Introducing Prism Embeddings: A New Family of Geometric Ontology Embeddings

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Abstract

One way of enhancing link prediction in knowledge graphs in an interpretable and trustworthy way is to use ontology embeddings. These embeddings translate a knowledge graph together with its background ontology into a vector space by representing concepts as convex sets and logical operators between these concepts as geometric operations between the resp. sets. This allows for using both geometric regularities and background knowledge for learning. Some fragments of the description logic $\mathcal{EL}^{++}$ such as $\mathcal{ELHO}(\circ)^{\perp}$ are particularly well-suited as a basis for an embedding, as they offer a good trade-off between expressiveness and complexity. One popular approach for embedding these ontologies is to interpret concepts as boxes in some $\mathbb{R}^n$. Although box embeddings proved to be particularly useful in this context, they are not able to represent every ontology correctly. In this work, we open the door to a new geometric framework by introducing prism embeddings. Firstly, we show that prisms do not suffer from the same restrictions as boxes. To illustrate this advantage, we present concrete ontology examples that are problematic for box embeddings but can be represented using prisms. Secondly, we show that prism embeddings extend box embeddings in the sense that each box interpretation of an $\mathcal{ELHO}(\circ)^{\perp}$ ontology induces a prism interpretation, though possibly at the cost of an increase in dimensionality of at most two. Finally, we discuss the usage of prism embeddings in practice by sketching adaptations of implementations of box-based embedding approaches to the prism case.

Keywords

Ontology Embeddings, Boxes, Triangular Prisms, Description Logics, Neuro-symbolic Reasoning

1. Introduction

Ontologies as a structured representation of knowledge are of increased importance as a means to provide background knowledge information for subsymbolic learning approaches. This increases the interpretability and trustworthiness of these approaches and paves the way towards neurosymbolic learning and reasoning approaches. For this purpose, the given data, e.g., in the form of a knowledge graph is interpreted as an Abox of an ontology, while its Tbox represents the background knowledge. Knowledge graphs, especially large-scale real-world ones such as DBpedia [1] and Wikidata [2] or other collections of facts tend to be highly incomplete. A learning approach now aims to infer the missing facts. These need to be in line with the ontology and thus with the specifics of world knowledge. Therefore, this inference task can be interpreted as the search for a model of the ontology. However, not every model of the ontology is equally suitable for modeling inferred information, as the inferred facts not only need to be ontologically correct, but also grounded in the actual world. Therefore, a neurosymbolic method is needed: The basic idea is to equip the model with subsymbolic information on similarity of concepts and individuals by considering an interpretation of the ontology over the domain of a real-valued vector space. Following otherwise the classical interpretation, individuals are then represented as points in this space, concepts as regions, and relations map points to points. This equips the model of the ontology with measures for the similarity of individuals or concepts and with the notion of convexity. In contrast to purely symbolic similarity measures, such as in [3], here subsymbolic similarity information is also used. This allows for representing concepts as convex objects. This is

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both cognitively grounded (e.g., based on Gärdenfors' Conceptual Spaces [4]) and computationally motivated. In this context, several different approaches emerged, representing concepts, e.g., as spheres [5], axis-aligned hyperrectangles [6, 7, 8, 9] or closed convex cones [10]. With the help of loss functions, a neural network is trained that maps individuals and concepts of the ontology to the vector space. These approaches rest on the assumption that similar concepts/similar individuals are mapped to similar places. Then, on the basis of this similarity, new facts can be inferred. There is a major tradeoff between learnability and representational capabilities: restricting the space of possible interpretations to those representing concepts as convex objects in a vector space leads to the fact that not necessarily all satisfiable ontologies have a model based on the specific representation. Or, to state it differently: there are ontologies that have a classical model but no model in this geometric semantics.

We are considering the $\mathcal{ELHO}(\circ)^{\perp}$-fragment due to its computational properties, as the subsumption is polynomial. Prominent examples of ontologies in $\mathcal{ELHO}(\circ)^{\perp}$ are, e.g., SNOMED [11] for clinical documentation and the Gene Ontology [12] for modeling genes and their interactions. For $\mathcal{ELHO}(\circ)^{\perp}$, box interpretations turned out to be particularly useful due to their good computational properties and, therefore, good learnability. These interpretations are learned based on a box embedding framework. These have proven to be useful in several practical use cases. Examples are next to knowledge base embeddings, query embeddings [13], hierarchy embeddings [14] and others. However, in [15], it has been shown that interpretations based on boxes suffer from a severe restriction: boxes fulfill the *Helly Property* [16] describing that if three boxes are pairwise intersecting, then all three are intersecting. It is easily possible to imagine a real-world ontology not being in line with this: assume, e.g., that the intake of each two out of three drugs is unproblematic, but all three together have severe side effects. A precise esteem on how often this problem arises in real world data is subject to future work, however, the simplicity of the property leads to the assumption that this property has practical relevance in many scenarios. For a visualization of this property, see Figure 2 (a,b).

Therefore, the aim of this paper is to present an interpretation based on a geometric structure that is as similar to boxes as possible to have (i) equally good computational properties and (ii) allow for reusing existing box embedding approaches with only small adaptations. For this purpose, we are using prisms. Concretely, we are looking at a Cartesian product of an axis-aligned equilateral triangle with intervals. A three-dimensional example can be seen in Figure 1 (b). It turns out that interpretations based on prisms are able to represent ontologies that are not in line with the Helly Property. We show that this does not come to the cost of expressivity: we prove that each ontology that has a model based on boxes in $\mathbb{R}^n$ also has a model based on prisms in at most $\mathbb{R}^{n+2}$, thus without a significant increase in dimensionality. Additionally, we sketch how existing box embedding approaches can be modified to be based on prisms, thus paving the way towards the practical application of these prism interpretations.

The paper is structured as follows: In Section 2, we give a short overview of the description logic $\mathcal{ELHO}(\circ)^{\perp}$. After that, in Section 3, an overview on boxes, box interpretations, and their restrictions is given, followed by the main part of the work, the introduction of prism interpretations in Section 4. In Section 5, the expressivity of these interpretations is discussed, both based on the Helly Property and their connection to box interpretations. The paper ends with a sketch on how existing box embedding approaches can be easily adapted to model prism embeddings in Section 6 and with a short conclusion and outlook in Section 7.

2. The Description Logic $\mathcal{ELHO}(\circ)^{\perp}$

During this work we consider the $\mathcal{ELHO}(\circ)^{\perp}$-fragment and the $\mathcal{EL}_{\perp}$-fragment of the well-known description logic $\mathcal{EL}^{++}$ [17]. For general details on syntax and semantics of description logics, see [18].

A DL vocabulary is given by a set of individual names $\mathbf{I}$, a set of role names $\mathbf{R}$ and concept names $\mathbf{C}$. The $\mathcal{ELHO}(\circ)^{\perp}$ concepts over $\mathbf{C} \cup \mathbf{R} \cup \mathbf{I}$ are written according to the syntax rules

$$C \longrightarrow A \,|\, \{a\} \,|\, \bot \,|\, \top \,|\, C \sqcap C \,|\, \exists R.C$$

where $A \in \mathbf{C}$, $a \in \mathbf{I}$, $\{a\}$ denotes a nominal concept, $R \in \mathbf{R}$ and C stands for arbitrary concept

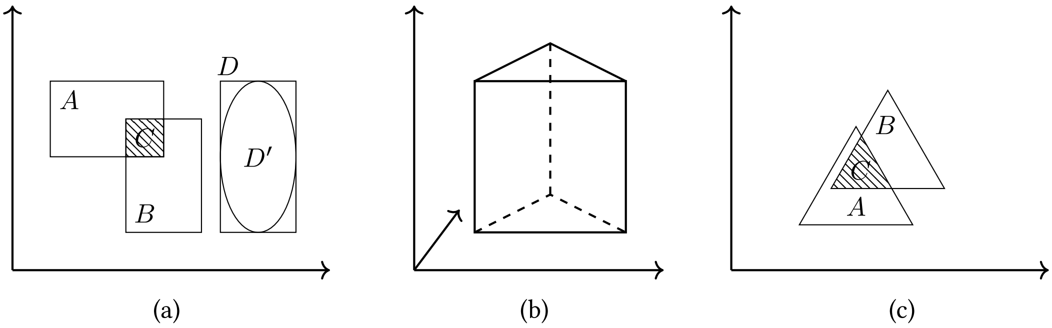


Figure 1: (a) A, B, C and D are examples for axis-aligned boxes. The intersection of two axis-aligned boxes (e.g., A and B) results always in an axis-aligned box (in this case C). $D = BoxHull(D')$, thus D is the box hull of D'; (b) An example for a three-dimensional triangular prism; (c) A, B and C are examples for axis-aligned equilateral triangles: The intersection of two of them results always in an axis-aligned equilateral triangle.

Name	Syntax	Semantics
top	$\top$	Δ
bottom	$\bot$	$\emptyset$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \in \Delta \mid \exists y \in \Delta : (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\}$
role concatenation	$(R_1 \circ R_2)^{\mathcal{I}}$	$\{(a,c) \mid \exists b \in \Delta : (a,b) \in R_1^{\mathcal{I}}, (b,c) \in R_2^{\mathcal{I}}\}$

Table 1
Syntax and semantics of $\mathcal{ELHO}(\circ)^{\bot}$ [17]

descriptions. A terminological box (Tbox) $\mathcal{T}$ consists of general inclusion axioms (GCIs) $C \sqsubseteq D$ with C, D being concept descriptions. An assertional box (Abox) $\mathcal{A}$ consists of individual assertions, namely facts of the form $a : C$ or $(a, b) : R$ with $a, b \in \mathbf{I}$, $R \in \mathbf{R}$ and C being a concept description. The Rbox $\mathcal{R}$ consists of *role inclusions* $R_1 \circ \cdots \circ R_k \sqsubseteq R$ and *role hierarchies* $R_1 \sqsubseteq R$. The $\mathcal{EL}_{\bot}$-fragment is a fragment of $\mathcal{ELHO}(\circ)^{\bot}$, neither allowing for nominals nor for role concatenations and role hierarchies. An ontology $\mathcal{O}$ is a triple of the form $(\mathcal{T}, \mathcal{R}, \mathcal{A})$.

The semantics of arbitrary concept descriptions for a given interpretation $\mathcal{I}$ is given in Table 1 (details can be found in [17]). In this paper, the focus lies on finite ontologies.

3. Box Embeddings for Ontology Completion

The aim of this work is to introduce prism interpretations as a variant of box interpretations with increased representative abilities. In this section, we describe the geometric properties of boxes, introduce boxes as basis of an interpretation of ontologies and discuss the restrictions of these interpretations. Real-world box embedding approaches based on box interpretations are discussed in Section 6.

3.1. Boxes

A **box** in some $\mathbb{R}^n$, for $n \in \mathbb{N}$, is defined as a Cartesian product of intervals $I_1 \times \cdots \times I_n$, where each $I_i \subseteq \mathbb{R}$ is a closed interval or a half line or the whole $\mathbb{R}$. Let $BoxHull(A)$ be the smallest box containing all elements of a set A. This can be defined as

$$BoxHull(A) = \{(x_1, \ldots, x_n)^T \mid x_i \in ConvHull(\{a_i \mid a \in A\}) \text{ for } 1 \leq i \leq n\}$$

where $ConvHull(X)$ is the convex hull of X and a_i is the value at the i-th dimension of the vector a. Let the set $\mathcal{B}^n = \{BoxHull(X) \mid X \subseteq \mathbb{R}^n\}$ be the set of all boxes in $\mathbb{R}^n$ including the whole space $\mathbb{R}^n$ and the empty set. One main advantage of axis-aligned boxes is their closure under intersection. The intersection of two axis-aligned boxes A and B is defined as follows:

$$A \cap B := I_1^A \cap I_1^B \times \cdots \times I_n^A \cap I_n^B.$$

In Figure 1 (a), an example of such an intersection of two boxes can be seen. Note here that arbitrary dimensional boxes are considered and that the examples are restricted to the two-dimensional case for representation reasons only.

3.2. Box Embeddings of Ontologies

Now, assume that an $\mathcal{ELHO}(\circ)^{\perp}$-ontology $\mathcal{O} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ is given. Its box embedding is based on a (classical) model of this ontology under some restrictions. Concepts are represented as boxes in some $\mathbb{R}^n$. As the domain equals the representation of the top-concept and is thus the representation of the largest concept, it can be represented as an n-dimensional box in $\mathbb{R}^n$ such that all other concept representations are subsets. As boxes are not necessarily finite, this is a generalization of representing the domain as $\mathbb{R}^n$.

Individuals are represented as points, and nominals as special type of boxes that include only one point. The relations are defined as general as possible. They are, however, not allowed to be arbitrary: as each concept needs to be represented as a convex object, also each pre-image of a box needs to result in a box (e.g., the representation of $\exists R.C$).

The following definition follows in principle the one in [15, Def.1], however, focuses on the fact that box interpretations are a special case of classical interpretations.

Definition 1. *A classical interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ is a* ***box interpretation*** *if*

- $\Delta \in \mathcal{B}^n$ *for some* $n \in \mathbb{N}$,
- $\cdot^{\mathcal{I}}$ *maps*
  - *each concept name* $A \in \mathbf{C}$ *to some box in* $\mathcal{B}^n$
  - *each nominal concept* $\{c\}$ *to the box* $\{c^{\mathcal{I}}\}$
  - *each role* $R \in \mathbf{R}$ *to a subset* $R^{\mathcal{I}} \subseteq \Delta \times \Delta$ *such that for every* $B \in \mathcal{B}^n$: $R^{\mathcal{I}^{-1}}(B) \in \mathcal{B}^n$.

For ease of discrimination, a box interpretation is denoted with ξ on the structure $(\Xi, \cdot^{\xi})$.

At first glance, it could seem that, in the role interpretation requirement, $\mathcal{B}^n$ is a too large set, and that we should consider only the boxes contained in Δ. However, since the intersection of two boxes is a box, the set of all boxes contained in Δ is exactly $\{B \cap \Delta : P \in \mathcal{B}^n\}$.

It is important to note that this definition of box embeddings is one focused on its relation to classical interpretations. For application settings, thus for the definition of actual embedding approaches, more flexible (but less classical) definitions are used. To avoid overfitting and to ease learning, it is, e.g., often not the case that arbitrary relations (as defined in Definition 1) are modeled but restricted ones that can be, e.g., modeled as translations. Whereas these would still be interpretable as classical interpretations, there are also many embedding approaches that change basic principles of interpretations, e.g., modeling a conjunction not as an intersection.

For interpretations as defined in Definition 1, by definition it is trivially the case that the existence of such a model of an ontology $\mathcal{O}$ implies that the ontology is satisfiable. As shown in [19], this is not the case for many real-world box embeddings, as they differ too widely from this classical interpretation. However, we focus on this definition because it is the most suited one for theoretical considerations and can be used as a starting point for relaxations to cover real-world approaches.

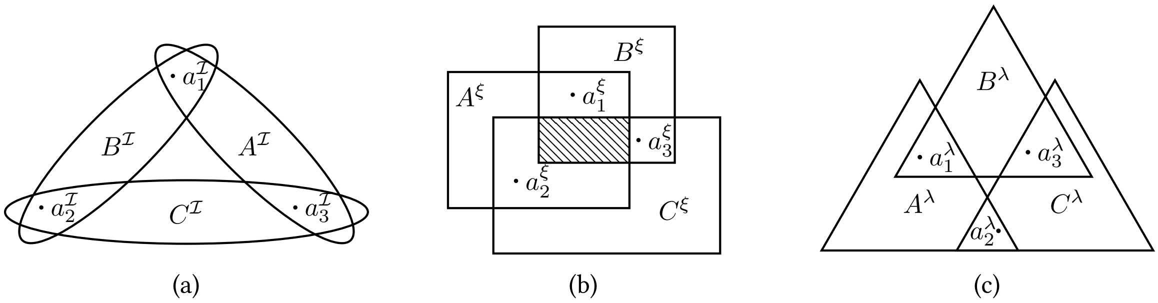


Figure 2: (a) a classical model of the ontology given in Example 3; (b) a box interpretation not fulfilling this ontology, exemplifying the Helly Property; (c) a prism model of the ontology, showing enhanced representation abilities compared to boxes.

3.3. Restrictions of Box Embeddings

Due to the fact that each box interpretation is a classical interpretation it is the case that the existence of a box model for an $\mathcal{ELHO}(\circ)^\perp$-ontology $\mathcal{O}$ implies satisfiability of $\mathcal{O}$. However, not every $\mathcal{ELHO}(\circ)^\perp$ ontology $\mathcal{O}$ has a box model. This has been shown in [15] and is based on the so-called "Helly Property".

Definition 2 (Helly Property, adapted from [20])**.** *A family B fulfills the **Helly Property** if it is the case that: $\bigcap_{b \in B} b \neq \emptyset$ if and only if for all $b_1, b_2 \in B$: $b_1 \cap b_2 \neq \emptyset$.*

Using the fact that boxes fulfill the Helly Property [20], it has been shown in [15] that there exists a classically satisfiable $\mathcal{ELHO}(\circ)^\perp$ ontology $\mathcal{O}$ such that no box interpretation in $\mathbb{R}^n$, for arbitrary $n \in \mathbb{N}$, satisfies $\mathcal{O}$. An example of such an ontology is given in the following.

Example 3. *Let $\mathcal{O} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ with $\mathcal{T} = \{A \sqcap B \sqcap C \sqsubseteq \bot\}$, $\mathcal{R} = \{\}$ and $\mathcal{A} = \{a_1 : A \sqcap B, a_2 : B \sqcap C, a_3 : A \sqcap C\}$. This ontology is clearly satisfiable (see Figure 2 (a) for a model). However, when trying to construct a box interpretation either one individual is incorrectly assigned or the Tbox axiom is not fulfilled (see Figure 2 (b) for an interpretation not in line with the ontology). This resembles the Helly Property. Note that this does not depend on the dimensionality.*

Whether there exists a box interpretation for all ontologies that do not include patterns showing the Helly Property is still an open problem. The aim of this paper is to find a geometric shape as basis for the embedding that is as close to boxes as possible to keep the computational advantages but to allow for avoiding the restrictions due to the Helly Property.

4. Prism Embeddings

In the previous section, we discussed that the Helly property represents a strong limitation of box semantics. As we can see in Figure 2 (c), simple shapes, such as axis-aligned triangles, do not suffer from the same restrictions. The idea is to define a class of interpretations that preserves the good properties of the box ones and that mitigates their geometrical constraints. To do that, we introduce embeddings where each concept is represented as the Cartesian product of one axis-aligned triangle and one arbitrary dimensional box.

In this section, we first define these geometrical objects that we call "prisms", then we show that like boxes these objects are closed under intersection and thus can act as the basis for an embedding approach that we define in Section 4.2.

4.1. Prisms

We define the objects that we call "prisms" as the Cartesian product of an equilateral axis-aligned triangle contained in $\mathbb{R}^2$ and an m-dimensional box. Hence a prism lives in a $\mathbb{R}^n$ space where $n = m + 2$. The first component, the triangular one, changes the combinatorial properties of the boxes, freeing them from the Helly Property. Moreover, in doing that, triangles do not compromise the closure under intersection.

Definition 4. *An* ***axis-aligned equilateral triangle*** T *is either a point in* $\mathbb{R}^2$ *or the whole of* $\mathbb{R}^2$ *or a closed equilateral triangle in* $\mathbb{R}^2$ *with one side parallel to the* x*-axis (called "axis-aligned" in the following) and with the top vertex pointing upwards. A* ***prism*** P *in* $\mathbb{R}^n$ *with* $n = 2 + m$*, is a set of the form* $T \times B$*, where* T *is a axis-aligned equilateral triangle and* B *is a box in* $\mathbb{R}^m$*. We denote by* $\mathcal{P}^n$ *the set of all prisms of* $\mathbb{R}^n$*. Notice that this set includes the whole space* $\mathbb{R}^n$ *and the empty set.*

We point out that, in the standard literature, prisms are objects of the form $P \times I \subset \mathbb{R}^3$, where P is a regular polygon and I an interval. Thus, we are doing a slight abuse of notation; on the one hand by calling our sets prisms instead of triangular prisms, and on the other hand by generalizing the notion to an arbitrary $\mathbb{R}^{m+2}$. An important feature of boxes is their closure under intersection. This property, that is essential for representing conjunction, is satisfied also by prisms.

Proposition 5. *If* P_1 *and* P_2 *are elements of* $\mathcal{P}^n$*, then* $P_1 \cap P_2$ *is an element of* $\mathcal{P}^n$*.*

Proof. If $P_1 \cap P_2 = \varnothing$ then it trivially belongs to $\mathcal{P}^n$. Otherwise, we have $P_1 \cap P_2 \neq \varnothing$ and can assume that $P_1 = T_1 \times B_1$ and $P_2 = T_2 \times B_2$. Then P_1, P_2 intersect in every dimension, so clearly also $T_1 \cap T_2 \neq \varnothing$ and $B_1 \cap B_2 \neq \varnothing$. By simple geometric reasoning (see Fig. 2 (c)) , one can prove that $T_1 \cap T_2$ is again an axis-aligned equilateral triangle and that $B_1 \cap B_2$ is a box. Hence $P_1 \cap P_2$ is a prism. □

Note here that this is the reason for using axis-aligned and equally oriented equilateral triangles. If one of these three features would be missing, then this closure under intersection would not be the case.

4.2. Prism Embeddings of Ontologies

We are now able to introduce a new family of embeddings based on prisms. We basically proceed as in the case of the boxes in Definition 1, but instead considering prisms. Thus, the domain is some prism, each concept is represented as a prism and the preimage of each prism based on a relation R is again a prism.

Definition 6. *A classical interpretation* $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ *is a* ***prism interpretation*** *if*

- $\Delta \in \mathcal{P}^n$ *for some* $n \in \mathbb{N}$*,*
- $\cdot^{\mathcal{I}}$ *maps*
  - *each concept name* $A \in \mathbf{C}$ *to some prism in* $\mathcal{P}^n$
  - *each nominal concept* $\{c\}$ *to the prism* $\{c^{\mathcal{I}}\}$
  - *each role* $R \in \mathbf{R}$ *to a subset* $R^{\mathcal{I}} \subseteq \Delta \times \Delta$ *such that for every* $P \in \mathcal{P}^n$*:* $R^{\mathcal{I}^{-1}}(P) \in \mathcal{P}^n$*.*

To easily distinguish box and prism interpretations, a prism interpretation is denoted as $\lambda = (\Lambda, \cdot^{\lambda})$. As pointed out after Definition 1, the closure under intersection of prisms guaranties that the previous definition is well posed. We stress that, since it is a classical interpretation, its semantics on general $\mathcal{ELHO}(\circ)^{\perp}$ concepts is the one given in Table 1. Next, we prove that, with this definition, the interpretation of a general $\mathcal{ELHO}(\circ)^{\perp}$ concept is a prism.

Proposition 7. *Let* $(\Lambda, \cdot^{\lambda})$ *be a prism interpretation with* $\Lambda \in \mathcal{P}^n$*. The interpretation of each* $\mathcal{ELHO}(\circ)^{\perp}$ *concept is an element of* $\mathcal{P}^n$*.*

Proof. We proceed by induction on concepts. By definition for all concept names or nominal concepts $A \in \mathbf{C}$ we have that A^{λ} is a prism and if $C \in \{\bot, \top\}$ then C^{λ} is a prism.

If $C := D \sqcap E$ for some (not necessarily atomic) concepts C and D and C^{λ}, D^{λ} are elements of $\mathcal{P}^n$ then by definition $C^{\lambda} = D^{\lambda} \cap E^{\lambda}$. Thus, Proposition 5 implies that $C^{\lambda} \in \mathcal{P}^n$.

If $C := \exists R.D$ for a relation $R \in \mathbf{R}$ and a (not necessarily atomic) concept D and $D^{\lambda} \in \mathcal{P}^n$ we have $(\exists R.D)^{\lambda} = R^{\lambda^{-1}}(D^{\lambda})$. By definition, $R^{\mathcal{I}^{-1}}(P) \in \mathcal{P}^n$ for every $P \in \mathcal{P}^n$, so we have $R^{\lambda^{-1}}(D^{\lambda}) \in \mathcal{P}^n$. □

Since we are working in $\mathcal{ELHO}(\circ)^{\perp}$, it is important to notice that prism embeddings behave well with respect to the composition of relations. It is particularly the case that if $R_1, \dots R_n$ are relations, then for each $P \in \mathcal{P}^n$ we have

$$(R_1 \circ \dots \circ R_n)^{\lambda^{-1}}(P) \in \mathcal{P}^n.$$

This allows for defining interpretations where $(R_1 \circ R_2)^{\lambda} = R_3^{\lambda}$.

5. The Expressivity of Prism Embeddings

In the following, it is shown that prism interpretations serve as a means for avoiding restrictions due to the Helly Property. Additionally, they are generalizations of box interpretations in the sense that each ontology $\mathcal{O}$ that has a box model also has a prism model, as will be shown in Theorem 9. This comes to the cost of a slight increase of two in dimensionality. In Theorem 14 it is shown that there are cases where an increase of the dimensionality of one is unavoidable.

5.1. Prism Interpretations and the Helly Property

As pointed out before, prisms (and particularly the equilateral triangles in its first two dimensions) do not fulfill the Helly Property.

Consider the two-dimensional space and three prism (thus, equilateral triangles) A, B, C. In Figure 2 (c), an example for a construction can be seen where all three triangles are pairwise intersecting but not all three, a contradiction to the Helly Property. It is straightforwardly possible to adapt this construction for arbitrary dimensionality n: Let $A_n = A \times B_A$, $B_n = B \times B_B$ and $C_n = C \times B_C$ with $B_A, B_B, B_C \in \mathcal{B}^n$. As two prisms are disjoint if they are disjoint in the first two dimensions, this construction still interferes with the Helly Property.

Based on this result, it is straightforward to show that there are ontologies that have a prism model but no box model.

Corollary 8. *There exists an ontology $\mathcal{O} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ that has a prism model in $\mathbb{R}^m$ but no box model in $\mathbb{R}^n$ for an arbitrary dimensions m, n.*

Proof. Consider again the ontology $\mathcal{O} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ with $\mathcal{T} = \{A \sqcap B \sqcap C \sqsubseteq \bot\}$, $\mathcal{R} = \{\}$ and $\mathcal{A} = \{a_1 : A \sqcap B, a_2 : B \sqcap C, a_3 : A \sqcap C\}$. As pointed out in [15] and sketched in Example 3 this ontology does not have a box model. However, it is possible to construct a prism model of that ontology, even in $\mathbb{R}^2$. One model of the ontology is depicted in Figure 2 (c). □

5.2. Translating Box Embeddings to Prism Embeddings

We now discuss the relation between box and prism embeddings. We show that the family of prism embeddings extends the family of box embeddings in the following sense: In Theorem 9, it is shown that for each box embedding in $\mathbb{R}^n$ there exists a prism embedding in $\mathbb{R}^{n+2}$ that satisfies exactly the same ontology axioms. In Theorem 14, we discuss the question of whether this increase in dimensionality is necessary. We provide a partial answer to the question by showing that for each $n \in \mathbb{N}$ there is an ontology $\mathcal{O}_n$ that has a box model in $\mathbb{R}^n$ but does not have any prism model in $\mathbb{R}^k$ for any $k \leq n$.

Theorem 9. *For each box interpretation $(\Xi, \cdot^{\xi})$ with $\Xi \in \mathcal{B}^n$ there exists a prism interpretation $(\Lambda, \cdot^{\lambda})$ with $\Lambda \in \mathcal{P}^{n+2}$ such that*

$$\xi \vDash \varphi \iff \lambda \vDash \varphi, \tag{1}$$

for all φ being arbitrary GCIs, individual assertions, or role hierarchy statements expressible in $\mathcal{ELHO}(\circ)^{\perp}$.

Proof. Let $\xi = (\Xi, \cdot^{\xi})$ be a box interpretation with $\Xi \in \mathcal{B}^n$ and let T_0 be a point in $\mathbb{R}^2$ that is by definition also a prism. The idea is to define a prism interpretation $(\Lambda, \cdot^{\lambda})$ in $\mathbb{R}^{2+n}$ in which every concept, individual and relation is represented as the Cartesian product of T_0 and their representation in the box interpretation. We let:

$$\Lambda = T_0 \times \Xi, \quad \bot^\lambda = \varnothing, \quad \top^\lambda = T_0 \times \top^\xi, \quad a^\lambda = T_0 \times a^\xi \text{ for all } a \in \mathbf{I}, \quad A^\lambda = T_0 \times A^\xi \text{ for all } A \in \mathbf{C},$$
$$R^\lambda = \{(T_0 \times x, T_0 \times y) : (x,y) \in R^\xi\} \text{ for all } R \in \mathbf{R}.$$

We point out a slight abuse of notation that we make by using the symbol $\times$ between T_0 and a point. For instance, with $T_0 \times a^\xi$ we mean the point $(t_0, t_1, a_0, \ldots, a_n)$ where $T_0 = (t_0, t_1)$ and $a^\xi = (a_0, \ldots, a_n)$ and not the singleton $\{(t_0, t_1, a_0, \ldots, a_n)\}$. To begin with, we have to verify that $(\Lambda, \cdot^\lambda)$ is a prism embedding. For this purpose, we only have to check that for each relation $R \in \mathbf{R}$ the R^λ-preimage of each $P \in \mathcal{P}^{n+2}$ is also an element of $\mathcal{P}^{n+2}$. Let $R \in \mathbf{R}$ and let $P \in \mathcal{P}^{n+2}$, we distinguish two cases. If $P = \varnothing$ then the R^λ-preimage of the empty set is the empty set. Otherwise $P = T \times B$, then we have that the R^λ-preimage of P^λ is $T_0 \times R^{\xi^{-1}}(B)$ that is an element of $\mathcal{P}^n$.

Next, we show that the interpretation λ maps each concept to a prism. For this purpose we prove in the following that

$$C^\lambda = T_0 \times C^\xi \text{ for all general } \mathcal{ELHO}(\circ)^\bot \text{ concepts.} \tag{2}$$

This can be proven by induction on concepts. The case of $A \in \mathbf{C} \cup \{\bot, \top\}$ is immediate from the definition of λ. Under the assumption that (2) holds for concepts D and E, the following two cases need to be considered:

If $C = D \sqcap E$ it follows that $(D \sqcap E)^\lambda = D^\lambda \cap E^\lambda = T_0 \times D^\xi \cap T_0 \times E^\xi = T_0 \times (D \sqcap E)^\xi$. For the case of $C = \exists R.D$, notice that with our definition of λ for each $X \subseteq \Lambda$ and each $R \in \mathbf{R}$, it holds that $R^{\lambda^{-1}}(X) = T_0 \times R^{\xi^{-1}}(X)$. Thus, we have $(\exists R.D)^\lambda = R^{\lambda^{-1}}(D^\lambda) = T_0 \times R^{\xi^{-1}}(D^\xi) = T_0 \times (\exists R.D)^\xi$. Hence (2) has been established and thus $(\Lambda, \cdot^\lambda)$ is a prism interpretation.

Now, we can prove (1). The cases of general inclusion axiom and individual assertion are straightforward from (2) and the definition of λ. If $\varphi = R_1 \circ \cdots \circ R_n \sqsubseteq R$, then we observe that the following is true:

$$(x,y) \in (R_1 \circ \cdots \circ R_n)^\xi \iff (T_0 \times x, T_0 \times y) \in (R_1 \circ \cdots \circ R_n)^\lambda$$

Therefore, we conclude the proof with

$$\xi \vDash R_1 \circ \cdots \circ R_n \sqsubseteq R \iff (R_1 \circ \cdots \circ R_n)^\xi \subseteq R^\xi \iff (R_1 \circ \cdots \circ R_n)^\lambda \subseteq R^\lambda \iff \lambda \vDash R_1 \circ \cdots \circ R_n \sqsubseteq R.$$

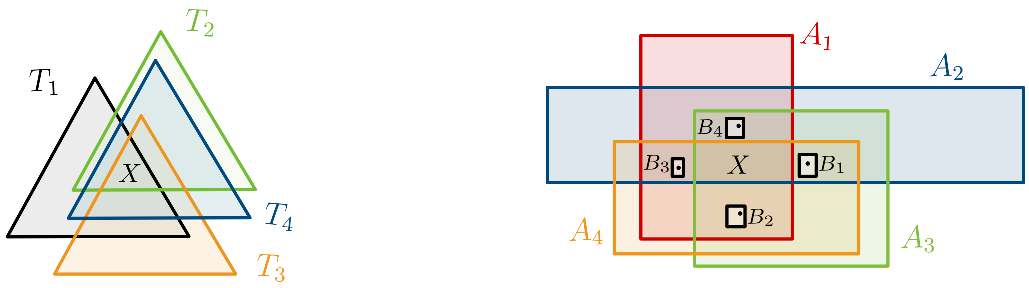
□

Thanks to Corollary 8 and Theorem 9 we know that the set of all ontologies with a prism model is strictly larger than the set of all ontologies with a box model. At this stage, it is natural to ask if the increment in the dimension for prism embeddings is necessary or if it can be avoided with more advanced constructions. In the following Theorem 14, we will give a partial answer to this question, proving that for each n there exists an $\mathcal{ELHO}(\circ)^\bot$ ontology $\mathcal{O}$ that has a box interpretation in $\mathbb{R}^n$ and does not have any prism interpretation in $\mathbb{R}^n$. To establish this result we introduce two combinatorial properties that distinguish the intersection of intervals from the intersection of axis-aligned triangles. We gather this results in the following two lemmas. The first lemma denotes that if the intersection of a family of intervals is non empty then the intersection of all intervals equals the intersection of two intervals of this family.

Lemma 10. *Let $\{J_i\}_{i \in I}$ be a finite family of intervals,*

$$\text{if } \bigcap_{i \in I} J_i \neq \varnothing, \text{ then} \bigcap_{i \in I} J_i = J_{i_1} \cap J_{i_2} \text{ for some } i_1, i_2 \in I.$$

Proof. If $\bigcap_{i \in I} J_i \neq \varnothing$ then $\bigcap_{i \in I} J_i = [a, b]$ for some $a, b \in \mathbb{R}$. Hence there are some intervals J_a, J_b of the family such that $J_a = [a, c]$ and $J_b = [d, b]$ with $c > b, d < a$. Therefore $\bigcap_{i \in I} J_i = J_a \cap J_b$. □



(a) (b)

Figure 3: (a) An example of an intersecting family of four axis-aligned triangles, where the intersection of the family coincides with $T_1 \cap T_2 \cap T_3$. (b) A box model of the ontology $\mathcal{O}$ in the proof of Lemma 12, depicting an ontology representable with a box model in $\mathbb{R}^2$ but not with a prism model in $\mathbb{R}^2$.

This particularly implies that each box in $\mathbb{R}^2$ can be represented as an intersection of at most four boxes (for details, see the proof of Lemma 13). In contrast, whenever an axis-aligned triangle can be written as the intersection of a family of axis-aligned triangles, it can also be written as the intersection of at most three triangles of the family.

Lemma 11. *Let $\{T_i\}_{i\in I}$ be a finite family of axis-aligned triangles,*

$$\text{if } \bigcap_{i\in I} T_i \neq \varnothing, \text{ then } \bigcap_{i\in J} T_i = \bigcap_{i\in I} T_i \text{ for some } J \subseteq I \text{ and } |J| \leq 3.$$

Proof. Let $T = \bigcap_{i\in I} T_i$ be non empty. Notice that T is an axis-aligned triangle, in particular it has sides s_1, s_2 and s_3 that lie on lines l_1, l_2 and l_3 respectively. This implies that there are three elements of the family T_1, T_2 and T_3 such that each of them has one side contained in one of the l_i. Therefore $T = T_1 \cap T_2 \cap T_3$. An intuition of this construction can be seen in Figure 3 (a). □

This difference is the basic reason for the higher dimension needed for prism interpretations: Roughly, it is the idea to construct an ontology that defines a concept X as intersection of (in the case of two dimensions) four concepts $A_1, \ldots, A_4$. All of these concepts should influence the concept X, thus, e.g., X should be strictly smaller than $A_1 \sqcap A_2 \sqcap A_3$. Due to Lemma 10 this is representable with the help of boxes but due to Lemma 11 this is not the case for prisms.

In the following, we will state our results using the description logic $\mathcal{EL}_\perp$, namely the sub-fragment of $\mathcal{ELHO}(\circ)^\perp$ without nominal concepts, composition of relations and hierarchies for relations to point out that this construction is not dependent on any special features of $\mathcal{ELHO}(\circ)^\perp$. Notice that each $\mathcal{EL}_\perp$ ontology is also a $\mathcal{ELHO}(\circ)^\perp$ ontology, and so the following results also hold for $\mathcal{ELHO}(\circ)^\perp$. The next lemma proves the existence of an $\mathcal{EL}_\perp$ ontology that has a box model in $n = 2$ but no prism model. This will be used as a basis for the proof of the general case in Theorem 14.

Lemma 12. *There exists an $\mathcal{EL}_\perp$ ontology $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ that has a box model in $\mathbb{R}^2$ but not a prism model in $\mathbb{R}^2$.*

Proof. Let $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ be the $\mathcal{EL}_\perp$ ontology with $\mathcal{A} := \{b_i : B_i\}_{i\in\{1,2,3,4\}}$ and

$$\mathcal{T} := \Big\{\bigsqcap_{i=1}^{4} A_i \equiv X\Big\} \cup \Big\{B_i \sqcap X \sqsubseteq \perp, B_i \sqsubseteq \bigsqcap_{j\neq i} A_j\Big\}_{i\in\{1,2,3,4\}}.$$

In Figure 3 (b) we see a box model for $\mathcal{O}$. We now prove that there are no prism models of $\mathcal{O}$ in $\mathbb{R}^2$. Assume by contradiction that λ is a prism model of $\mathcal{O}$. Then A_i^λ for $i \in \{1,2,3,4\}$ are axis-aligned triangles in $\mathbb{R}^2$ and their intersection is the triangle X^λ. By Lemma 12, we can assume wlog that $X^\lambda = A_1^\lambda \cap A_2^\lambda \cap A_3^\lambda$. By assumption $B_4^\lambda \subseteq A_1^\lambda \cap A_2^\lambda \cap A_3^\lambda$ hence $B_4^\lambda \subseteq X^\lambda$. Therefore $b_4^\lambda \in B_4^\lambda \cap X^\lambda$, and therefore $\lambda \not\models B_4 \sqcap X \sqsubseteq \bot$, a contradiction to the assumption. □

To prove the general case, the following lemma is needed. It extends the case of triangles in Lemma 11 to prisms. This is done by combining it with the results of the interval case in Lemma 10 and leads to the fact that every prism in $\mathbb{R}^n$ that is the intersection of a family of prisms, can also be written as the intersection of at most $2n-1$ prisms of the family. As with Lemma 10 it follows that boxes can be represented based on at most $2n$ boxes, this fact directly leads to the increase of the dimensionality of one.

Lemma 13. *Let $\{P_i\}_{i\in I}$ be a finite family of axis-aligned prisms in $\mathbb{R}^n$ with $n = 2 + m$.*

$$\text{If } \bigcap_{i\in I} P_i \neq \varnothing, \text{ then } \bigcap_{i\in I} P_i = \bigcap_{i\in J} P_i \text{ for some } J \subseteq I \text{ and } |J| \leq 2n-1.$$

Proof. We assume that $P_i = T_i \times I_i^1 \times \cdots \times I_i^m$ for each $i \in I$. If $\bigcap_{i\in I} P_i \neq \varnothing$ then $\bigcap_{i\in I} T_i \neq \varnothing$ and $\bigcap_{i\in I} I_i^k \neq \varnothing$ for each $0 \leq k \leq m$. By Lemma 11, we know that there is a subset J of I such that $|J| \leq 3$ and $\bigcap_{i\in I} T_i = \bigcap_{j\in J} T_j$. Similarly, by Lemma 10, we know that for each k with $0 \leq k \leq m$ there is a subset J_k of I such that $|J_k| \leq 2$ and $\bigcap_{i\in I} I_i^k = \bigcap_{j\in J_k} I_j^k$. Let $J^P = J \cup \bigcup_{k\leq m} J_k$ we have that

$$\bigcap_{i\in J} P_i = \bigcap_{i\in J^P} P_i \text{ with } J^P \subseteq I \text{ and } |J^P| \leq 3 + \sum_{k\leq m} |J_k| \leq 3 + 2\cdot m = 2n-1.$$

□

This directly leads to the theorem:

Theorem 14. *For each n there exists an $\mathcal{EL}_\bot$ ontology $\mathcal{O} = (\mathcal{T}, \mathcal{A})$ that has a box model ξ with $\Xi \subseteq \mathbb{R}^n$ but does not have a prism model λ with $\Lambda \subseteq \mathbb{R}^n$.*

Proof. Let $\mathcal{O}_n = (\mathcal{T}_n, \mathcal{A}_n)$ be the $\mathcal{EL}_\bot$ ontology defined analogously to the two-dimensional case in Lemma 12 with $\mathcal{A}_n := \{b_i : B_i\}_{i\in\{1,\dots,2n\}}$ and

$$\mathcal{T}_n := \Big\{\prod_{i=1}^{2n} A_i \equiv X\Big\} \cup \Big\{B_i \sqcap X \sqsubseteq \bot, B_i \sqsubseteq \prod_{j\neq i} A_j\Big\}_{i\in\{1,\dots,2n\}}$$

The construction of a box embedding of $\mathcal{O}_n$ follows trivially from the case of $\mathbb{R}^2$ discussed in Lemma 12.

We now prove that there is no prism model of $\mathcal{O}_n$ in $\mathbb{R}^n$. Assume by contradiction that λ is a model of $\mathcal{O}_n$. Then A_i^λ for $i \in \{1,\dots,2n\}$ are prisms in $\mathbb{R}^n$ and their intersection is the prism X^λ. With Lemma 13, we can assume wlog that $X^\lambda = \bigcap_{i=1}^{2n-1} A_i^\lambda$. By assumption $B_{2n}^\lambda \subseteq \bigcap_{i=1}^{2n-1} A_i^\lambda$ hence $B_{2n}^E \subseteq X^E$. Therefore $b_{2n} \in B_{2n}^\lambda \cap X^\lambda$ and therefore $\lambda \not\models B_{2n} \sqcap X \sqsubseteq \bot$, a contradiction. □

This result shows that prism interpretations have the ability to represent each ontology in a dimension at most two larger than for box interpretations. This increase is marginal compared to usual embedding dimensions of, e.g., 20 or larger. Additionally, at least the construction of the counterexample enforces a seemingly unnatural conjunction of concepts. Whereas this could occur for smaller dimensions, thus, e.g., four concepts in $\mathbb{R}^2$, it seems to be not as relevant for, e.g., twenty dimensions where this special type of intersection for 40 concepts would be necessary. It is future work to figure out whether there are other ontologies that lead to an increased dimension when modeling them with prisms instead of boxes, but for the moment, these results show that prisms have larger representational abilities than boxes to the cost of only a small increase in dimensionality that seems to be restricted to some special ontologies. In the next section, it is pointed out that these prism interpretations can also straightforwardly be used as a basis for prism embedding approaches. Though we are focusing here on $\mathcal{ELHO}(\circ)^\bot$-ontologies, prism embeddings could also be an interesting starting point for considerations on other, more expressive description logics, including, e.g., universal quantification or temporal information.

6. Prisms in the Wild

In the last sections, we have pointed out that prism interpretations exceed box interpretations regarding their representational abilities, at the cost of only a slight increase in dimensionality. In this section, it is discussed how this construction can be used in practice and how it can be experimentally evaluated. Whereas an experimental evaluation is left for future work, we are showing in the following that some of the existing box embedding techniques can be straightforwardly extended to prism embeddings. This directly enables us to use existing implementations with only slight adaptations and thus underlines the usefulness of the prism embeddings. Prisms are based on boxes and for both prisms and for boxes, operations such as intersection and translation are considered dimension-wise (resp. in $\mathbb{R}^2$ for the triangle part). Therefore, existing box embedding approaches can be directly adapted to the prism case by adding a workflow for the first two dimensions, thus the triangle part in $\mathbb{R}^2$. This makes it necessary to define new loss functions for the learning, however, only for the two-dimensional case. Therefore the calculation of, e.g., concept subsumption or intersection is also for triangles only slightly more complex than for boxes. The main question for real-world box embeddings is how to model relations. When relations are modeled as translations or other simple transformations, e.g., as translation combined with scaling as in [6], an adaptation is straightforwardly possible. As prisms are closed under translation, all box embedding approaches that are modeling relations as translations are directly able to be interpreted as prism embeddings. For approaches such as TransBox [9], a definition for the prism case is more complicated: The basic idea of TransBox is to model the relations itself as boxes. Thus, $(a, b) : R$ is represented in the embedding if $b^E \in a^E + R^E$ and $\exists R.C \sqsubseteq D$ is the case if $C^E + R^E \subseteq D^E$. Adapting this to the prism case would mean that the relation is represented as a prism. As, however, equilateral triangles are not closed under pointwise addition, it would be necessary to adapt the approach slightly, e.g., by approximating the resulting triangle. This still depicts an increased representativity compared to the original TransBox case and is therefore a promising candidate for testing. This short discussion points out that it is, with some adaptations, possible to use existing box embedding approaches straightforwardly for prism embeddings. This underlines the usability of these prism interpretations.

7. Conclusion and Outlook

In this paper, we have presented a relaxation of box embeddings to prism embeddings to circumvent restrictions introduced by the Helly Property. Many existing box embedding approaches can straightforwardly be extended to the prism case, allowing thus for a relatively simple conduction of experiments. As a next step, we plan to extend some existing box embedding approaches to prism embeddings and test them on real-world data. Here, it is particularly interesting not only to focus on cases with the Helly Property but also to consider in general whether the increased flexibility of modeling could improve the results. Next, we are interested in determining the expressive bounds of the prism embeddings. Although they can represent more ontologies correctly than box embeddings can do, the exact fragment of ontologies they can represent remains unknown. Additionally, the question needs to be tackled whether the bound on an increase of two dimensions to model a box embedding as a prism embedding is tight or whether it is possible to find a construction that allows for doing so by only increasing the dimension by one. Prisms are only one step towards a more comprehensive generalization of boxes. It would also be possible, for instance, to consider Cartesian products of several equilateral triangles (and not of a triangle and an m-dimensional box). It would even be possible to consider a mixture of different shapes, depending on the use case. Therefore, this study can be seen as a starting point for considering alternative shapes for embedding concepts. Next to enhanced representative abilities for $\mathcal{ELHO}(\circ)^{\perp}$, it is also possible to use prisms as a starting point for DLs with temporal semantics, thus to be able to represent not only a conceptual but also a temporal dimension geometrically.

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Declaration on Generative AI

The authors have not employed any Generative AI tools.

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