (17) is applied, then $insCl\_X(\vec{o}) = insCl\_\exists R(o)$ for some role R and constant o. Furthermore, we have $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_R(o, o')$ for some constant o', which implies $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_\exists R(o)$ by Rule (3). Thus, $\exists R(o) = X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$. ☐

Thus, we have $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$. Moreover, by Lemma 8, we know that add and del are disjoint. Hence, $X(\vec{o}) \in (cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \setminus \mathtt{del})$.

In the "if" direction, it suffices to show that $X(\vec{o}) \in (cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \setminus \mathtt{del})$ implies $\langle \mathcal{T}, \mathcal{A}^* \rangle \models X(\vec{o})$. Since $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$, we have $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_X(\vec{o})$ by Lemma 5. Furthermore, since $X(\vec{o}) \notin \mathtt{del}$, we have $\mathcal{R}_\mathcal{T}, D \not\models delCl\_X(\vec{o})$ by Rule (14), which implies $\langle \mathcal{T}, \mathcal{A}^* \rangle \models X(\vec{o})$ by Lemma 9. ☐

Now, we are ready to show that $\mathcal{R}_\mathcal{T}$ computes the prioritized update semantics (modulo equivalence).

Theorem 11. *If $\mathcal{R}_\mathcal{T}, D \not\models incompatible()$, then $\mathcal{A} \xrightarrow{\prec, \mathcal{T}}_\mathcal{U} \mathcal{A}'$.*

Proof. We show that $\mathcal{A}' = (cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \setminus \mathtt{del})$ is a maximal subset of $cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$ that satisfies conditions PU1, PU2, and PU3.

PU1: Assume the contrary that $\mathcal{A}'$ is inconsistent. Hence, by Lemma 2, we have $\bot(o) \in cl_\mathcal{T}(\mathcal{A}')$. Thus, either (i) *funct* $R \in \mathcal{T}$, $R(o, o')$ and $R(o, o'')$ are in $\mathcal{A}'$, and $o' \neq o''$; or (ii) $\bigsqcap_{i=1}^{k} B_i \sqsubseteq \bot \in cl(\mathcal{T})$ and $B_1(o), \ldots, B_k(o)$ are in $cl_\mathcal{T}(\mathcal{A}')$.

We observe that for (i), since $\langle \mathcal{T}, \mathcal{A} \rangle$ is consistent, we cannot have $\{R(o, o'), R(o, o'')\} \subseteq \mathcal{A}$. Moreover, $R(o, o') \in \mathcal{A}$ and $R(o, o'') \in \mathcal{A}_\mathcal{U}^{\mathtt{ins}}$ would lead to $delCl\_R(o, o')$ and $del\_R(o, o')$ being derived by Rules (8) and 14 (and vice versa). However, $\{R(o, o'), R(o, o'')\} \subseteq \mathcal{A}_\mathcal{U}^{\mathtt{ins}}$ implies that $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathtt{ins}} \rangle$ is inconsistent, which contradicts the assumption that $\mathcal{R}_\mathcal{T}, D \not\models incompatible()$ by Lemma 6.

Suppose that (ii) occurs in $\langle \mathcal{T}, \mathcal{A}' \rangle$. Then, by Lemma 1, there is $\tau = \bigsqcap_{i=1}^{q} B'_i \sqsubseteq \bot \in \mathcal{T}$ such that $B'_i(o) \in cl_\mathcal{T}(\mathcal{A}')$ for all $i \in \{1, \ldots, q\}$. Since $\mathcal{R}_\mathcal{T}, D \not\models incompatible()$, there is $j \in \{1, \ldots, q\}$ such that $\mathcal{R}_\mathcal{T}, D \not\models ApCl\_B'_j(o)$ by Rule (12). This means that there is a B'_i with $\mathcal{R}_\mathcal{T}, D \models min\_B'_i\_in\_\tau(o)$ by Rule (10). Moreover, by definition of $\mathcal{A}'$, we have $B'_j(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}^{\mathtt{ins}})$ for all $j \in \{1, \ldots, q\}$. Thus, $\mathcal{R}_\mathcal{T}, D \models delCl\_B'_i(o)$ by Rule (11) and $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_B'_i(o)$ by Lemma 5. Hence, by Lemma 9, $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models B'_i(o)$, which implies $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models B'_i(o)$ by Lemma 10. However, this contradicts the fact that $B'_i(o) \in cl_\mathcal{T}(\mathcal{A}')$.

PU2: Assume that there is $X(\vec{o}) \in \mathcal{A}_\mathcal{U}^{\mathtt{del}}$ with $\langle \mathcal{T}, \mathcal{A}' \rangle \models X(\vec{o})$. We show that this leads to a contradiction. Clearly, $X(\vec{o}) \in \mathcal{A}_\mathcal{U}^{\mathtt{del}}$ implies $Am\_X(\vec{o}) \in D$ by definition. Hence, we get $\mathcal{R}_\mathcal{T}, D \models delCl\_X(\vec{o})$ by Rule (5). Moreover, by definition of $\mathcal{A}'$, we have $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$. Thus, $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X(\vec{o})$ by Lemma 9. We obtain $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models X(\vec{o})$ by Lemma 10, which contradicts our assumption.

For $X(\vec{o}) \in \mathcal{A}_\mathcal{U}^{\mathtt{ins}}$, we have $Ap\_X(\vec{o}) \in D$. Thus, by Rule (19), $\mathcal{R}_\mathcal{T}, D \models ins\_X(\vec{o})$, which implies $X(\vec{o}) \in \mathtt{ins}$. This means that $\langle \mathcal{T}, \mathcal{A}^* \rangle \models X(\vec{o})$ as $\mathcal{A}^* = (\mathcal{A} \setminus \mathtt{del}) \cup \mathtt{ins}$. Thus, since $\mathcal{A}^*$ and $\mathcal{A}'$ are logically equivalent w.r.t. $\mathcal{T}$ by Lemma 10, we have $\langle \mathcal{T}, \mathcal{A}' \rangle \models X(\vec{o})$.

PU3: For $\tau = \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$, we show that if $\langle \mathcal{T}, \mathcal{A}' \rangle \not\models B(o)$ and $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}} \rangle \models B_i(o)$ for all $1 \leq i \leq k$, then $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models \min_\tau \left(\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \right)$. By Lemma 9, we have $\mathcal{R}_\mathcal{T}, D \models delCl\_B(o)$. Since $delCl\_\bot(o)$ is true by default, it suffices to consider $B \neq \bot$ in the following. By Rule (12), Lemma 5, and the assumption that $\mathcal{R}_\mathcal{T}, D \not\models incompatible()$, we have $\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \neq \emptyset$. Thus, there exists $B_i(o) = \min_\tau \left(\mathit{left}(\tau)(o) \setminus cl_\mathcal{T}(\mathcal{A}^{\mathtt{ins}}) \right)$. By Lemma 7, we have $\mathcal{R}_\mathcal{T}, D \models min\_B_i\_in\_\tau(o)$. By Rule 11, we obtain that $\mathcal{R}_\mathcal{T}, D \models delCl\_B_i(o)$, and hence $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models B_i(o)$ by Lemma 9.

Maximal subset: As the last step, we show that every set $\mathcal{A}'' \subseteq cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$ that satisfies Definition 1 must also satisfy the following property: for all $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}})$ such that $\mathcal{R}_\mathcal{T}, D \models delCl\_X(\vec{o})$, we have $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models X(\vec{o})$. Then, we know that $\mathcal{A}'' \subseteq cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathtt{ins}}) \setminus \mathtt{del} =$

$\mathcal{A}'$ by Rule (5), and thus $\mathcal{A}'$ is the unique maximal such set. We show the property by induction on the number of applications m of $T_{\mathcal{R}_\mathcal{T}}$ such that $delCl\_X(\vec{o}) \in T^m_{\mathcal{R}_\mathcal{T}}(D)$.
Inductive base: $m = 0$ implies that $delCl\_X(\vec{o}) \in D$, which is impossible.
Inductive step: We consider $delCl\_X(\vec{o}) \in T^{m+1}_{\mathcal{R}_\mathcal{T}}(D)$, which is derived by Rule (5), (6), (7), (8), or (11).

For Rule (5), we obtain that $X(\vec{o}) \in \mathcal{A}^{\mathrm{del}}_\mathcal{U}$, and thus $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models X(\vec{o})$ by PU2.

For Rule (6), we have $X(\vec{o}) = R^-(o', o)$, $delCl\_R(o, o') \in T^m_{\mathcal{R}_\mathcal{T}}(D)$, and $R(o, o') \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}^{\mathrm{ins}}_\mathcal{U})$ by Rule A3, and thus by induction we know that $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models R(o, o')$. By Rule A3, we conclude that $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models R^-(o', o)$.

For Rule (7), we have $X(\vec{o}) = R(o, o')$, $delCl\_\exists R(o) \in T^m_{\mathcal{R}_\mathcal{T}}(D)$, and $\exists R(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}^{\mathrm{ins}}_\mathcal{U})$ by Rule A4. By induction, we get $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models \exists R(o)$, and thus $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models R(o, o')$ by Rule A4.

For Rule (8), we have $X(\vec{o}) = R(o, o'')$, R is functional, $\mathcal{R}_\mathcal{T}, D \models ApCl\_R(o, o')$, and $R(o, o'') \in \mathcal{A}$ with $o'' \neq o'$. By Lemma 5, we obtain $R(o, o') \in cl_{\mathcal{T}^+}(\mathcal{A}^{\mathrm{ins}}_\mathcal{U})$, which implies $\langle \mathcal{T}, \mathcal{A}'' \rangle \models R(o, o')$ by PU2. Thus, $\langle \mathcal{T}, \mathcal{A}'' \rangle \models R(o, o'')$ is impossible, since this would imply that $\langle \mathcal{T}, \mathcal{A}'' \rangle$ is inconsistent (because R is functional), which would contradict PU1.

For Rule (11), there is $\tau = \bigsqcap_{i=1}^k X_i \sqsubseteq X' \in \mathcal{T}^*$ and an index $i \in \{1, \ldots, k\}$ s.t. $delCl\_X'(\vec{o})$, $min\_X_i\_in\_\tau(\vec{o})$, $AOrApCl\_X_i(\vec{o})$, …, $AOrApCl\_X_k(\vec{o}) \in T^m_{\mathcal{R}_\mathcal{T}}(D)$. Moreover, by Lemma 7, we have $X(\vec{o}) = X_i(\vec{o}) = \min_\tau \big(\mathit{left}(\tau)(\vec{o}) \setminus cl_\mathcal{T}(\mathcal{A}^{\mathrm{ins}}_\mathcal{U})\big)$. By Rule (10), we also know that $ApCl\_X_1(\vec{o})$, …, $ApCl\_X_{i-1}(\vec{o}) \in T^m_{\mathcal{R}_\mathcal{T}}(D)$, and thus $X_1(\vec{o}), \ldots, X_k(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}^{\mathrm{ins}}_\mathcal{U})$ by Lemma 5. By Rules A2 and A5, we know that $X'(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}^{\mathrm{ins}}_\mathcal{U})$, and hence we can apply induction to obtain $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models X'(\vec{o})$. If $X(\vec{o}) = X_i(\vec{o}) = R(o, o')$ and $X'(\vec{o}) = R'(o, o')$ are role assertions and $\tau = R \sqsubseteq R'$, then we must have $k = i = 1$, and thus $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models R(o, o')$. If $X(\vec{o}) = X_i(\vec{o}) = B(o)$ and $X'(\vec{o}) = B'(o)$ are concept assertions, then PU3 implies that $\langle \mathcal{T}, \mathcal{A}'' \rangle \not\models B(o)$.

This concludes the proof of the property of $\mathcal{A}''$, and we conclude that $\mathcal{A} \xrightarrow{\prec, \mathcal{T}}_\mathcal{U} \mathcal{A}'$. □

Since all rules in $\mathcal{R}_\mathcal{T}$ contain at most 3 variables, the program can be evaluated over D in polynomial time (even in combined complexity) [24]. Thus, the updated ABox $\mathcal{A}'$ or its smaller, equivalent representation $\mathcal{A}^*$ can also be computed in polynomial time. Statically checking whether $\mathcal{U}$ is compatible with $\mathcal{T}$ (regardless of $\mathcal{A}$) can also be done using $\mathcal{R}_\mathcal{T}$ by considering $\mathcal{A} = \emptyset$ and checking whether $\mathcal{R}_\mathcal{T}, D \models \mathit{incompatible}()$, for which all rules for the predicates with the prefixes $AOrApCl$, del, $preInsCl$, $insCl$, and ins can be omitted.

6. Conclusion

Our new prioritized coherence update semantics generalizes the one for *DL-Lite*$^{(\mathcal{HF})}_{core}$ [14] by the Condition PU3, which ensures unique update results via the prioritization of conjuncts in $\bigsqcap_{i=1}^k B_i$ for necessary deletions. This new condition may lead to redundant deletions, since the conjunction may already become unsatisfied due to other deletion operations that eliminate a non-minimal conjunct B_i. However, it allows us to ensure the uniqueness, syntax independence, and minimal change properties. Moreover, we can compute the updated ABox in polynomial time via a stratified Datalog$^\neg$ program.

In contrast to [14], where $cl(\mathcal{T})$ was used for the rewriting instead of $\mathcal{T}^*$, our program $\mathcal{R}_\mathcal{T}$ uses recursion, and it is currently unknown whether there also exists an FO rewriting (as for the coherence update semantics in *DL-Lite*$^{(\mathcal{HF})}_{core}$ [14] and for query answering [22]). Additionally, we would like to extend our result to full *DL-Lite*$^{(\mathcal{HF})}_{horn}$ with role disjointness, (a)symmetry, and (ir)reflexivity axioms. Ultimately, our aim is to use the new update semantics for planning, which involves finding sequences of ABox updates to entail a goal query starting from an initial ABox [16, 18, 19].

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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A. Appendix

Lemma 1. *For a basic role or basic concept X, if $\bigsqcap_{i=1}^{k} X_i \sqsubseteq X \in cl(\mathcal{T})$ and $X_i(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq i \leq k$, then there is $\bigsqcap_{j=1}^{p} X'_j \sqsubseteq X \in \mathcal{T}^*$ such that $X'_j(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq j \leq p$.*

Proof. We do induction on the number m of rule applications from Figure 1 needed to infer $\bigsqcap_{i=1}^{k} X_i \sqsubseteq X$ from $\mathcal{K}$.

Inductive base: $m = 0$ implies $\bigsqcap_{i=1}^{k} X_i \sqsubseteq X$ in either $\mathcal{T}$, which is in $\mathcal{T}^*$. Thus, the claim is trivially satisfied.

Inductive step: Assume that up to n, if $\bigsqcap_{i=1}^{q} X'_i \sqsubseteq X'$ is derived in n rule applications and $X'_i(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq i \leq q$ implies there is $\bigsqcap_{i=1}^{s} X''_i \sqsubseteq X' \in \mathcal{T}^*$ s.t. $X''_i(o) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $1 \leq i \leq s$.

Now assume that $\tau = \bigsqcap_{i=1}^{k} X_i \sqsubseteq X$ is inferred after $n + 1$ rule applications, which implies the application of T1–T7.

If T1 or T7 is applied, then τ is a tautology of the form $R \sqsubseteq R$, $\bigsqcap_{i=1}^{k} B_i \sqsubseteq D$ where $D \in \{B_1, \ldots, B_k\}$, or $\bot \in D$. Since these tautologies are in $\mathcal{T}^*$ by definition and *left*$(\tau)(o) \subseteq cl_{\mathcal{T}}(\mathcal{A})$, the claim is trivially satisfied.

If T6 is applied, then τ is of the form $M \sqsubseteq B \subseteq B$ where $B(o) \in cl_{\mathcal{T}}(\mathcal{A})$. Thus, since $B \sqsubseteq B \in \mathcal{T}^*$ by T1, the claim is trivially satisfied.

If T3 is applied, then the claim follows directly from the inductive assumption and the application of Rule A3.

If T5 is applied, then τ is $\bigsqcap_{i=1}^{k} B_i \sqsubseteq B$, and $\bigsqcap_{i=1}^{k} B_i \sqsubseteq N$ and $N \sqsubseteq B$ are derived in at most m rounds, for some $N = \bigsqcap_{j=1}^{k_2} B'_j$. Thus, by the inductive assumption, for all $1 \le j \le k_2$, there is $\bigsqcap_{i=1}^{k_j} B_i^j \sqsubseteq B'_j \in \mathcal{T}^*$ s.t. $\{B_1^j(o), \ldots, B_{k_j}^j(o)\} \subseteq cl_{\mathcal{T}}(\mathcal{A})$. Hence, $B'_j(o) \in cl_{\mathcal{T}}(\mathcal{A})$ by A5, for all $1 \le j \le k_2$. Similarly, for $\bigsqcap_{j=1}^{k_2} B'_j \sqsubseteq B$, by induction we obtain $\bigsqcap_{i=1}^{k_3} B''_i \sqsubseteq B \in \mathcal{T}^*$ s.t. $\{B''_1(o), \ldots, B''_{k_3}(o)\} \subseteq cl_{\mathcal{T}}(\mathcal{A})$.

If T2 is applied, then τ is $R_1 \sqsubseteq R$, and $R_1 \sqsubseteq R_2$ and $R_2 \sqsubseteq R$ are inferred in n rounds of rule applications. The rest of the argument for T2 is similar to that of T5 w.r.t. the application of A2.

If T4 is applied, then $\tau = \exists R_1 \sqsubseteq \exists R_2$, $R_1 \sqsubseteq R_2$ is inferred after n rounds of rule applications, and $\exists R_1(o) \in cl_{\mathcal{T}}(\mathcal{A})$. We first show an intermediate result:

Claim. *$R \sqsubseteq R' \in cl(\mathcal{T})$ implies the existence of a chain of axioms $R \sqsubseteq R_1 \sqsubseteq \ldots R_q \sqsubseteq R' \in \mathcal{T}^*$.*

Proof of claim. Proof via induction on the number of rule applications from Figure 1 that infers $R \sqsubseteq R'$. In the inductive base, the claim is trivial since $R \sqsubseteq R' \in \mathcal{T}$. In the inductive step, assume that up to n, $R_1 \sqsubseteq R_2$ inferred in n rounds of rule applications implies the existence of a chain of axioms $R_1 \sqsubseteq R'_1 \sqsubseteq \cdots \sqsubseteq R'_q \sqsubseteq R_2 \in \mathcal{T}^*$. For $R \sqsubseteq R'$ inferred after $n+1$ rounds, it is derivable from the following rules from Figure 1: T1, T2, and T3. For T1, the claim is trivial by the definition of $\mathcal{T}^*$. For T2 and T3, the claim follows directly from the inductive assumption. $\square$

By the claim above, there is a chain of axioms $R_1 \sqsubseteq R'_1 \sqsubseteq \cdots \sqsubseteq R'_q \sqsubseteq R_2 \in \mathcal{T}^*$. Thus, we have $\exists R'_q \sqsubseteq \exists R_2 \in \mathcal{T}^*$ due to T4. Moreover, since $\exists R_1(o) \in cl_{\mathcal{T}}(\mathcal{A})$, by the successive application of T4 and A5 to the chain in observation, we obtain $\exists R'_q(o) \in cl_{\mathcal{T}}(\mathcal{A})$. Thus, $\exists R'_q \sqsubseteq \exists R_2 \in \mathcal{T}^*$ with $\exists R'_q(o) \in cl_{\mathcal{T}}(\mathcal{A})$ satisfy our claim. $\square$

Lemma 2. *$\mathcal{K}$ is inconsistent iff $\bot(o) \in cl_{\mathcal{T}}(\mathcal{A})$ for some $o \in \mathbf{I}$. If $\mathcal{K}$ is consistent, then for any axiom τ and any assertion α, the following hold:*

1. *$\mathcal{T} \models \tau$ iff $\tau \in cl(\mathcal{T})$.*
2. *$\mathcal{K} \models \alpha$ iff $\alpha \in cl_{\mathcal{T}}(\mathcal{A})$.*

Proof. For the "if"-directions (soundness), it is straightforward to verify that any interpretation satisfying all premises and side conditions of a rule also satisfies its conclusions. In particular, if $\bot(o)$ is derived, then $\mathcal{K}$ cannot have any models.

For the "only if"-directions (completeness), we assume that $\bot(o) \notin cl_{\mathcal{T}}(\mathcal{A})$ for any o and construct the interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ as follows:

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{d_o \mid o \in \mathbf{I}\} \cup \{d_M \mid M \text{ is a conjunction over } \mathcal{T} \text{ where } M \sqsubseteq \bot \notin cl(\mathcal{T})\} \\
A^{\mathcal{I}} &= \{d_o \mid A(o) \in cl_{\mathcal{T}}(\mathcal{A})\} \cup \{d_M \mid M \sqsubseteq A \in cl(\mathcal{T})\} \\
P^{\mathcal{I}} &= \{(d_o, d_{o'}) \mid P(o, o') \in cl_{\mathcal{T}}(\mathcal{A})\} \cup \\
&\quad \{(d_o, d_{\exists R'^-}) \mid \exists R'(o) \in cl_{\mathcal{T}}(\mathcal{A}), R' \sqsubseteq P \in cl(\mathcal{T}), \forall o' \in \mathbf{I} : R'(o, o') \notin cl_{\mathcal{T}}(\mathcal{A})\} \cup \\
&\quad \{(d_{\exists R'^-}, d_o) \mid \exists R'(o) \in cl_{\mathcal{T}}(\mathcal{A}), R' \sqsubseteq P^- \in cl(\mathcal{T}), \forall o' \in \mathbf{I} : R'(o, o') \notin cl_{\mathcal{T}}(\mathcal{A})\} \cup \\
&\quad \{(d_M, d_N) \mid N = \textstyle\bigsqcap_{i=1}^{k} \exists R_i^- \text{ where } M \sqsubseteq \exists R_i \in cl(\mathcal{T}), R_i \sqsubseteq P \in cl(\mathcal{T}) \text{ for all } 1 \le i \le k\} \cup \\
&\quad \{(d_N, d_M) \mid N = \textstyle\bigsqcap_{i=1}^{k} \exists R_i^- \text{ where } M \sqsubseteq \exists R_i \in cl(\mathcal{T}), R_i \sqsubseteq P^- \in cl(\mathcal{T}) \text{ for all } 1 \le i \le k\}
\end{aligned}
$$

Observe that the above equation for $P^{\mathcal{I}}$ also holds if we replace the role name P by a basic role R, because the definition is symmetric due to Rule A3. We show that $\mathcal{I}$ is a *canonical model* of $\mathcal{K}'$, i.e., it satisfies exactly the assertions and axioms in $\mathcal{K}'$, and thus it is also a model of $\mathcal{K} \subseteq \mathcal{K}'$.

Claim I: $R(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$ iff $(d_o, d_{o'}) \in R^{\mathcal{I}}$

This follows directly from the definition of $R^{\mathcal{I}}$ (using Rule A3).

Claim II: $D(o) \in cl_{\mathcal{T}}(\mathcal{A})$ iff $d_o \in D^{\mathcal{I}}$

For $D = A$, the claim follows directly from the definition of $A^{\mathcal{I}}$.

For $D = \exists R$, $d_o \in (\exists R)^{\mathcal{I}}$ implies that there is $d' \in \Delta^{\mathcal{I}}$ s.t. $(d_o, d') \in R^{\mathcal{I}}$. If d' is $d_{o'}$ where $o' \in \mathbf{I}$, then by Claim I, we obtain $R(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$. Thus, by A4 we have $\exists R(o) \in cl_{\mathcal{T}}(\mathcal{A})$. If $d' = d_{\exists R'^-}$, then $(d_o, d_{\exists R'^-}) \in R^{\mathcal{I}}$ implies that $\exists R'(o) \in cl_{\mathcal{T}}(\mathcal{A})$, $R' \sqsubseteq R \in cl(\mathcal{T})$ by definition. Thus, by T4 we also have $\exists R' \sqsubseteq \exists R \in cl(\mathcal{T})$. By Lemma 1 and A5, we also have $\exists R(o) \in cl_{\mathcal{T}}(\mathcal{A})$. Conversely, assume that $\exists R(o) \in cl_{\mathcal{T}}(\mathcal{A})$. If there is $R(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$ for some $o' \in \mathbf{I}$, then by Claim I we obtain $(d_o, d_{o'}) \in R^{\mathcal{I}}$, which implies $d_o \in (\exists R)^{\mathcal{I}}$. We now assume that there is no $R(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$ for any $o' \in \mathbf{I}$. Since $\exists R(o) \in cl_{\mathcal{T}}(\mathcal{A})$ and $\bot(o) \notin cl_{\mathcal{T}}(\mathcal{A})$, we have $\exists R^- \sqsubseteq \bot \notin cl(\mathcal{T})$ and thus $d_{\exists R^-} \in \Delta^{\mathcal{I}}$. Thus, by T1 we have $R \sqsubseteq R \in cl(\mathcal{T})$, from which we obtain $(d_o, d_{\exists R^-}) \in R^{\mathcal{I}}$ by definition and $d_o \in (\exists R)^{\mathcal{I}}$.

Claim X: $M \sqsubseteq B \in cl(\mathcal{T})$ iff $d_M \in B^{\mathcal{I}}$

If $B = A$, the claim follows directly from the definition of $A^{\mathcal{I}}$.

If $B = \exists R$ and $d_M \in (\exists R)^{\mathcal{I}}$, then there exists $(d_M, d) \in R^{\mathcal{I}}$. Thus, according to the definition of $R^{\mathcal{I}}$, either (a) $d = d_N$, $N = \bigsqcap_{i=1}^{k} \exists R_i^-$, and for all $1 \le i \le k$: $M \sqsubseteq \exists R_i, R_i \sqsubseteq R \in cl(\mathcal{T})$; (b) $d = d_N$, $M = \bigsqcap_{i=1}^{k} \exists R_i^-$, and for all $1 \le i \le k$: $N \sqsubseteq \exists R_i$, $R_i \sqsubseteq R^- \in cl(\mathcal{T})$; or (c) $d = d_o$ for some $o \in \mathbf{I}$, $M = \exists R'^-$, $\exists R'(o) \in cl_{\mathcal{T}}(\mathcal{A})$, $R' \sqsubseteq R^- \in cl(\mathcal{T})$, $\forall o' \in \mathbf{I}$: $R'(o,o') \notin cl_{\mathcal{T}}(\mathcal{A})$.

In Case (a), by Rules T5 and T4, we also have $\exists R_i \sqsubseteq \exists R$ for all $1 \le i \le k$ and $M \sqsubseteq \exists R$ in $cl(\mathcal{T})$. In Case (b), by Rules T4 and T3, we get $R_i{}^- \sqsubseteq R$, $\exists R_i{}^- \sqsubseteq \exists R$ for all $1 \le i \le k$, and $M \sqsubseteq \exists R$ in $cl(\mathcal{T})$. In Case (c), by Rules T3 and T4, we directly get $\exists R'^- \sqsubseteq \exists R$ and thus $M \sqsubseteq \exists R$. Conversely, if $M \sqsubseteq \exists R \in cl(\mathcal{T})$, then $R \sqsubseteq R \in cl(\mathcal{T})$ by Rule T1, and thus $(d_M, d_{\exists R^-}) \in R^{\mathcal{I}}$, which shows that $d_M \in (\exists R)^{\mathcal{I}}$.

If $B = \bot$, then $d_M \in \bot^{\mathcal{I}}$ is equivalent to $d_M \notin \Delta^{\mathcal{I}}$, and thus to $M \sqsubseteq \bot \in cl(\mathcal{T})$.

Claim IIIA: $M \sqsubseteq D \in cl(\mathcal{T})$ iff $\mathcal{I} \models M \sqsubseteq D$

In the "only if" direction, for any $d \in M^{\mathcal{I}}$, we show that $d \in D^{\mathcal{I}}$. If $d = d_o$ for some $o \in \mathbf{I}$, then $d_o \in B^{\mathcal{I}}$ for all $B \in M$. Thus, by Claim II, $B(o) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $B \in M$. Moreover, since $M \sqsubseteq D \in cl(\mathcal{T})$, by Lemma 1, there is $M' \sqsubseteq D \in \mathcal{T}^*$ s.t. $B'(o) \in cl_{\mathcal{T}}(\mathcal{A})$ for all $B' \in M'$. Hence, by the application of A5, we obtain $D(o) \in cl_{\mathcal{T}}(\mathcal{A})$, and thus $d_o \in D^{\mathcal{I}}$ by Claim II.

For a conjunction M', if $d = d_{M'}$ and $d_{M'} \in B^{\mathcal{I}}$ for all $B \in M$, then by Claim X we have $M' \sqsubseteq B \in cl(\mathcal{T})$ for each $B \in M$, i.e., $M' \sqsubseteq M \in cl(\mathcal{T})$. Applying Rule T5 to $M' \sqsubseteq M$ and $M \sqsubseteq D$, we get $M' \sqsubseteq D \in cl(\mathcal{T})$, and finally $d_{M'} \in D^{\mathcal{I}}$ by Claim X.

In the "if" direction, if $\mathcal{I} \models M \sqsubseteq D$, then by Rule T6 we have $M \sqsubseteq B \in cl(\mathcal{T})$ for all $B \in M$, and thus $d_M \in M^{\mathcal{I}}$ by Claim X. Hence, by the assumption, we also have $d_M \in B^{\mathcal{I}}$. Again, by Claim X, this shows that $M \sqsubseteq D \in cl(\mathcal{T})$.

Claim IIIB: $M \sqsubseteq \bot \in cl(\mathcal{T})$ iff $\mathcal{I} \models M \sqsubseteq \bot$

Consider $d \in M^{\mathcal{I}}$, then $M \sqsubseteq \bot \notin cl(\mathcal{T})$ by definition of $\Delta^{\mathcal{I}}$ and for any $N \subseteq M$, we have $N \sqsubseteq \bot \notin cl(\mathcal{T})$ since otherwise T6 and T4 would derive $M \sqsubseteq \bot$. Hence, d is d_o for some o that appears in $cl_{\mathcal{T}}(\mathcal{A})$ due to the definition of $\Delta^{\mathcal{I}}$. By Lemma 1, since $M \sqsubseteq \bot \in cl(\mathcal{T})$, there is $\tau = M' \sqsubseteq \bot \in \mathcal{T}^*$ where $\mathit{left}(\tau)(o) \subseteq cl_{\mathcal{T}}(\mathcal{A})$. Thus, $\bot(o) \in cl_{\mathcal{T}}(\mathcal{A})$ by rule A5.

Conversely, we show that $M \sqsubseteq \bot \notin cl(\mathcal{T})$ implies $\mathcal{I} \not\models M \sqsubseteq \bot$. Since $M \sqsubseteq \bot \notin cl(\mathcal{T})$, by definition of $\Delta^{\mathcal{I}}$, we have $d_M \in \Delta^{\mathcal{I}}$. For $M = \bigsqcap_{i=1}^{k} B_i$, by Rule T6, we have $M \sqsubseteq B_i \in cl(\mathcal{T})$ for all $1 \le i \le k$. Hence, by Claim X, $d_M \in (B_i)^{\mathcal{I}}$ for all $1 \le i \le k$, which means $d_M \in M^{\mathcal{I}}$ and thus $\mathcal{I} \not\models M \sqsubseteq \bot$.

Claim IV: $R_1 \sqsubseteq R_2 \in cl(\mathcal{T})$ iff $\mathcal{I} \models R_1 \sqsubseteq R_2$ Consider the "only if" direction. Recall the auxiliary Claim from Lemma 1, that $R_1 \sqsubseteq R_2 \in cl(\mathcal{T})$ implies the existence of a chain $c = R_1 \sqsubseteq R_1' \sqsubseteq \cdots \sqsubseteq R_k' \sqsubseteq R_2 \in \mathcal{T}^*$. Consider now $(d, d') \in R_1^{\mathcal{I}}$. We make a case distinction between types of d, d', based on the definition of $R_1^{\mathcal{I}}$.

- If $d = d_o$ and $d' = d_{o'}$ for some $o, o' \in \mathbf{I}$, then $R_1(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$ by the definition of $R_1^{\mathcal{I}}$ and Rule A3. By the application of Rule A2 to c, we obtain $R_2(o,o') \in cl_{\mathcal{T}}(\mathcal{A})$, and thus also

$(d_o, d_{o'}) \in R_2^{\mathcal{I}}$.

- If $d = d_o$ and $d' = d_{\exists R^-}$ for some $o \in \mathbf{I}$ and basic role R, then $\exists R(o) \in cl_{\mathcal{T}}(\mathcal{A})$ and $R \sqsubseteq R_1 \in cl(\mathcal{T})$ and there is no $o' \in \mathbf{I}$ with $R(o, o') \in cl_{\mathcal{T}}(\mathcal{A})$. Since $R_1 \sqsubseteq R_2 \in cl(\mathcal{T})$, by Rule T2 we also have $R \sqsubseteq R_2 \in cl(\mathcal{T})$, and thus $(d_o, d_{\exists R^-}) \in R_2^{\mathcal{I}}$.
- If $d = d_M$ and $d' = d_N$, where N is a conjunction of existential restrictions $\exists R^-$ with $M \sqsubseteq \exists R$ and $R \sqsubseteq R_1$ in $cl(\mathcal{T})$, then by Rule T2, we again obtain $R \sqsubseteq R_2 \in cl(\mathcal{T})$, and thus also $(d_M, d_N) \in R_2^{\mathcal{I}}$.
- The remaining two cases are identical to the previous two cases, since we also have $R_1^- \sqsubseteq R_2^- \in cl(\mathcal{T})$ by Rule T3.

For the "if" direction, we know that $(d_{\exists R_1}, d_{\exists R_1^-}) \in R_1^{\mathcal{I}}$ by Rules T1 and T4. Thus, $\mathcal{I} \models R_1 \sqsubseteq R_2$ implies $(d_{\exists R_1}, d_N) \in R_2^{\mathcal{I}}$. By the definition of $R_2^{\mathcal{I}}$, this means that we either directly have $R_1 \sqsubseteq R_2 \in cl(\mathcal{T})$, or $R_1^- \sqsubseteq R_2^- \in cl(\mathcal{T})$, which also implies the former axiom by Rule T3.

Claim V: *funct* $R \in cl(\mathcal{T})$ implies $\mathcal{I} \models$ *funct* R

For all $o \in \mathbf{I}$, since $\perp(o) \notin cl_{\mathcal{T}}(\mathcal{A})$, there is at most one assertion $R(o, o') \in \mathcal{A}$ (i.e. Rule A1 did not apply). Thus, $d_{o'}$ is only named $R^{\mathcal{I}}$-successor of d_o by definition. If there is another pair $(d_o, d') \in R^{\mathcal{I}}$, then d' must be of the form $d_{\exists R'^-}$. Since R does not occur on the right-hand side of any axiom besides $R \sqsubseteq R$ and $\exists R \sqsubseteq \exists R$, this implies that $R' = R$ and for all $o^* \in \mathbf{I}$: $R(o, o^*) \notin cl_{\mathcal{T}}(\mathcal{A})$ according to the definition of $R^{\mathcal{I}}$. Subsequently, either $(d_o, d_{\exists R^-})$ or $(d_o, d_{o'})$ is the only pair of elements in $R^{\mathcal{I}}$ for any $o \in \mathbf{I}$, which shows that $R^{\mathcal{I}}$ is functional.

Suppose now there is $(d_M, d_N) \in R^{\mathcal{I}}$ s.t. $N = \bigsqcap_{i=1}^k \exists R_i^-$ where $M \sqsubseteq \exists R_i^-$, and $R_i \sqsubseteq R$ for all $1 \leq i \leq k$. Since R is functional, by T1 and T4 we only have $R \sqsubseteq R$ and $\exists R \sqsubseteq \exists R$ where R is only the right-hand side. Thus, M, N can only be $\exists R$ and $\exists R^-$, respectively, which also satisfies the claim.

By Claims I–V, $\mathcal{I}$ is a model of $\mathcal{K}'$ and thus is a model of $\mathcal{K}$. Now, if $\mathcal{K} \models \alpha$, then $\mathcal{I} \models \alpha$ (and $\mathcal{I}' \models \alpha$), which implies $\alpha \in cl_{\mathcal{T}}(\mathcal{A})$. Similarly, if $\mathcal{K} \models \tau$, then unless $\tau =$ *funct* R, we have $\mathcal{I} \models \tau$, which implies $\tau \in cl(\mathcal{T})$. For $\tau =$ *funct* R, we know by our assumption on $\mathcal{T}$ that $\mathcal{K} \models$ *funct* R implies *funct* $R \in \mathcal{T} \subseteq cl(\mathcal{T})$. ☐

Lemma 5. *For every basic assertion $X(\vec{o})$, the following hold:*

1. $\mathcal{R}_{\mathcal{T}}, D \models AOrApCl\_X(\vec{o})$ *iff* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{ins})$.
2. $\mathcal{R}_{\mathcal{T}}, D \models ApCl\_X(\vec{o})$ *iff* $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A}_{\mathcal{U}}^{ins})$.

Proof. Claim 1: In the "only if" direction, we show the claim via induction on m, where m is the least number of applications of $T_{\mathcal{R}_{\mathcal{T}}}$ that infer $AOrApCl\_X(\vec{o})$.

In the inductive base, the claim is trivially satisfied since $\mathcal{R}_{\sqcap}, D \not\models AOrApCl\_X(\vec{o})$ for $m = 0$. In the inductive step, the inductive assumption is that up to n, $AOrApCl\_X'(\vec{o}) \in T_{\mathcal{R}_{\mathcal{T}}}^n(D)$ implies $X'(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$.

For $m = n+1$, $AOrApCl\_X(\vec{o}) \in T_{\mathcal{R}_{\mathcal{T}}}^{n+1}(D)$ implies that it is derived from either Rules (1), (2), or (3), (4).

If Rule (1) or (2) is applied, then $X(\vec{o})$ or $ApCl\_X(\vec{o})$ are in D, which implies the membership of $X(\vec{o})$ in either $\mathcal{A}$ or $\mathcal{A}_{\mathcal{U}}^{\text{ins}}$ by definition of D. Hence, $X(\vec{o})$ $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$.

If Rule (3) is applied, then $AOrApCl\_R(o, o') \in T_{\mathcal{R}_{\mathcal{T}}}^n(D)$. Since $X(\vec{o})$ is either $\exists R(o)$, $\exists R^-(o')$, or $R^-(o', o)$, the claim follows directly from the inductive assumption.

If Rule (4) is applied, there are $\bigsqcap_{i=1}^k X_i \sqsubseteq X \in \mathcal{T}^*$ and $AOrApCl\_X_i(\vec{o}) \in T_{\mathcal{R}_{\mathcal{T}}}^n(D)$ for all $1 \leq i \leq k$. By the inductive assumption, we have $X_i(\vec{o}) \in cl_{\mathcal{T}}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$. Thus, either Rule A2 or A5 from Figure 1 can be applied to obtain $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$.

In the "if" direction, we do induction on the number of rule applications m from Figure 1 for deriving $X(\vec{o})$ from $\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}}$.

In the inductive base, $m = 0$ implies $X(\vec{o}) \in \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}}$. Thus, $X(\vec{o}) \in D$ or $Ap\_X(\vec{o}) \in D$. Hence, either Rule (2) or Rule (1) derives $AOrApCl\_X(\vec{o})$. In the inductive step, assume that up to n, $X'(\vec{o})$

derived in n rounds of rule applications from Figure 1 implies that $\mathcal{R}_\mathcal{T}, D \models AOrApCl\_X'(\vec{o})$. We show the claim for $X(\vec{o})$ derived after $n+1$ rule applications to $(\mathcal{A} \cup \mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ via a case distinction of the last rule applied. It suffices to consider only rules A2–A5, as $\mathcal{T}^+$ contains only positive axioms. We use the auxiliary results below to show the rest of the claim:

Claim. *For each $pred \in \{ApCl, AOrApCl\}$, if $\bigsqcap_{i=1}^k X_i \sqsubseteq X \in \mathcal{T}^*$ and $\mathcal{R}_\mathcal{T}, D \models pred\_X_i(\vec{o})$ for all $1 \leq i \leq k$, then $\mathcal{R}_\mathcal{T}, D \models pred\_X(\vec{o})$*

Proof of claim. We do induction on the number m of rule applications from Figure 1 that derives $\bigsqcap_{i=1}^k X_i \sqsubseteq X$ in $\mathcal{T}^*$.

In the inductive base, $m = 0$ implies $\bigsqcap_{i=1}^k X_i \sqsubseteq X \in \mathcal{T} \subseteq \mathcal{T}^*$. Thus, Rule (4) derives $pred\_B(o)$.

In the inductive step, Assume that for any $\bigsqcap_{i=1}^p X_i' \sqsubseteq X'$ inferred by at most n rule applications from Figure 1, we have $\mathcal{R}_\mathcal{T}, D \models pred\_X_i'(\vec{o})$ for all $1 \leq i \leq p$ implies $\mathcal{R}_\mathcal{T}, D \models pred\_X'(\vec{o})$. If $\bigsqcap_{i=1}^k X_i \sqsubseteq X$ is inferred after $n+1$ rule applications, then it is derived by one of the Rules T1, T3, T4, and T7.

For T1 and T7, the claim is trivial since we either directly have $\mathcal{R}_\mathcal{T}, D \models pred\_B(o)$ or $pred\_\bot(o)$ is impossible in $\mathcal{R}_\mathcal{T}, D$.

For T4, Rule (4) derives $pred\_R_1(\vec{o})$, which implies $\mathcal{R}_\mathcal{T}, D \models pred\_\exists R_1(o)$ by Rule (3). Moreover, since $\exists R_1 \sqsubseteq \exists R_2 \in \mathcal{T}^*$, we again obtain $pred\_\exists R_2(o)$ by Rule (4).

The argument for T3 is similar to that of T4, since $\exists R_1^- \sqsubseteq \exists R_2^- \in \mathcal{T}^*$. ☐

If Rule A2 or A5 is applied, then the claim follows from the auxiliary results and that $\mathcal{R}_\mathcal{T}$ is also defined over $\mathcal{T}^*$.

For A3 and A4, the claim follows from the auxiliary result and the construction of Rule (3).

Claim 2: The proof is analogous to Claim 1, as $ApCl$ is propagated similarly in $\mathcal{R}_\mathcal{T}$. ☐

Lemma 6. *$\mathcal{R}_\mathcal{T}, D \models incompatible()$ iff $\mathcal{U}$ is not compatible with $\mathcal{T}$.*

Proof. We first recall from Theorem 3 that $\mathcal{U}$ is not compatible with $\mathcal{T}$ iff $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$ is inconsistent or $\mathcal{A}_\mathcal{U}^{\mathrm{del}} \cap cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}}) \neq \emptyset$.

In the "only if" direction, we observe that due to the construction of Rules (9), (12), (13), and Lemma 1, it is easy to see that $\mathcal{R}_\mathcal{T}, D \models incompatible()$ implies that one of the following occurs:

(iu1) There is $\bigsqcap_{i=1}^k B_i \sqsubseteq \bot \in cl(\mathcal{T})$ where $\mathcal{R}_\mathcal{T}, D \models ApCl\_B_i(o)$ for all $1 \leq i \leq k$,

(iu2) There is *funct* $R \in \mathcal{T}$ where $\mathcal{R}_\mathcal{T}, D \models \{Ap\_R(o, o'), Ap\_R(o, o'')\}$ and $o' \neq o''$, and

(iu3) There is $X(\vec{o})$ s.t. $\mathcal{R}_\mathcal{T}, D \models \{ApCl\_X(\vec{o}), delCl\_X(\vec{o})\}$.

For (iu1), we have $B_i(o) \in cl_{\mathcal{T}^+}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ by Lemma 5, which implies $B_i(o) \in cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ for all $1 \leq i \leq k$. Thus, (iu1) corresponds to the case where $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$ is inconsistent. Similarly, (iu2) also corresponds to inconsistency of $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$, as R can not be the consequence of other assertions due to *funct* $R \in \mathcal{T}$. Hence, (iu3) corresponds to $\mathcal{A}_\mathcal{U}^{\mathrm{del}} \cap cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}}) \neq \emptyset$.

In the "if" direction, if $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$ is inconsistent, then either (i) *funct* $R \in \mathcal{T}$, $R(o, o')$ and $R(o, o'')$ are in $\mathcal{A}_\mathcal{U}^{\mathrm{ins}}$, and $o' \neq o''$; or (ii) $\bigsqcap_{i=1}^k B_i \sqsubseteq \bot \in cl(\mathcal{T})$ and $B_1(o), \ldots, B_k(o)$ are in $cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$. Thus, (i) and (ii) correspond to (iu1) and (iu2), respectively. Suppose now $\langle \mathcal{T}, \mathcal{A}_\mathcal{U}^{\mathrm{ins}} \rangle$ is consistent and $\mathcal{A}_\mathcal{U}^{\mathrm{del}} \cap cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}}) \neq \emptyset$. Then, this implies that $\mathcal{A}_\mathcal{U}^{\mathrm{del}} \cap cl_{\mathcal{T}^+}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}}) \neq \emptyset$. Thus, by Claim 2 of Lemma 5 and Rule (5), we directly obtain (iu3), which concludes the proof. ☐

Lemma 7. *For $\tau \in \mathcal{T}^*$, we have $\mathcal{R}_\mathcal{T}, D \models min\_X_i\_in\_\tau(\vec{o})$ iff $X_i(\vec{o}) = \min_\tau \left(left(\tau)(\vec{o}) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{ins}) \right)$.*

Proof. Since such facts can only be derived by Rule (10), we have $\mathcal{R}_\mathcal{T}, D \models min\_X_i\_in\_\tau(\vec{o})$ iff $ApCl\_X_1(\vec{o}), \ldots, ApCl\_X_{i-1}(\vec{o})$ are entailed by $\mathcal{R}_\mathcal{T}, D$ while $ApCl\_X_i(\vec{o})$ is not. By Claim 2 of Lemma 5, this is equivalent to $X_1(\vec{o}), \ldots, X_{i-1}(\vec{o}) \in cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$ and $X_i(\vec{o}) \notin cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})$, which in turn is equivalent to $\min(left(\tau)(\vec{o}) \setminus cl_\mathcal{T}(\mathcal{A}_\mathcal{U}^{\mathrm{ins}})) = X_i(\vec{o})$. ☐

Lemma 9. *Let $X(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{ins})$. Then, $\mathcal{R}_{\mathcal{T}}, D \models delCl\_X(\vec{o})$ iff $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X(\vec{o})$.*

Proof. "$\Rightarrow$": Proof by contraposition: Suppose $\langle \mathcal{T}, \mathcal{A}^* \rangle \models X(\vec{o})$, we show that $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_X(\vec{o})$ via induction on the number of rule applications m from Figure 1 to $\langle \mathcal{T}, \mathcal{A}^* \rangle$ that infers $X(\vec{o})$.
Inductive base: $m = 0$ implies $X(\vec{o}) \in \mathcal{A}^*$. Hence, $X(\vec{o}) \notin \mathtt{del}$, meaning that $\mathcal{R}_{\mathcal{T}}, D$ does not entail $del\_X(\vec{o})$ and, subsequently $delCl\_X(\vec{o})$ by Rules (14).
Inductive step: Assume that up to n, $X'(\vec{o})$ inferred in n rounds of rule applications implies that $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_X'(\vec{o})$. Moreover, since $\mathcal{A}^* = (\mathcal{A} \setminus \mathtt{del}) \cup \mathtt{add}$ and $\mathtt{add}$ and $\mathtt{del}$ are disjoint, it is easy to see that $\langle \mathcal{T}, \mathcal{A}^* \rangle \models X'(\vec{o})$ implies $X'(\vec{o}) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$. Suppose now $X(\vec{o})$ is inferred after $n+1$ rounds of rule applications, then it is derived from either A2, A3, A4, or A5.

If A2 is applied, then there is $X'(\vec{o})$ derived in n rounds of rule applications and $\tau = X' \sqsubseteq X \in \mathcal{T}^*$. By the inductive assumption, we have $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_X'(\vec{o})$. Observe that in Rule (11) and we have $delCl\_X'(\vec{o}) \leftarrow delCl\_X(\vec{o}),\ min\_X'\_in\_\tau(\vec{o}),\ AOrApCl\_X'(\vec{o})$. Since $\mathcal{R}_{\mathcal{T}}, D \models min\_X'\_in\_\tau(\vec{o})$, by (iu3) of Lemma 6, we know that $\mathcal{R}_{\mathcal{T}}, D \models delCl\_X(\vec{o})$ would infer $delCl\_X'(\vec{o})$. Thus, $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_X(\vec{o})$.

If A3 is applied, then the claim follows directly from the inductive assumption and Rule (6).

If A4 is applied, then $X(\vec{o}) = \exists R(o)$ for some constant o and $R(o, o')$ is inferred in n rounds of rule applications. By construction of Rule (7), we have $delCl\_R(o, o') \leftarrow AOrApCl\_R(o, o'),\ delCl\_\exists R(o)$. Hence, similar to A2, it follows from the inductive assumption that $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_\exists R(o)$.

If A5 is applied, then $X(\vec{o}) = B(o)$ and there is $\tau = \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$ s.t. $B_i(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$ for all $1 \leq i \leq k$. By construction of Rule (11), for each $i \in \{1, \ldots, k\}$, we have

$$delCl\_B_i(Y) \leftarrow delCl\_X(Y),\ min\_B_i\_in\_\tau(Y),\ AOrApCl\_B_i(Y), \ldots, AOrApCl\_B_k(Y)$$

By Rule (12), Lemma 5, and the assumption that $\mathcal{R}_{\mathcal{T}}, D \not\models incompatible()$, there must be a B_i such that $\mathcal{R}_{\mathcal{T}}, D \models min\_B_i\_in\_\tau(o)$. Hence, analogously to A4 and A2, the rest of the proof for this direction follows directly from the inductive assumption.

"$\Leftarrow$": Proof via induction on the number of rule applications m from Figure 1 that derive $X(\vec{o})$ from $\langle \mathcal{T}^+,\ \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}} \rangle$.
Inductive base: If $m = 0$, then $X(\vec{o}) \in \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}}$. If $X(\vec{o}) \in \mathcal{A}$, then since $\mathcal{A}^* = (\mathcal{A} \setminus \mathtt{del}) \cup \mathtt{add}$ and $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X(\vec{o})$, we must have $X(\vec{o}) \in \mathtt{del}$, which implies $\mathcal{R}_{\mathcal{T}}, D \models delCl\_X(\vec{o})$ due to Rule (14). It is impossible that $X(\vec{o}) \in \mathcal{A}_{\mathcal{U}}^{\text{ins}} \setminus \mathcal{A}$, since that would imply $X(\vec{o}) \in \mathtt{ins}$ by Rule (19), which contradicts the assumption that $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X(\vec{o})$.
Inductive step: The inductive assumption is that up to n, if $X'(\vec{o})$ is inferred in n rounds of rule applications (w.r.t. $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}} \rangle$) and $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models X'(\vec{o})$, then $\mathcal{R}_{\mathcal{T}}, D \models delCl\_X'(\vec{o})$. Below, we suppose the contrary that $\mathcal{R}_{\mathcal{T}}, D \not\models delCl\_X(\vec{o})$ and show that this leads to a contradiction. Consider the derivation of $X(\vec{o})$ in round $n+1$, which implies the application of either rule A2, A3, A4, or A5.

For A5, we have $X(\vec{o}) = B(o)$ for some basic concept B and there is $\tau = \bigsqcap_{i=1}^{k} B_i \sqsubseteq B \in \mathcal{T}^*$ where $B_i(o)$ is derived in at most n rounds of rule applications, for all $1 \leq i \leq k$. By Lemma 5, $B_i(o) \in cl_{\mathcal{T}^+}(\mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\text{ins}})$ implies $\mathcal{R}_{\mathcal{T}}, D \models AOrApCl\_B_i(o)$ for all $1 \leq i \leq k$. Since $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models B(o)$, there is a B_i such that $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models B_i(o)$. By the inductive assumption, we have $\mathcal{R}_{\mathcal{T}}, D \models delCl\_B_i(o)$.

Recall that $\mathcal{R}_{\mathcal{T}}$ contains Rule (15) and (16):

$$preInsCl\_\tau(Y) \leftarrow AOrApCl\_B_1(Y), \ldots, AOrApCl\_B_k(Y),\ \neg delCl\_B(Y)$$
$$insCl\_B(Y) \leftarrow delCl\_B_i(Y),\ preInsCl\_\tau(Y)$$

Thus, we have $\mathcal{R}_{\mathcal{T}}, D \models preInsCl\_\tau(o)$ and $\mathcal{R}_{\mathcal{T}}, D \models insCl\_B(o)$. Hence, $\mathcal{R}_{\mathcal{T}}, D \models ins\_B(o)$ by Rule (18), which means that $B(o) \in \mathtt{ins} \subseteq \mathcal{A}^*$, which is again a contradiction.

For A2, the argument is analogous to that of A5.

For A3, we have $X = R^-(o', o)$ and that $R(o, o')$ is derived in n rounds of rule applications from Figure 1. By the inductive assumption, we have $\mathcal{R}_{\mathcal{T}}, D \models delCl\_R(o, o')$, from which we obtain $delCl\_R^-(o', o)$ via Rule (6).

Last, if A4 is applied, then there is $R(o, o')$ derived in n rounds of rule application from Figure 1 with $X = \exists R$. Thus, $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models \exists R(o)$ implies $\langle \mathcal{T}, \mathcal{A}^* \rangle \not\models R(o, o')$. Hence, by the inductive assumption, we have $\mathcal{R}_{\mathcal{T}}, D \models delCl\_R(o, o')$. Next, recall Rule (17):

$$insCl\_\exists R(Y_1) \leftarrow delCl\_R(Y_1, Y_2),\ AOrApCl\_R(Y_1, Y_2),\ \neg delCl\_\exists R(Y_1)$$

Since $\langle \mathcal{T}^+, \mathcal{A} \cup \mathcal{A}_{\mathcal{U}}^{\mathrm{ins}} \rangle \models R(o, o')$, we have $\mathcal{R}_{\mathcal{T}}, D \models AOrApCl\_R(o, o')$ by Claim 1 of of Lemma 5. The rest of the argument is similar to that of A5. $\square$