

The More the Merrier: Combining Properties for ABox Abduction under Repair Semantics in $\mathcal{EL}_\perp$

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Abstract

Abduction is a central approach to explain missing entailments from a knowledge base by providing a hypothesis, that would, if added to the knowledge base, make the missing entailment become true. Abduction under repair semantics has recently been investigated in detail, where several desirable properties and optimality criteria were considered, such as signature-restrictions and minimality in size and of introduced conflicts. Naturally, hypotheses that satisfy more than one of these properties or combine a property with an optimality criterion would be even more desirable for applications. So far, such hypotheses have not been investigated in the literature. In the present paper, we consider the ABox abduction problem for hypotheses satisfying more than one property or additional optimality criteria, for $\mathcal{EL}_\perp$ under brave and AR semantics. Our main observation is that often requiring additional properties for hypotheses does not lead to an increase of complexity.

Keywords

Abduction, Repair semantics, Inconsistency-tolerant reasoning

1. Introduction

Explainability remains a core feature of knowledge-based AI systems, including those using description logic ontologies. Developing methods that can also practically explain decisions made by such systems is crucial in order to exploit this feature. While there is a lot of work on explaining reasoning results under classical semantics [1, 2, 3], this problem is less investigated for non-classical semantics. In this paper, we consider *repair-based semantics*, which are a family of well-understood semantics that allow to define meaningful entailments from inconsistent knowledge bases [4]. As the name suggests, these semantics are based on *repairs*, which are maximal consistent subsets of the ABox. Depending on the chosen semantics, we may then consider those entailments that hold in at least one of the repairs (*brave semantics*), or only those entailments that hold in all of the repairs (*AR semantics*).

Since inconsistencies can easily occur in realistic data, inconsistency-tolerant semantics are often indispensable. Unfortunately, they also make it more challenging to understand entailments from a user perspective. To explain why an assertion is entailed, we may need to provide explanations for the different repairs at the same time. For explaining *missing* entailments, the situation is even more complex: Indeed, under repair-based semantics, two things can contribute to the non-entailment of an assertion α : 1) some other assertions “block” the entailment of α , because they create a conflict with required assertions, or 2) assertions that are needed for the entailment are missing. While explanations based on Situation 1 have been investigated in detail [5, 6], we focus here on explanations based on Situation 2. Interestingly, no one so far considered explanations that combine Situations 1) and 2), i.e., where assertions can both be added and removed. We prefer to first fully understand abductive reasoning under repair-based semantics, and investigate more advanced notions of explanations in the future. Focusing on Situation 2, this corresponds to explaining missing entailments using abductive reasoning, which is

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We recently proposed and investigated a range of desirable properties for abductive hypotheses under repair-based semantics. Some of these have been considered also under classical semantics, such as *non-triviality* (the hypothesis should not contain the observation) [7], *subset* and *cardinality* minimality [9], and restrictions on the concept and role names to be used (*signature-restricted hypotheses*) [12]. On the other hand, the typical requirement of consistency of the hypothesis with the knowledge base, which is also needed to avoid far-fetched explanations, makes less sense in an inconsistency-tolerant setting, where the knowledge base may already be inconsistent from the start. Instead, we may require the hypothesis to be *conflict-confining* [10], which means that it does not introduce any new conflicts, or at least *conflict-minimal*, which means it does not introduce more conflicts than necessary. In a recent conference publication [17], complementing the research by Du et al. [10] that focused on conflict-confining hypotheses under IAR semantics, we analysed the computational complexity of abduction for each of these properties, considering brave and AR semantics for DL-Lite and $\mathcal{EL}_{\perp}$. In

particular, we looked at the verification problem (is a given set of assertions a hypothesis satisfying the requirement), and the existence problem (is there any such hypothesis). An overview of our earlier results [17] is shown in Figure 1. This investigation considered the different properties in isolation, and left open the question of what happens if one desires more than one property to hold, for instance, if the hypothesis should be both non-trivial and conflict-confining, or subset-minimal among all the signature-restricted hypothesis. These and other combinations are very natural, which is why we have a closer look at them in this paper. This time, we focus on $\mathcal{EL}_\perp$. Specifically, we again investigate the theoretical complexity of the verification and the existence problem for hypotheses that satisfy two of the properties in Figure 1, and that combine one property with one of the minimality requirements.

For several results, we only provide proof ideas due to space constraints. The full proof details can be found in the technical report available online [18].

2. Preliminaries

For a general introduction to description logics, we refer the reader to Baader et al. (2017). In this paper, *ABox assertions* refer to *concept assertions* of the form $A(a)$ and *role assertions* of the form $r(a, b)$ for a concept name A , role name r and individuals a, b . In other words, we only use concept assertions without complex concepts, which are also known as *flat* or *simple* assertions. We assume familiarity with computational complexity [20], in particular with the complexity classes NL, P, NP, coNP, Σ_2^P and Π_2^P , as well as DP, the class of decision problems representable as the intersection of a problem in NP and a problem in coNP.

2.1. Repair Semantics

If a knowledge base is inconsistent, repair semantics can “restore” consistent versions and admit meaningful reasoning again. We focus here on ABox repairs and define these as well as two common kinds of repair semantics next. We will often use the following notions of consistency and inconsistency w.r.t. a TBox $\mathcal{T}$. An ABox $\mathcal{A}$ is $\mathcal{T}$ -consistent, if $\langle \mathcal{T}, \mathcal{A} \rangle \not\models \perp$, and $\mathcal{T}$ -inconsistent otherwise. Further, we say that $\mathcal{A}$ is a $\mathcal{T}$ -support of a concept assertion α , if $\langle \mathcal{T}, \mathcal{A} \rangle \models \alpha$.

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an inconsistent knowledge base and α be a Boolean (conjunctive) query. A *repair* of $\mathcal{K}$ is a $\mathcal{T}$ -consistent subset $\mathcal{R} \subseteq \mathcal{A}$ and subset-maximal with this property, i.e., there is no $\mathcal{T}$ -consistent subset $\mathcal{R}' \subseteq \mathcal{A}$ that is a strict superset of $\mathcal{R}$. The somewhat dual notion is a *conflict* or *conflict set* $\mathcal{C}$, which is a $\mathcal{T}$ -inconsistent subset of the ABox and subset-minimal with this property. We denote by $\text{Conf}(\mathcal{K})$ the set of conflicts of $\mathcal{K}$. We recall entailment under Brave [21] and AR semantics [22]:

- $\mathcal{K} \models_{\text{Brave}} \alpha$ iff there is a repair $\mathcal{R}$ of $\mathcal{K}$ s.t. $\langle \mathcal{T}, \mathcal{R} \rangle \models \alpha$.
- $\mathcal{K} \models_{\text{AR}} \alpha$ iff $\langle \mathcal{T}, \mathcal{R} \rangle \models \alpha$ for every repair $\mathcal{R}$ of $\mathcal{K}$.

The complexity of query entailment under repair semantics is well understood [4]: For $\mathcal{EL}_\perp$, checking entailment of a concept assertion under Brave semantics is NP-complete in combined complexity, while it is coNP-complete under AR semantics.

2.2. ABox Abduction for Inconsistent KBs

We now introduce the necessary notions for ABox abduction under repair semantics, as introduced in [17]. We begin with the central definitions of abduction problems and hypotheses in this setting.

Definition 2. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be an inconsistent KB, α a concept assertion (called an *observation*) and $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$ such that $\mathcal{K} \not\models_{\mathcal{S}} \alpha$. Then, the pair $\langle \mathcal{K}, \alpha \rangle$ is called an $\mathcal{S}$ -abduction problem. A solution for such a problem, called $\mathcal{S}$ -hypothesis, is an ABox $\mathcal{H}$ using only individuals occurring in $\mathcal{K}$ and α s.t. $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\mathcal{S}} \alpha$.

Additionally, for a set Σ (signature) containing individual, concept and role names, we define the triple $\langle \mathcal{K}, \alpha, \Sigma \rangle$ to be a Σ -restricted $\mathcal{S}$ -abduction problem. A solution for such a problem is an ABox $\mathcal{H}$ using only symbols from Σ that is an $\mathcal{S}$ -hypothesis for $\langle \mathcal{K}, \alpha \rangle$.

Note also that we do not admit fresh individuals beyond the specified signature as it is done for example in [12]. Abduction with fresh individuals easily becomes ExpTime-hard in $\mathcal{EL}_\perp$ [12]. Because there are at most exponentially many repairs, the complexity is in this case dominated by the complexity of (classical) abduction, so such a setting might make our complexity analysis less insightful.

To obtain hypotheses that are meaningful for explanation purposes, a number of additional properties and minimality criteria for hypotheses that yield *preferred hypotheses* have been introduced. For two sets S_1, S_2 we may abbreviate $|S_1| \leq |S_2|$ by $S_1 \leq S_2$. We next provide the definition of these properties and notions of minimality.

Definition 3. Let $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$, $\langle \mathcal{K}, \alpha \rangle$ an $\mathcal{S}$ -abduction problem, and $\preceq \in \{\subseteq, \leq\}$. An ABox $\mathcal{H}$ is

1. *conflict-confining* for $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, provided that $\text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle) = \text{Conf}(\mathcal{K})$.

If $\mathcal{H}$ is an $\mathcal{S}$ -hypothesis for $\langle \mathcal{K}, \alpha \rangle$, we call it

2. *non-trivial*, if $\alpha \notin \mathcal{H}$,
3. $\preceq$ -*minimal*, if there is no $\mathcal{S}$ -hypothesis $\mathcal{H}'$ for $\langle \mathcal{K}, \alpha \rangle$ s.t. $\mathcal{H}' \prec \mathcal{H}$, and
4. $\preceq_c$ -*minimal*, if there is no $\mathcal{S}$ -hypothesis $\mathcal{H}'$ for $\langle \mathcal{K}, \alpha \rangle$ s.t. $\text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle) \prec \text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle)$.

Conflict-confinement can be equivalently defined by requiring that $\langle \mathcal{T}, \mathcal{R} \cup \mathcal{H} \rangle \not\models \perp$ for every repair $\mathcal{R}$ of $\mathcal{K}$. Note that hypotheses introducing new conflicts can be desirable, as erroneous facts might need to implicitly be replaced by new facts, leading to conflicts when considering both together.

We use the terms *subset-minimal* for $\subseteq$ -minimal and *cardinality-minimal* for $\leq$ -minimal. Further, we also consider minimal variants of hypotheses with additional properties: For example, a $\subseteq$ -minimal conflict-confining AR-hypothesis need only be $\subseteq$ -minimal among all conflict-confining AR-hypotheses.

We investigate the following reasoning problems for a given (Σ -restricted) $\mathcal{S}$ -abduction problem.

Definition 4 (Reasoning Problems). Given a (Σ -restricted) $\mathcal{S}$ -abduction problem $\mathfrak{A} = \langle \mathcal{K}, \alpha \rangle$ ($\mathfrak{A} = \langle \mathcal{K}, \alpha, \Sigma \rangle$),

1. the *existence problem* asks whether $\mathfrak{A}$ has a solution, and
2. the *verification problem* asks whether a given ABox $\mathcal{H}$ is a hypothesis for $\mathfrak{A}$ (over Σ).

We restate the following two results that were established in [17].

Lemma 5. Given a KB $\mathcal{K}$ and an ABox $\mathcal{H}$, checking whether $\mathcal{H}$ is conflict-confining for $\mathcal{K}$ is coNP-complete.

Lemma 6. Let $\langle \mathcal{K}, \alpha \rangle$ be an AR-abduction problem. The following are equivalent: 1. there is an AR-hypothesis for $\langle \mathcal{K}, \alpha \rangle$, 2. $\{\alpha\}$ is a conflict-confining AR-hypothesis for $\langle \mathcal{K}, \alpha \rangle$.

3. Complexity Results for Combinations of Multiple Properties

We will now establish the complexity of existence and verification of hypotheses for combinations of two of the properties of being Σ -restricted, non-trivial and conflict-confining, as well as of one of these properties together with one of the minimality criteria. Note that the combination of conflict-confining with one of the minimalities regarding the set of conflicts does not make sense, as the set of conflicts for a conflict-confining hypothesis is always minimal.

Regarding the combination of Σ -restriction with other properties or minimality criteria, we immediately obtain the following general observation: any hardness result for the individual properties directly carries over to the case where Σ -restriction is required additionally. To see this, observe that one can always set Σ to the (full) signature of the given abduction problem, ensuring that Σ does not actually restrict the solutions.

Observation 7. *Let P be a property of hypotheses considered in this paper. Then, the existence (resp., verification) of $\mathcal{S}$ -hypotheses with property P can be reduced in polynomial time to the existence (resp., verification) of Σ -restricted $\mathcal{S}$ -hypotheses with property P for $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$.*

3.1. Existence

Here, we consider the existence problem for combinations of the three main properties of hypotheses. Note that minimality criteria are not interesting for existence, as there is a minimal hypothesis with certain properties iff there is any such hypothesis.

First, we briefly motivate the combination of Σ -restriction and non-triviality. One main motivation for using Σ -restriction can be to avoid trivial hypotheses. However, a non-trivial hypothesis might still use all the symbols from the observation, just in different combinations. Consequently, we might lose relevant hypotheses if we use this trick. The following example illustrates such a situation:

Example 8. Let $\mathcal{K} := \langle \mathcal{T}, \mathcal{A} \rangle$ be a KB with $\mathcal{T} := \{A \sqcap B \sqsubseteq C, D \sqcap \exists r.C \sqsubseteq A\}$ and $\mathcal{A} := \{B(m), r(m, n)\}$. Moreover, let $\alpha := C(m)$ be the observation and $\Sigma := \{C, D, m, n\}$ be the signature. Here, the trivial hypothesis $\{C(m)\}$ is in fact also Σ -restricted. The hypothesis $\mathcal{H}_1 := \{A(m)\}$ is non-trivial but outside our signature. Moreover, $\mathcal{H}_2 := \{C(n), D(m)\}$ is non-trivial and uses both C and m , that is, both the concept name and the individual from the observation.

Turning to the complexity results, the combination of signature restriction and non-triviality does not affect the overall complexity, as the individual cases exhibit the same computational behavior.

Corollary 9. *The existence problem for non-trivial Σ -restricted $\mathcal{S}$ -hypotheses is (i) NP-complete, for $\mathcal{S} = \text{Brave}$, and (ii) Σ_2^P -complete, for $\mathcal{S} = \text{AR}$.*

Proof. We recall the membership for the non-trivial case (Figure 1). For Brave semantics, existence of a non-trivial Brave-hypothesis for an instance $\langle \langle \mathcal{T}, \mathcal{A} \rangle, \alpha \rangle$ can be checked by guessing a candidate hypothesis $\mathcal{H}$ over the signature of $\langle \mathcal{K}, \alpha \rangle$, as well as a candidate repair $\mathcal{R}$ of $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle$ and verifying in polynomial time that $\mathcal{H}$ is non-trivial, $\mathcal{R}$ is a repair, and $\langle \mathcal{T}, \mathcal{R} \rangle \models \alpha$. For AR semantics, we guess a hypothesis $\mathcal{H}$ as before, and verify that $\mathcal{H}$ is non-trivial and $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{AR}} \alpha$. The latter is AR-entailment, which can be handled by querying an NP oracle.

For the combination, note that both sketched approaches are guess-and-check algorithms, so we can simply include a check for Σ -restriction of $\mathcal{H}$. Hence, we get the same complexity upper bounds as for the non-trivial setting. The hardness in both cases follows from the lower bound for existence of non-trivial hypotheses combined with Observation 7. $\square$

Interestingly, adding signature restriction, or non-triviality, on top of conflict-confinement, yields two cases in which both semantics incur the same “higher” complexity.

Theorem 10. *The existence problem for conflict-confining non-trivial $\mathcal{S}$ -hypotheses is Σ_2^P -complete for $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$. This also holds for the case of conflict-confining Σ -restricted $\mathcal{S}$ -hypotheses and the case of conflict-confining non-trivial Σ -restricted $\mathcal{S}$ -hypotheses.*

Proof idea. For membership, we guess a hypothesis $\mathcal{H}$ over the signature of $\mathcal{K}$ and verify that $\mathcal{H}$ is non-trivial, $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\mathcal{S}} \alpha$, and $\mathcal{H}$ is conflict-confining. The first check can easily be done in polynomial time, while the last two checks can be performed via NP-oracle calls, yielding Σ_2^P -membership. In particular, $\mathcal{S}$ -entailment has NP (resp., coNP) complexity for $\mathcal{S} = \text{Brave}$ ($\mathcal{S} = \text{AR}$), while the complexity for checking conflict-confinement was stated in Lemma 5.

When adding Σ -restriction, we guess $\mathcal{H}$ over the given signature Σ (instead of the signature of $\langle \mathcal{K}, \alpha \rangle$), and if we drop the requirement for non-triviality, the corresponding check is dropped. In both cases, we obtain Σ_2^P -membership.

For hardness, we first consider the case of brave semantics. The existence of conflict-confining Brave-hypotheses is Σ_2^P -complete (Figure 1) and we observe that the reduction easily extends to our present case. For non-trivial, the proof follows by observing that the employed reduction constructs

a hypothesis that is always non-trivial. When adding Σ -restriction (with or without non-triviality), hardness is retained due to Observation 7.

For AR semantics, we observe that the existence problem for non-trivial hypotheses (Figure 1) is Σ_2^P -complete. We argue that the reduction employed for hardness in that result extends to the combination of non-triviality with conflict-confinement. This requires us to prove that the considered hypothesis in the reduction is indeed conflict-confining. We find it convenient to present the reduction here for completeness. For hardness, we reduce from the validity problem for $\exists\forall$ -QBFs. Let $\Psi = \exists Y \forall Z \psi$ be an $\exists\forall$ -QBF, where ψ is in DNF, and let $X = Y \cup Z$. We construct a TBox $\mathcal{T}$ using concept names $N = \{A_x, A_{\bar{x}} \mid x \in X\} \cup \{C, A_\psi\}$. Intuitively, $\mathcal{T}$ expresses that for any term $t \in \psi$, the conjunction of concepts corresponding to the literals in t entails the concept A_ψ , which represents satisfaction of ψ . To this end, we define $\mathcal{K} := \langle \mathcal{T}, \mathcal{A} \rangle$, where

$$\mathcal{T} := \{ A_x \sqcap A_{\bar{x}} \sqsubseteq \perp \mid x \in X \} \cup \left\{ C \sqcap \bigcap_{\ell \in t} A_\ell \sqsubseteq A_\psi \mid t \in \psi \right\},$$

$$\mathcal{A} := \{ A_z(m), A_{\bar{z}}(m) \mid z \in Z \}$$

for an individual name m . Now, $\langle \mathcal{K}, A_\psi(m) \rangle$ is the desired abduction problem. Observe that $\mathcal{H}_{\text{triv}} := \{A_\psi(m)\}$ is the trivial AR-hypothesis for $\langle \mathcal{K}, \alpha \rangle$. However, we are considering existence of a non-trivial hypotheses here, i.e., one that does not contain $A_\psi(m)$. Intuitively, any non-trivial hypothesis corresponds to an assignment satisfying the formula $\forall Z \psi$, containing assertions for all literals over Y satisfied by the assignment. It is not hard to see that adding such hypotheses to $\mathcal{K}$ does not introduce new conflicts, and that there is no need to include additional assertions, allowing us to prove the following claim.

Claim 1. *Ψ is true iff $\langle \mathcal{K}, \alpha \rangle$ admits a non-trivial conflict-confining AR-hypothesis.*

Hardness when adding Σ -restriction to either of the cases again follows from Observation 7. $\square$

3.2. Verification

We now turn towards verification. We first argue that the combinations of the main properties (without minimality criteria) can be handled similar as for existence. Observe that the verification problem for either signature-restricted, or non-trivial $\mathcal{S}$ -hypotheses admits the same complexity as $\mathcal{S}$ -entailment for $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$. Given an instance $\langle \mathcal{K}, \alpha \rangle$ of $\mathcal{S}$ -abduction problem with $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, verifying that an ABox $\mathcal{H}$ is a non-trivial $\mathcal{S}$ -hypothesis requires that (1) $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\mathcal{S}} \alpha$, and (2) $\mathcal{H}$ is non-trivial. Here, the complexity is dominated by the first check, as the second step can be performed in polynomial time. Similarly, for Σ -restriction, step (2) above can be replaced by a check that $\mathcal{H}$ is Σ -restricted, which again takes polynomial time. Further, the same is still true when combining non-triviality with Σ -restriction. The lower bound follows from hardness of verification for (general) $\mathcal{S}$ -hypotheses and non-trivial $\mathcal{S}$ -hypotheses combined with Observation 7, yielding the following corollary.

Corollary 11. *Verification of non-trivial $\mathcal{S}$ -hypotheses and of Σ -restricted $\mathcal{S}$ -hypotheses is (i) NP-complete for $\mathcal{S} = \text{Brave}$, and (ii) coNP-complete for $\mathcal{S} = \text{AR}$. Moreover, the same is true when combining both properties, i.e., for verification of non-trivial Σ -restricted $\mathcal{S}$ -hypotheses.*

It is already known that adding conflict-confinement increases the complexity for verification, but only for brave semantics (Figure 1). We next prove that adding non-triviality or Σ -restriction does not increase the complexity beyond that of checking that a given ABox is a conflict-confining hypothesis.

Theorem 12. *Verification of non-trivial conflict-confining $\mathcal{S}$ -hypotheses is (i) DP-complete for $\mathcal{S} = \text{Brave}$, and (ii) coNP-complete for $\mathcal{S} = \text{AR}$. The complexity remains the same when non-triviality is replaced by Σ -restriction, and when requiring all three properties.*

Proof idea. The hardness applies due to the case of verification for conflict-confining hypotheses since the considered hypotheses in each case are indeed non-trivial. Hardness when adding Σ -restriction to either of the cases again follows from Observation 7. We give a bit more detail for the case of conflict-confining non-trivial hypotheses. For brevity, we only outline the proof for Brave semantics, which utilises a reduction from the combination of entailment and non-entailment under Brave semantics to verification of non-trivial conflict-confining Brave-hypotheses. Given an instance $\langle \mathcal{K}, \alpha_1, \alpha_2 \rangle$ for some inconsistent KB $\mathcal{K}$, the problem asks whether $\mathcal{K} \models_{\text{Brave}} \alpha_1$ and $\mathcal{K} \not\models_{\text{Brave}} \alpha_2$. This problem is DP-complete because the first question is NP-complete and the second question is coNP-complete under Brave semantics: Membership follows immediately, while only using a single KB in the problem is a slight obstacle to showing hardness. To address this, reduce from the variant of the problem with two KBs by first ensuring that the first and the second part of the problem use disjoint sets of names, and then defining the new single KB as the disjoint union of both KBs. For the reduction to the abduction problem at hand, assume w.l.o.g. that α_1 and α_2 both use the same individual, but different concept names, and let $\alpha_1 = A(a)$, $\alpha_2 = B(a)$, with $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$. We consider fresh concepts C and D to construct a KB $\mathcal{K}' := \langle \mathcal{T}', \mathcal{A} \rangle$, for

$$\mathcal{T}' := \mathcal{T} \cup \{C \sqcap A \sqsubseteq D, C \sqcap B \sqsubseteq \perp\},$$

an observation $\alpha := D(a)$, and a hypothesis $\mathcal{H} := \{C(a)\}$. It can now be shown that $\mathcal{H}$ is a Brave-hypothesis for $\langle \mathcal{K}', \alpha \rangle$ iff $\mathcal{K} \models_{\text{Brave}} A(a)$, and conflict-confining for $\mathcal{K}'$ iff $\mathcal{K} \not\models_{\text{Brave}} B(a)$. Observing that $\alpha \notin \mathcal{H}$, we obtain the following claim, proving correctness.

Claim 2. $\mathcal{H}$ is a non-trivial conflict-confining Brave-hypothesis for α in $\mathcal{K}'$ iff $\mathcal{K} \models_{\text{Brave}} \alpha_1$ and $\mathcal{K} \not\models_{\text{Brave}} \alpha_2$.

We next prove membership in each case. Let $\langle \mathcal{K}, \alpha \rangle$ be an $\mathcal{S}$ -abduction problem with $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ and $\mathcal{H}$ an ABox.

We begin with Brave semantics. Observe that $\mathcal{H}$ is a non-trivial conflict-confining Brave-hypothesis iff (1) $\mathcal{H}$ is non-trivial, (2) $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{Brave}} \alpha$, and (3) $\text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle) = \text{Conf}(\langle \mathcal{T}, \mathcal{A} \rangle)$. Here, (1) can be easily checked in polynomial time. Moreover, (2) is instance checking for $\mathcal{EL}_\perp$ under brave semantics and hence in NP, while (3) can be checked in coNP, since the complement can be checked by guessing a conflict $\mathcal{C}$ s.t. $\mathcal{C} \in \text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle)$ and $\mathcal{C} \notin \text{Conf}(\langle \mathcal{T}, \mathcal{A} \rangle)$. Concretely, this amounts to guessing a $\mathcal{C} \subseteq \mathcal{A} \cup \mathcal{H}$ such that: (a) $\mathcal{C} \not\subseteq \mathcal{A}$, (b) $\langle \mathcal{T}, \mathcal{C} \rangle \models \perp$, and (c) $\langle \mathcal{T}, \mathcal{C} \setminus \{\gamma\} \rangle \not\models \perp$ for any $\gamma \in \mathcal{C}$. Checking whether $\mathcal{C}$ satisfies (a)–(c) can be done in polynomial time. Altogether, (a) and (b) together are in NP, while (c) is in coNP. Hence, the problem is contained in DP.

The membership proof for brave semantics can be adapted to the cases where Σ -restriction is added as a requirement, or replaces non-triviality, by noting that checking Σ -restriction on top or instead of non-triviality can also be done in polynomial time.

We now turn to membership for AR-semantics. First, check non-triviality of $\mathcal{H}$ in polynomial time. If this is the case, $\mathcal{H}$ is a non-trivial conflict-confining AR-hypothesis iff (1) $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{AR}} \alpha$, and (2) $\mathcal{H}$ is conflict-confining in $\mathcal{K}$, i.e., $\text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle) = \text{Conf}(\langle \mathcal{T}, \mathcal{A} \rangle)$. As a result, one can guess a candidate repair $\mathcal{R}$ and a candidate conflict $\mathcal{C}$ of $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle$ simultaneously, as a counter example to $\mathcal{H}$ being an AR-hypothesis ($\mathcal{R}$) and conflict-confining ($\mathcal{C}$). More precisely, one needs to check that $\mathcal{R}$ is a subset-maximal $\mathcal{T}$ -consistent subset of $\mathcal{A}$ (hence a repair) and $\langle \mathcal{T}, \mathcal{R} \rangle \not\models \alpha$, as well as that $\mathcal{C} \notin \text{Conf}(\langle \mathcal{T}, \mathcal{A} \rangle)$, but $\mathcal{C} \in \text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle)$. All these checks can be performed in polynomial time. The claimed coNP-membership follows by observing that $\mathcal{H}$ is a conflict-confining hypothesis for α in $\mathcal{K}$ iff both these checks return false. Membership when adding Σ -restriction to the conflict-confining and the conflict-confining non-trivial case again follows by noting that Σ -restriction can be checked separately in polynomial time. $\square$

Combination of Properties with Minimality Criteria

We next turn towards combinations involving minimality criteria, beginning with $\sqsubseteq$ -minimality. Here, the following general observation regarding our main properties is very helpful.

Observation 13. *The properties of being conflict-confining, non-trivial, and Σ -restricted are downwards-closed. More precisely, let $\langle \mathcal{K}, \alpha \rangle$ be an $\mathcal{S}$ -abduction problem for $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$, $\mathcal{H}$ an ABox and $\mathcal{H}' \subseteq \mathcal{H}$. If $\mathcal{H}$ is conflict-confining for $\mathcal{K}$, then so is $\mathcal{H}'$. Similarly, if $\alpha \notin \mathcal{H}$, then $\alpha \notin \mathcal{H}'$, and if $\mathcal{H}$ is an ABox over a signature Σ , then so is $\mathcal{H}'$.*

This implies that an ABox $\mathcal{H}$ being a $\subseteq$ -minimal conflict-confining $\mathcal{S}$ -hypotheses is equivalent to $\mathcal{H}$ being a $\subseteq$ -minimal $\mathcal{S}$ -hypothesis and being conflict-confining (as a separate requirement). The same observation applies to the other two properties as well as combinations of properties.

Using the established results on verification of $\subseteq$ -minimal hypotheses and the above observation, we obtain the following complexities when additionally requiring other properties.

Theorem 14. *Verification of $\subseteq$ -minimal non-trivial $\mathcal{S}$ -hypotheses is (i) DP-complete, for $\mathcal{S} = \text{Brave}$, and (ii) Π_2^P -complete, for $\mathcal{S} = \text{AR}$. The complexity remains same when we replace non-triviality by Σ -restriction, and when combining both properties.*

Proof idea. The hardness in each case follows from the corresponding result on verification of $\subseteq$ -minimal $\mathcal{S}$ -hypotheses. Observe that the reduction in each case uses a non-trivial hypothesis, thus the hardness applies to verification of non-trivial subset-minimal hypotheses. Again, when adding Σ -restriction (either instead of or in combination with non-triviality), hardness is retained by Observation 7. We outline the reduction for DP-hardness with Brave semantics as it is also used in later results. We reduce again from a combination of entailment and non-entailment under Brave semantics to verification of $\subseteq$ -minimal Brave-hypotheses, similar to hardness under Brave semantics in the proof of Theorem 12. Given an instance $\langle \mathcal{K}, \alpha_1, \alpha_2 \rangle$ for some inconsistent KB $\mathcal{K}$, let $\alpha_1 = A(a)$, $\alpha_2 = B(a)$, and $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$. We now construct a KB $\mathcal{K}'$, an observation α , and an ABox $\mathcal{H}$ as follows. Let $\mathcal{K}' := \langle \mathcal{T}', \mathcal{A} \rangle$ with

$$\mathcal{T}' := \mathcal{T} \cup \{B \sqsubseteq C, C \sqcap D \sqcap A \sqsubseteq Q\},$$

$\alpha := Q(a)$, and $\mathcal{H} := \{C(a), D(a)\}$ for fresh concepts C, D, Q . Intuitively, $\mathcal{H}$ is a Brave-hypothesis for $\langle \mathcal{K}', \alpha \rangle$ iff $\mathcal{K} \models_{\text{Brave}} A(a)$ and $\mathcal{H}$ is $\subseteq$ -minimal iff $\mathcal{K} \not\models_{\text{Brave}} B(a)$. To see the latter, note that $\subseteq$ -minimality is violated iff $\{D(a)\}$ is also a Brave-hypothesis for $\langle \mathcal{K}', \alpha \rangle$, which is the case iff $\mathcal{K} \models_{\text{Brave}} B(a)$. The following claim shows correctness.

Claim 3. *$\mathcal{H}$ is a $\subseteq$ -minimal non-trivial hypothesis for $\langle \mathcal{K}', \alpha \rangle$ iff $\mathcal{K} \models_{\text{Brave}} \alpha_1$ and $\mathcal{K} \not\models_{\text{Brave}} \alpha_2$.*

Membership follows from the the upper bound for verification of $\subseteq$ -minimal $\mathcal{S}$ -hypotheses together with Observation 13: By the observation, it is sufficient to check that $\mathcal{H}$ is a $\subseteq$ -minimal $\mathcal{S}$ -hypotheses, and separately checking that it is non-trivial or Σ -restricted. Since the latter two properties can be checked in P, we obtain the desired upper bounds. For this, note that both DP and Π_2^P are closed under intersection with languages from P. □

Verification of $\subseteq$ -minimal conflict-confining hypotheses enjoys the same complexity for Brave-hypotheses, but the precise complexity of AR-hypotheses remains open for now.

Theorem 15. *Verification of $\subseteq$ -minimal conflict-confining $\mathcal{S}$ -hypotheses is (i) DP-complete for $\mathcal{S} = \text{Brave}$, while it is (ii) in Π_2^P and coNP-hard for $\mathcal{S} = \text{AR}$. The same holds when additionally requiring non-triviality, Σ -restriction, or both.*

Proof idea. Membership for both semantics follows from the membership of verification of $\subseteq$ -minimal $\mathcal{S}$ -hypotheses, again using Observation 13: By the observation, it is sufficient to verify that $\mathcal{H}$ is a $\subseteq$ -minimal $\mathcal{S}$ -hypothesis, and separately checking that it is conflict-confining. As the latter is in coNP and both DP as well as Π_2^P are closed under intersection with languages from coNP, we obtain the desired upper bounds. As both non-triviality and Σ -restriction can even be checked in P, the same upper bound still holds when adding one or both of these properties.

DP-hardness for Brave semantics follows due to the same complexity of verification of $\subseteq$ -minimal Brave-hypotheses. We note that the considered hypothesis in the reduction employed for this case

in Theorem 14 is actually conflict-confining. As a result, Claim 3 can be reused after replacing non-triviality by conflict-confinement. As argued, the constructed hypothesis is in fact both non-trivial and conflict-confining, thus the hardness also applies when adding non-triviality in the combination. Adding Σ -restriction can again be handled using Observation 7.

Finally, coNP-hardness for AR semantics follows from the reduction employed in the coNP-hardness proof for verification of conflict-confining AR-hypotheses [17]. Precisely, reduce from AR-entailment as follows. Given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ and a concept assertion $A(a)$, define the new KB $\mathcal{K}' = \langle \mathcal{T}', \mathcal{A} \rangle$, where $\mathcal{T}' := \mathcal{T} \cup \{X \sqcap A \sqsubseteq C\}$, and let $\alpha := C(a)$ and $\mathcal{H} := \{X(a)\}$. We have $\mathcal{K} \models_{\text{AR}} A(a)$ iff $\langle \mathcal{T}', \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{AR}} \alpha$. Furthermore, it is easy to see that $\mathcal{H}$ is conflict-confining and minimality follows from $|\mathcal{H}| = 1$. Further, $\mathcal{H}$ is non-trivial, while Σ -restriction can again be handled using Observation 7, yielding hardness when adding one or both of these properties. $\square$

Cardinality-minimality behaves similar to subset-minimality, if we consider either non-trivial or Σ -restricted hypotheses. It was shown in our earlier work [17] that $\leq$ -minimal $\mathcal{S}$ -hypotheses are of size 1, since there exists some $\mathcal{S}$ -hypothesis iff the (singleton) trivial hypothesis is one. This does not apply when considering $\leq$ -minimal non-trivial (resp., Σ -restricted) hypotheses, as the trivial hypothesis is now disallowed (resp., can be disallowed). This explains the increased complexity in both cases when compared to the base case (verification of $\leq$ -minimal $\mathcal{S}$ -hypotheses).

Theorem 16. *Verification of $\leq$ -minimal non-trivial $\mathcal{S}$ -hypotheses is (i) DP-complete for $\mathcal{S} = \text{Brave}$, and (ii) Π_2^P -complete for $\mathcal{S} = \text{AR}$. The complexity remains the same when we replace non-triviality by Σ -restriction, and when combining both properties.*

Proof. Brave semantics: For membership, one can guess a set $\mathcal{H}'$ of assertions over the signature of $\mathcal{K}$ such that $\mathcal{H}'$ is a counter-witness to $\mathcal{H}$ being a $\leq$ -minimal $\mathcal{S}$ -hypothesis for $\langle \mathcal{K}, \alpha \rangle$. Precisely, $\mathcal{H}$ is a $\leq$ -minimal non-trivial Brave-hypothesis iff (1) it is a non-trivial Brave-hypothesis, and (2) there is no $\mathcal{H}'$ over the signature of $\mathcal{K}$ with $\alpha \notin \mathcal{H}'$ and $|\mathcal{H}'| < |\mathcal{H}|$ s.t. $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle \models_{\text{Brave}} \alpha$. Here, (1) can be verified in NP, while for (2), we guess as a counter-example both a set of assertions $\mathcal{H}'$ of size at most $|\mathcal{H}| - 1$ over the signature of $\mathcal{K}$ as well as a candidate repair $\mathcal{R} \subseteq \mathcal{A} \cup \mathcal{H}'$. We then verify that $\alpha \notin \mathcal{H}'$, $\mathcal{R}$ is a repair, and $\langle \mathcal{T}, \mathcal{R} \rangle \models \alpha$, which can all be checked in polynomial time. This shows that (2) can be checked in coNP, yielding DP-membership in total. The membership also applies when requiring Σ -restriction instead of (or in addition to) non-triviality, as verification of (non-trivial) Σ -restricted Brave-hypotheses is again in NP, and instead of (or in addition to) checking non-triviality $\mathcal{H}'$, we can check that $\mathcal{H}'$ only uses names from Σ in polynomial time.

For hardness, we reconsider the reduction employed in the case of verification of $\subseteq$ -minimal Brave-hypotheses (Theorem 14). We observe that the considered hypothesis $\mathcal{H}$ is $\subseteq$ -minimal iff $\mathcal{H}$ is $\leq$ -minimal. The reduction already applies for the non-trivial case as $\mathcal{H}$ is non-trivial. The case of Σ -restriction can again be handled using Observation 7.

AR semantics: The Π_2^P -membership can be shown similarly to DP-membership for Brave semantics above. Precisely, $\mathcal{H}$ is a non-trivial $\leq$ -minimal AR-hypothesis iff (1) it is a non-trivial AR-hypothesis, and (2) there is no $\mathcal{H}'$ over the signature of $\mathcal{K}$ with $\alpha \notin \mathcal{H}'$ and $|\mathcal{H}'| < |\mathcal{H}|$ s.t. $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle \models_{\text{AR}} \alpha$. Here, (1) can be checked in coNP, while $\mathcal{H}'$ can be guessed as before. The main difference is that we do not guess a candidate repair, and instead check $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle \models_{\text{AR}} \alpha$ using a coNP oracle. The cases obtained by replacing non-triviality by Σ -restriction or combining both can be handled in the same way as above for Brave semantics.

For Π_2^P -hardness, we reduce from the complement of the existence problem for non-trivial AR-hypotheses, which is Σ_2^P -complete. Let $\langle \mathcal{K}, \alpha \rangle$ be an AR-abduction problem over some signature $\Sigma_{\mathcal{K}}$ and $\alpha := C(m)$. We consider $\mathcal{A}_{\Sigma} := \{A(a) \mid A, a \in \Sigma_{\mathcal{K}}\} \cup \{r(a, b) \mid r, a, b \in \Sigma_{\mathcal{K}}\}$ as the set of all the ABox assertions over $\Sigma_{\mathcal{K}}$. It holds that $\mathcal{H} \subseteq \mathcal{A}_{\Sigma}$ for any AR-hypothesis $\mathcal{H}$ for $\langle \mathcal{K}, \alpha \rangle$. As a result, any such hypothesis for α must have its size bounded by $N = |\mathcal{A}_{\Sigma}|$. To complete the reduction, we add fresh concept names $\{X_i \mid i \leq N + 1, X_i \notin \Sigma_{\mathcal{K}}\}$, and define the new KB $\mathcal{K}' := \langle \mathcal{T}', \mathcal{A} \rangle$, where $\mathcal{T}' := \mathcal{T} \cup \{\bigwedge_{1 \leq i \leq N+1} X_i \sqsubseteq C\}$. Then, we use the observation $\alpha := C(m)$ as before and define $\mathcal{H} := \{X_i(m) \mid i \leq N + 1\}$.

For correctness, we observe that $\mathcal{H}$ is a non-trivial AR-hypothesis for $\langle \mathcal{K}', C(m) \rangle$ of size $N + 1$, whereas any non-trivial AR-hypothesis for $\langle \mathcal{K}, \alpha \rangle$ has size $\leq N$ (since $|\mathcal{A}_\Sigma| = N$). This implies the following equivalences: $\mathcal{H}$ is not a $\leq$ -minimal AR-hypothesis for $\langle \mathcal{K}', C(m) \rangle$ iff there is a non-trivial AR-hypothesis for $\langle \mathcal{K}', \alpha \rangle$ of size $\leq N$ iff there is a non-trivial AR-hypothesis for $\langle \mathcal{K}, \alpha \rangle$. This yields the mentioned Π_2^P -hardness for the AR semantics. The case of Σ -restriction can again be handled using Observation 7. $\square$

Considering conflict-confinement with $\leq$ -minimality, the complexity under AR semantics stays the same as for $\leq$ -minimality alone, which follows from Lemma 6. In contrast, the complexity again increases for Brave semantics. Similar to before, this comes from the fact that we do not have the trivial size bound of 1 in this case, and is closely related to existence of conflict-confining Brave-hypotheses.

Theorem 17. *Verification of $\leq$ -minimal conflict-confining $\mathcal{S}$ -hypotheses is (i) Π_2^P -complete for $\mathcal{S} = \text{Brave}$, and (ii) coNP-complete for $\mathcal{S} = \text{AR}$.*

Proof idea. For AR-semantics, Lemma 6 implies that a given AR-hypothesis is $\leq$ -minimal iff it is of size 1. The complexity now readily follows from coNP-completeness of AR entailment.

We now turn to Brave semantics. For membership, we need to determine whether (1) $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{Brave}} \alpha$, (2) $\mathcal{H}$ is conflict-confining, and (3) there is no $\mathcal{H}'$ over the signature of $\mathcal{K}$ with $|\mathcal{H}'| < |\mathcal{H}|$ that is conflict-confining for $\mathcal{K}$ and satisfies $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle \models_{\text{Brave}} \alpha$. Here, one can guess the ABox $\mathcal{H}'$ as a counter-witness, and handle all the remaining checks needed in (1)–(3) via oracle calls. Consequently, we obtain membership in Π_2^P .

For hardness, we reduce from the complement of the existence problem for conflict-confining Brave-hypotheses. To this aim, we extend and reuse the reduction for this case from our earlier work [17]. The trick we employ here is to use candidate hypothesis $\mathcal{H}$ of very large size following a similar idea as in the proof of Theorem 16 for the case of AR semantics. However, this trick cannot be applied as directly: We know that there might exist a conflict-confining hypothesis even though the observation itself is not conflict-confining. In this case, $\mathcal{H}$ would also not be conflict-confining, interfering with the rest of the construction. Nevertheless, we invented a way to construct a hypothesis that avoids new conflicts [17]. We apply the same strategy to our considered hypothesis $\mathcal{H}$ here, which requires slightly changing the newly added TBox axiom in the proof of Theorem 16. This yields the mentioned Π_2^P -hardness. $\square$

Finally, we consider $\subseteq_c$ -minimality. Recall that the combination of $\subseteq_c$ -minimality and conflict-confining is not interesting, since conflict-confining hypotheses are always $\subseteq_c$ -minimal. We consider the remaining cases. Here, we have to determine whether a given non-trivial (or Σ -restricted) hypothesis is $\subseteq_c$ -minimal among all *such* hypotheses. Considering AR semantics, although we know that the observation α has to be a conflict-confining AR-hypothesis in order for there to exist any AR-hypothesis by Lemma 6, we do not know whether and how non-trivial (Σ -restricted) hypotheses create conflicts. Concretely, while it might appear counter-intuitive, AR-hypotheses are not necessarily conflict-confining. The following example highlights this observation.

Example 18. Let $\mathcal{K} := \langle \mathcal{T}, \mathcal{A} \rangle$ where,

$$\begin{aligned} \mathcal{T} &:= \{A_1 \sqcap B_1 \sqsubseteq \perp, \quad A_1 \sqcap B_2 \sqsubseteq \perp, \quad A_2 \sqcap B_1 \sqsubseteq \perp, \quad A_2 \sqcap B_2 \sqsubseteq \perp, \\ &\quad A_1 \sqcap A_2 \sqsubseteq C, \quad B_1 \sqcap B_2 \sqsubseteq C\}, \\ \mathcal{A} &:= \{A_1(m), B_1(m)\}. \end{aligned}$$

Now, let $\alpha := C(m)$ be our observation. Then, the ABox $\mathcal{H} := \{A_2(m), B_2(m)\}$ is not conflict-confining in $\mathcal{K}$ since, for example, $\mathcal{H}$ itself is a new conflict in $\mathcal{K}$. Nevertheless, $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle \models_{\text{AR}} \alpha$, as the only repairs of $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle$ are $\{A_1(m), A_2(m)\}$ and $\{B_1(m), B_2(m)\}$. Further, any non-trivial AR-hypothesis for $\langle \mathcal{K}, C(m) \rangle$ is a superset of $\mathcal{H}$, highlighting that non-trivial AR-hypotheses might need to introduce new conflicts.

Theorem 19. *Verification of $\subseteq_c$ -minimal non-trivial $\mathcal{S}$ -hypotheses is Π_2^P -complete for $\mathcal{S} \in \{\text{Brave}, \text{AR}\}$. The complexity remains the same when we replace non-triviality by Σ -restriction or combine both.*

Proof. We begin with Brave semantics. For membership, one can guess as a counter-witness a set of assertions $\mathcal{H}'$ over the signature of $\langle \mathcal{K}, \alpha \rangle$ and verify that $\mathcal{H}'$ is a non-trivial Brave-hypothesis for α and $\mathcal{H}'$ admits fewer new conflicts than $\mathcal{H}$ does. Precisely, $\mathcal{H}$ is a $\subseteq_c$ -minimal non-trivial Brave-hypothesis for $\langle \mathcal{K}, \alpha \rangle$ iff (1) $\mathcal{H}$ is a non-trivial Brave-hypothesis for $\langle \mathcal{K}, \alpha \rangle$, and (2) there is no ABox $\mathcal{H}'$ over the signature of $\langle \mathcal{K}, \alpha \rangle$ with $\alpha \notin \mathcal{H}'$ and $\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle \models_{\text{Brave}} \alpha$ s.t. $\text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H}' \rangle) \subsetneq \text{Conf}(\langle \mathcal{T}, \mathcal{A} \cup \mathcal{H} \rangle)$. For (2), we can guess a counter-witness $\mathcal{H}'$, while the remaining checks can be handled via oracle calls. To compare the sets of conflicts, we can check that every conflict in the first set is contained in the second and that there is a conflict in the second set that does not occur in the first, which can be checked in coNP and NP, respectively. This results in Π_2^P -membership. For Σ -restriction, we check whether $\mathcal{H}$ and $\mathcal{H}'$ only use names over Σ instead of (or in addition to) checking non-triviality.

For hardness we reuse the same reduction as for the case of verification of $\subseteq_c$ -minimal hypotheses (Figure 1). Here, we observe that the reduction employs a non-trivial hypothesis. Adding Σ -restriction can again be handled using Observation 7.

We now turn to AR semantics. Membership follows by the same argument as for Brave semantics, since the verification of non-trivial AR-hypotheses and the additional checks can again be handled via oracle calls after guessing a counter-witness $\mathcal{H}'$. For hardness, we incorporate the idea from Example 18 into our reduction for the existence of non-trivial conflict-confining AR-hypotheses from Theorem 10. Let $\langle \mathcal{K}, \alpha \rangle$ be the AR-abduction problem constructed from an $\exists\forall$ -QBF Ψ as in the proof for Theorem 10. We let A_1, A_2, B_1, B_2 denote fresh concepts and consider the KB $\mathcal{K}_e := \langle \mathcal{T}_e, \mathcal{A}_e \rangle$ as constructed in Example 18. To complete the reduction, we set $\mathcal{K}' := \langle \mathcal{T}', \mathcal{A}' \rangle$ where $\mathcal{T}' := \mathcal{T} \cup \mathcal{T}_e$ and $\mathcal{A}' := \mathcal{A} \cup \mathcal{A}_e$ and use the same observation α . Finally, we take the hypothesis $\mathcal{H}_e := \{A_2(m), B_2(m)\}$ as in the example, which creates precisely three new conflicts $\{A_1(m), B_2(m)\}$, $\{A_2(m), B_1(m)\}$, and $\mathcal{H}_e$. Then, the correctness follows from the following claim.

Claim 4. Ψ is true iff $\mathcal{H}_e$ is not a non-trivial $\subseteq_c$ -minimal AR-hypothesis for $\langle \mathcal{K}', \alpha \rangle$.

Proof of Claim. Recall that Ψ is true iff $\langle \mathcal{K}, \alpha \rangle$ admits a non-trivial conflict-confining AR-hypothesis (Claim 1). We observe that $\langle \mathcal{K}, \alpha \rangle$ admits a non-trivial conflict-confining AR-hypothesis iff $\langle \mathcal{K}', \alpha \rangle$ admits a non-trivial conflict-confining AR-hypothesis (the same one) iff $\mathcal{H}_e$ is not a $\subseteq_c$ -minimal non-trivial AR-hypothesis for $\langle \mathcal{K}', \alpha \rangle$.

Indeed, if Ψ is false, there is no AR-hypothesis over the signature of $\mathcal{K}$ that entails α . Hence, the only way to obtain the AR-entailment of α is via using new axioms in $\mathcal{K}'$. This renders $\mathcal{H}_e$ to be the only AR-hypothesis for α in $\mathcal{K}'$, and hence makes it trivially $\subseteq_c$ -minimal. ■

□

The membership in Theorem 19 only applies to $\subseteq_c$ -minimality and does not seem to extend to $\leq_c$ -minimality. While in the case of the former, the necessary comparison of the sets of conflicts introduced by $\mathcal{H}$ and the guessed ABox $\mathcal{H}'$ can be handled using an NP oracle, this is not the case when we need to compare the cardinalities of these sets. Indeed, the following example illustrates that a relatively small hypothesis can trigger exponentially many new conflicts, hinting at potential #P-hardness of counting the number of conflicts. How this affects the complexity for $\leq_c$ -minimality is left for future work.

Example 20. Let $\mathcal{K} := \langle \mathcal{T}, \mathcal{A} \rangle$ where,

$$\begin{aligned} \mathcal{T} &:= \{C \sqsubseteq D\} \cup \{A_i \sqsubseteq X_i, B_i \sqsubseteq X_i \mid i \leq n\} \cup \{D \sqcap X_1 \sqcap \dots \sqcap X_n \sqsubseteq \perp\}, \\ \mathcal{A} &:= \{A_i(m), B_i(m) \mid i \leq n\}. \end{aligned}$$

Now, let $\alpha := D(m)$ be our observation. Then, the ABox $\mathcal{H} := \{C(m)\}$ is a Brave hypothesis for $\langle \mathcal{K}, \alpha \rangle$ of size one. Nevertheless, $\mathcal{H}$ triggers exponentially many new conflicts in $\mathcal{K}$, each of the form $\{C(m), Y_1(m), Y_2(m), \dots, Y_n(m)\}$ where $Y_i \in \{A_i, B_i\}$ for all i . Note that $\mathcal{K}$ is a consistent KB to begin with.

4. Conclusion and Outlook

We observe that combining basic properties (excluding minimality criteria) and potentially subset-minimality does not increase the complexity for either existence or verification of $\mathcal{S}$ -hypotheses in $\mathcal{EL}_\perp$: The complexity for all considered cases is governed by the highest complexity among the individual properties involved. In contrast, we often see an increase in the complexity, when combining cardinality-minimality or conflict-minimality with other properties.

Observe that the hardness proofs for some of our results (e.g., Theorems 10, 14 and 15) do not use role assertions. Consequently, several of our results already hold for propositional Horn logic. Nevertheless, we leave a detailed complexity characterization for this fragment as future work.

It will be interesting to complete the picture by also extending the study to conflict-minimality when considering the number of conflicts (instead of subset-minimality), and closing the gap for the case of $\subseteq$ -minimal conflict-confining AR-hypotheses, which both remain open for now.

Further, we previously also obtained complexity results for ABox abduction in the description logic DL-Lite [17]. Preliminary work for combinations of properties in this setting shows that in some cases the complexity is again governed by the highest complexity among the individual properties, but there is a notable exception. It turns out that in case of DL-Lite, conflict-confining AR-hypotheses have a much simpler structure: Every conflict-confining AR-hypothesis contains a *singleton* AR-hypothesis, which is of course again conflict-confining. This often makes abduction problems *easier* when adding conflict-confinement in addition to some other property (or minimality condition) compared to the same problem without requiring conflict-confinement. It will be interesting to fully understand the complexity landscape for combinations of properties in this setting.

It should be noted that in case of expressive DLs such as $\mathcal{ALC}$ and its extensions, the complexity of the considered abduction problems seems significantly less interesting. The reason is that for these DLs, entailment is often already ExpTime-complete. Consequently, the complexity of abduction tends to be dominated by the complexity of entailment.

Beside considering different DLs, another important direction for the future is to consider other repair semantics. Most prominently, IAR semantics are currently missing from our study. While there has been previous work on abduction under IAR semantics, it was more practical in nature, obtaining an upper bound for one specific case in DL-Lite. The complexity landscape for different (combinations of) properties remains widely unexplored.

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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