Finding New Boxes for the Diamonds: On the Behavior of Convex Modal Operators for Temporal Description Logics

Stefan Borgwardt, Alisa Kovtunova

Institute of Theoretical Computer Science, Technische Universität Dresden, Germany

Abstract

Temporal extensions of description logics often lead to a substantial increase in the complexity of reasoning. Nevertheless, tractable extensions of $\mathcal{EL}$ have been developed by restricting the occurrences of temporal concepts. One can even add so-called *convex* metric temporal diamond operators to $\mathcal{EL}$, if they are only allowed to occur on the left-hand side of concept inclusions. In this paper, we consider the dual convex box operators for the first time, and investigate the behavior of complex temporal formulas in which convex diamond and box operators can be combined via composition, union and intersection. We demonstrate the intricate interplay between these operators, and present decision procedures for checking subsumption between complex temporal formulas.

Keywords

temporal operators, convexity, modal logic

1. Introduction

Temporal description logics have been studied for several decades now, with the aim of reasoning with dynamic dependencies between concepts and roles. Many complexity results have been obtained both for classical reasoning problems such as satisfiability [1, 2, 3] as well as for temporal query answering [4]. In the context of the $\mathcal{EL}$ family of DLs, it was investigated how tractability can be preserved [5, 6, 7, 8]. Reasoning in temporal extensions of $\mathcal{EL}$ easily becomes undecidable, in particular in the presence of temporalized roles [9, 6], but under some restrictions, tractability can be regained [6, 7, 8]. In particular, restricting the occurrences of temporal operators to the left-hand side of concept inclusions preserves the convexity of the entailment relation and allows classification via completion rules and temporal query answering using a single universal model, as in standard $\mathcal{EL}$ [10, 11]. Apart from classical temporal operators of *linear temporal logic (LTL)* such as $\diamondminus$ (stating that something has happened in the past), research has also considered *metric* temporal DLs that can use operators like $\diamondminus_{[0,5]}$ to restrict the temporal reach of such operators (in this case, to 5 time points into the past). However, including such expressive operators leads to even harder reasoning problems [12, 13].

In this paper, we reconsider the *convex temporal operators* $\diamondplus_n$ introduced for the temporal logic $\mathcal{TELH}_{\bot}^{\diamondplus,\mathsf{lhs}}$ [7], which are special metric temporal operators that preserve convexity when only employed on the left-hand side of concept inclusions. Intuitively, the concept $\diamondplus_2$ Alarm represents all objects that have intermittent alarms with gaps of at most 2 time points (see Figure 1). This allows us to smooth over gaps that are at most 2 time points long, and can be used as a kind of low-pass filter on incoming time-stamped data. As such, they can be seen as a means of dealing with imperfect, incomplete data. However, the $\diamondplus_n$ operators only represent one half of a low-pass filter, because they do not remove spurious *peaks* in the data, where an Alarm is present only at a few time points, but then disappears again. To deal with this, it is natural to consider the dual operators $\boxplus_n$ that can be obtained as in classical modal logic from $\diamondplus_n$ via negation. For example, the effect of $\boxplus_2$ is to remove all peaks that last for at most 2 time points (see Figure 1). These operators are also convex, but before we can add them to a logic like $\mathcal{TELH}_{\bot}^{\diamondplus,\mathsf{lhs}}$, we first need to fully understand how they behave when combined with other temporal operators. For example, in contrast to [7], where it was shown that the

DL 2026: 39th International Workshop on Description Logics, July 17–19, 2026, Lisbon, Portugal
stefan.borgwardt@tu-dresden.de (S. Borgwardt); alisa.kovtunova@tu-dresden.de (A. Kovtunova)
https://lat.inf.tu-dresden.de/~stefborg/ (S. Borgwardt); https://lat.inf.tu-dresden.de/~alisa/ (A. Kovtunova)
0000-0003-0924-8478 (S. Borgwardt); 0000-0001-9936-0943 (A. Kovtunova)

CEUR Workshop Proceedings ceur-ws.org ISSN 1613-0073
published 2026-07-25

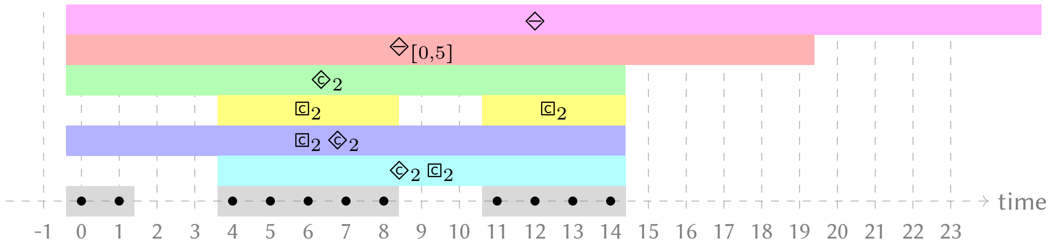


Figure 1: The behavior of temporal operators over $\mathbb{Z}$. The dots and gray boxes at the bottom represent the time points at which some concept (e.g., Alarm) holds for some individual, and the colored boxes enclose the (integer) time points resulting from applying various temporal operators to this concept.

Table 1
Semantics of temporal diamond and box operators. The input set $M \subseteq \mathbb{N}$ represents all time points at which a given property holds. In our semantics, "past" and "future" always include the current time point i.

$\triangle \in \mathfrak{D} \cup \mathfrak{B}$	$i \in \triangle M$ iff	The property described by M holds (relative to i) ...
$\diamondplus$	$\exists j.\ j \geq i \land j \in M$	eventually in the future
$\diamondminus$	$\exists j.\ j \leq i \land j \in M$	eventually in the past
$\diamondpm$	$\exists j.\ j \in M$	eventually in the past or the future
$\diamondc_\infty$	$\exists j, k.\ j \leq i \leq k \land j, k \in M$	eventually in the past and the future ...
$\diamondc_n$	$\exists j, k.\ j \leq i \leq k \land j, k \in M \land k - j \leq n + 1$	... separated by at most n time points
$\boxplus$	$\forall j.\ j \geq i \rightarrow j \in M$	always in the future
$\boxminus$	$\forall j.\ j \leq i \rightarrow j \in M$	always in the past
$\boxpm$	$\forall j.\ j \in M$	always in the past and the future
$\boxc_\infty$	$(-\infty, i] \subseteq M \lor [i, \infty) \subseteq M$	always in the past or the future
$\boxc_n$	$\exists j, k.\ j \leq i \leq k \land [j, k] \subseteq M \land k - j \geq n$	in the past and the future, for at least n time points

composition of several $\diamondc$ operators is commutative and equivalent to a single $\diamondc$ operator, if we add $\boxc$ into the mix, it makes a difference whether we consider the composition $\boxc_2 \diamondc_2$ (first apply $\diamondc_2$, and then apply $\boxc_2$ to the result) or $\diamondc_2 \boxc_2$. In Figure 1, we can see that $\diamondc_2 \boxc_2$ removes the short peak on the left, but also removes the gap between the two longer intervals.

In this paper, we present first results towards characterizing combinations of $\boxc_n$ and $\diamondc_n$ operators, not just via composition, but also point-wise intersection and union. These operations are relevant for reasoning in logics such as $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$. We show that subsumption between such complex operators can be decided in double exponential time in general, and in polynomial time when restricted to sequences of compositions. This paves the way towards extending reasoning procedures for $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$ [7] with the new $\boxc_n$ operators.

2. Preliminaries

We first recall the logic $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$, and in particular the temporal $\diamondc_n$ operators [7]. The set $\mathfrak{D} := \mathfrak{D}_{\mathsf{inf}} \cup \mathfrak{D}_{\mathsf{fin}}$ of *diamond operators* is composed of the classical operators $\mathfrak{D}_{\mathsf{inf}} := \{\diamondplus, \diamondminus, \diamondpm\}$ and the convex operators $\mathfrak{D}_{\mathsf{fin}} := \{\diamondc_n \mid n \in \mathbb{N} \cup \{\infty\}\}$. Their semantics is based on infinite sets of time points $M \subseteq \mathbb{Z}$ over the integer timeline. The idea is that M contains all time points at which a certain property X holds. For each operator $\Diamond \in \mathfrak{D}$, the set $\Diamond M$ then contains all time points from $\mathbb{Z}$ at which $\Diamond X$ holds, as defined in the upper half of Table 1 (cf. Figure 1). We slightly adapt the definitions from [7] to make them more intuitive in the new setting with box operators; in particular, $\diamondc_n$ now *removes all gaps of length* $\leq n$. More formally, given the input set M, after applying the diamond operator, $\diamondc_n M$ includes all time points i that lie between elements $j, k \in M$ with a difference of at most $n + 1$, i.e., j, k surround a gap of at most n time points that are not included in M. The special case $\diamondc_\infty$ removes all *finite* gaps from M. The classical operator $\diamondplus$ removes all finite gaps and *infinite gaps* of the form $(-\infty, k]$, and similarly for $\diamondminus$ and $\diamondpm$; the latter in particular maps $\varnothing$ to itself and all other sets to $\mathbb{Z}$.

We consider several ways of combining arbitrary diamond operators $\Diamond, \diamonddot \in \mathfrak{D}$ as follows:

- *Composition* $\Diamond \circ \diamonddot$, defined by $(\Diamond \circ \diamonddot)M := \Diamond(\diamonddot\, M)$, also written as $\Diamond \diamonddot$
- *Union* $\Diamond \cup \diamonddot$, defined by $(\Diamond \cup \diamonddot)M := (\Diamond\, M) \cup (\diamonddot\, M)$
- *Intersection* $\Diamond \cap \diamonddot$, defined by $(\Diamond \cap \diamonddot)M := (\Diamond\, M) \cap (\diamonddot\, M)$

We also consider the partial order $\subseteq$ (*subsumption*) on $\mathfrak{D}$ defined by $\Diamond \subseteq \diamonddot$ if $\Diamond\, M \subseteq \diamonddot\, M$ holds for all $M \subseteq \mathbb{Z}$, and define *equivalence* $\Diamond \equiv \diamonddot$ by $\Diamond \subseteq \diamonddot$ and $\diamonddot \subseteq \Diamond$. When we use the terms "largest/smallest" over a set of such operators, we refer to the order $\subseteq$. We recall the basic properties of these operators [7]. In particular, $\mathfrak{D}$ is closed under $\circ$, $\cup$, and $\cap$, the convex operators are totally ordered by $\subseteq$, all of them are extensive and monotone, and $\Diamond \cup \diamonddot$ and $\Diamond \circ \diamonddot$ are always equivalent (for $\Diamond, \diamonddot \in \mathfrak{D}$).

Lemma 1 ([7]). *For all $\Diamond, \diamonddot \in \mathfrak{D}$, $n, m \in \mathbb{N} \cup \{\infty\}$, and $M, M' \subseteq \mathbb{Z}$, we have*

(a) $M = \diamondc_0\, M$

(b) $\diamondc_n \subseteq \diamondc_m$ *iff* $n \leq m$

(c) $\diamondc_\infty \subseteq \diamondplus \subseteq \diamondpm$ *and* $\diamondc_\infty \subseteq \diamondminus \subseteq \diamondpm$*, but neither* $\diamondplus \subseteq \diamondminus$ *nor* $\diamondminus \subseteq \diamondplus$

(d) $M \subseteq \Diamond\, M$

(e) $M \subseteq M'$ *implies* $\Diamond\, M \subseteq \Diamond\, M'$

(f) $\Diamond\, \varnothing = \varnothing$ *and* $\Diamond\, \mathbb{Z} = \mathbb{Z}$

(g) $\Diamond \cap \diamonddot$ *is equivalent to the largest* $\diamondcircle \in \mathfrak{D}$ *with* $\diamondcircle \subseteq \Diamond$, $\diamondcircle \subseteq \diamonddot$

(h) $\Diamond \cup \diamonddot \equiv \Diamond \circ \diamonddot$ *is equivalent to the smallest* $\diamondcircle \in \mathfrak{D}$ *with* $\Diamond \subseteq \diamondcircle$, $\diamonddot \subseteq \diamondcircle$

In particular, from Lemma 1(b), (g), and (h), it follows that, for $n, m \in \mathbb{N} \cup \{\infty\}$, we have $\diamondc_n \cap \diamondc_m = \diamondc_{\min\{n,m\}}$ and $\diamondc_n \cup \diamondc_m = \diamondc_n \circ \diamondc_m = \diamondc_{\max\{n,m\}}$.

The temporal DL $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$ is based on *temporal roles* $r ::= s \mid \Diamond\, r$ and *temporal concepts* $C ::= A \mid \top \mid \bot \mid C \sqcap C \mid \exists r.C \mid \Diamond\, C$, where s is a role name, $\Diamond \in \mathfrak{D}$, and A is a concept name. It supports *concept inclusions* $C \sqsubseteq D$ and *role inclusions* $r \sqsubseteq s$, where s and D do not contain temporal operators. *Concept assertions* are of the form $A(a, i)$ and *role assertions* are of the form $s(a, b, i)$, where a, b are individual names and $i \in \mathbb{Z}$. The semantics is based on *temporal interpretations* $\mathfrak{I} = (\Delta^\mathfrak{I}, (\mathcal{I}_i)_{i \in \mathbb{Z}})$, where each $\mathcal{I}_i$ is a classical interpretation over the shared domain $\Delta^\mathfrak{I}$. The interpretation functions are defined as usual for atemporal concepts and roles, and additionally

$$(\Diamond\, r)^{\mathcal{I}_i} := \{(d, e) \in \Delta^\mathfrak{I} \times \Delta^\mathfrak{I} \mid i \in \Diamond\{j \in \mathbb{Z} \mid (d, e) \in r^{\mathcal{I}_j}\}\}$$
$$(\Diamond\, C)^{\mathcal{I}_i} := \{d \in \Delta^\mathfrak{I} \mid i \in \Diamond\{j \in \mathbb{Z} \mid d \in C^{\mathcal{I}_j}\}\}$$

for temporal roles and concepts, i.e., the time points at which a domain element $d \in \Delta^\mathfrak{I}$ is included in $\Diamond\, C$ are determined by applying $\Diamond$ to the temporal extension $\{j \in \mathbb{Z} \mid d \in C^{\mathcal{I}_j}\}$ of C. A temporal interpretation $\mathfrak{I}$ satisfies a concept inclusion $C \sqsubseteq D$ if $C^{\mathcal{I}_i} \subseteq D^{\mathcal{I}_i}$ holds for all $i \in \mathbb{Z}$, a concept assertion $A(a, i)$ if $a \in A^{\mathcal{I}_i}$ (using the standard name assumption), and similarly for role inclusions and -assertions.

Consistency and entailment for $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$ knowledge bases can be decided by means of completion rules similar to those for classical $\mathcal{EL}$ [10]. Since the focus of this paper is on the temporal operators rather than the temporal DL, we only consider a small example to motivate our research questions. The two sound rules

$$\mathsf{T4}\ \frac{\Diamond\, A_1 \sqsubseteq A_2 \qquad \diamonddot\, A_2 \sqsubseteq A_3}{(\diamonddot \circ \Diamond)A_1 \sqsubseteq A_3} \qquad \mathsf{T6}\ \frac{\Diamond\, A \sqsubseteq A_1 \qquad \diamonddot\, A \sqsubseteq A_2 \qquad A_1 \sqcap A_2 \sqsubseteq B}{(\Diamond \cap \diamonddot)A \sqsubseteq B}$$

illustrate how compositions and intersections of temporal operators naturally arise in the context of reasoning in temporal DLs. Moreover, if two rule applications derive different axioms of the form $\Diamond A \sqsubseteq B$ and $\mathord{\Diamond\mkern-11mu\cdot}\, A \sqsubseteq B$, they can be equivalently summarized by the axiom $(\Diamond \cup \mathord{\Diamond\mkern-11mu\cdot})A \sqsubseteq B$ by using the union of temporal operators. However, in contrast to $\mathcal{TELH}_{\bot}^{\mathord{\Diamond\mkern-11mu{\scriptstyle c}},\mathsf{lhs}}$, where the combination of two operators from $\mathfrak{D}$ again yields an operator from $\mathfrak{D}$, when adding temporal $\boxed{\scriptstyle c}_n$ operators, we have to deal with more complex expressions. Moreover, it is not straightforward to decide whether a subsumption or equivalence holds between two such expressions.

2.1. Adding Boxes

We now consider the dual *box operators* $\mathfrak{B} := \mathfrak{B}_{\mathsf{inf}} \cup \mathfrak{B}_{\mathsf{fin}}$ with the classical operators $\mathfrak{B}_{\mathsf{inf}} := \{\boxplus, \boxminus, \boxed{\scriptstyle\pm}\}$ and the convex operators $\mathfrak{B}_{\mathsf{fin}} := \{\boxed{\scriptstyle c}_n \mid n \in \mathbb{N} \cup \{\infty\}\}$, whose semantics is given in the lower half of Table 1. Dually to the diamond operators, $\boxed{\scriptstyle c}_n$ *removes all peaks of length* $\leq n$ from M, i.e., intervals $[j,k] \subseteq M$ with $j-1 \notin M$, $k+1 \notin M$, and $k-j < n$. Again, $\boxed{\scriptstyle c}_\infty$ removes all *finite* intervals from M, i.e., $\boxed{\scriptstyle c}_\infty M$ can only contain intervals of the forms $[j,\infty)$, $(-\infty,k]$, or $(-\infty,\infty)$. The operator $\boxplus$ additionally removes infinite intervals of the form $(-\infty,k]$, and similarly for $\boxminus$ and $\boxed{\scriptstyle\pm}$; the latter in particular maps $\mathbb{Z}$ to itself and any other set to $\varnothing$.

It is easy to see that the operators in $\mathfrak{D}_{\mathsf{inf}}$ and $\mathfrak{B}_{\mathsf{inf}}$ are duals of each other, in the sense that $\overline{\Box M} = \Diamond \overline{M}$, where $\bar{\cdot}$ denotes the complement w.r.t. $\mathbb{Z}$, $\Box \in \mathfrak{B}_{\mathsf{inf}}$, and $\Diamond$ is the corresponding operator from $\mathfrak{D}_{\mathsf{inf}}$. We verify that the same is also true for $\mathfrak{D}_{\mathsf{fin}}$ and $\mathfrak{B}_{\mathsf{fin}}$.

Lemma 2. *For every $\boxed{\scriptstyle c}_n \in \mathfrak{B}_{\mathsf{fin}}$ and every $M \subseteq \mathbb{Z}$, it holds that $\overline{\boxed{\scriptstyle c}_n M} = \mathord{\Diamond\mkern-11mu{\scriptstyle c}}_n \overline{M}$.*

Proof. We have $i \in \mathord{\Diamond\mkern-11mu{\scriptstyle c}}_\infty \overline{M}$ iff there are $j,k \notin M$ with $j \leq i \leq k$, which is equivalent to neither $(-\infty,i]$ nor $[i,\infty)$ being contained in M, i.e., to $i \in \overline{\boxed{\scriptstyle c}_\infty M}$. Similarly, for finite $n \in \mathbb{N}$, we have $i \in \mathord{\Diamond\mkern-11mu{\scriptstyle c}}_n \overline{M}$ iff there are $j,k \notin M$ with $j \leq i \leq k$ and $k-j \leq n+1$, which is equivalent to the non-existence of an interval $[j',k'] \subseteq M$ that includes i and is of length $k'-j' \geq n$, i.e., to $i \in \overline{\boxed{\scriptstyle c}_n M}$. □

Due to Lemma 2, all known properties from Lemma 1 also hold for the operators in $\mathfrak{B}$, but with the order reversed, i.e., with "smallest" swapped with "largest", $\subseteq$ swapped with $\supseteq$, and $\cup$ swapped with $\cap$:

Corollary 1. *For all $\Box, \boxdot \in \mathfrak{B}$, $n, m \in \mathbb{N} \cup \{\infty\}$, and $M, M' \subseteq \mathbb{Z}$, we have*

(a) $M = \boxed{\scriptstyle c}_0 M$

(b) $\boxed{\scriptstyle c}_n \subseteq \boxed{\scriptstyle c}_m$ *iff* $m \leq n$

(c) $\boxed{\scriptstyle\pm} \subseteq \boxplus \subseteq \boxed{\scriptstyle c}_\infty$ *and* $\boxed{\scriptstyle\pm} \subseteq \boxminus \subseteq \boxed{\scriptstyle c}_\infty$, *but neither* $\boxplus \subseteq \boxminus$ *nor* $\boxminus \subseteq \boxplus$

(d) $\Box M \subseteq M$

(e) $M \subseteq M'$ *implies* $\Box M \subseteq \Box M'$

(f) $\Box \varnothing = \varnothing$ *and* $\Box \mathbb{Z} = \mathbb{Z}$

(g) $\Box \cup \boxdot$ *is equivalent to the smallest* $\boxed{\circ} \in \mathfrak{B}$ *with* $\Box \subseteq \boxed{\circ}$, $\boxdot \subseteq \boxed{\circ}$

(h) $\Box \cap \boxdot \equiv \Box \circ \boxdot$ *is equivalent to the largest* $\boxed{\circ} \in \mathfrak{B}$ *with* $\boxed{\circ} \subseteq \Box$, $\boxed{\circ} \subseteq \boxdot$

This also proves that these operators are *convex*, i.e., whenever an intersection ψ of box operators is subsumed by a union ϕ of box operators, then ψ is already subsumed by *one* of the operators in ϕ. To see an example of the order $\subseteq$ on $\mathfrak{D}_{\mathsf{fin}}$ and $\mathfrak{B}_{\mathsf{fin}}$, consider the leftmost arrows in Figure 2.

We now investigate the combined set of operators $\mathfrak{T} := \mathfrak{D} \cup \mathfrak{B}$, which does not satisfy all the nice properties in Lemma 1 and Corollary 1. We denote by $\mathfrak{T}^*$ the set of all finite sequences (compositions) of such operators, and by $\mathfrak{T}^\times$ the set of all formulas that can be constructed from $\mathfrak{T}$ by applying $\circ$, $\cup$, and $\cap$, whose semantics is defined exactly as above. We also define the relations $\subseteq$, $\equiv$ on $\mathfrak{T}^\times$ as before.

We consider the empty sequence $\epsilon \in \mathfrak{T}^*$ to represent the identity function on $2^{\mathbb{Z}}$, which is equivalent to $\boxc_0$ and $\diamondc_0$ by Lemma 1(a) and Corollary 1(a).

Our goal in this paper is to decide $\sigma \sqsubseteq \tau$ for any formulas $\sigma, \tau \in \mathfrak{T}^\times$, in order to be able to use these complex operators for reasoning in an extension $\mathcal{TELH}_\bot^{\diamondc,\boxc,\mathsf{lhs}}$ of $\mathcal{TELH}_\bot^{\diamondc,\mathsf{lhs}}$.

3. Operator Sequences

As a first step, we prove that all sequences from $\mathfrak{T}^*$ can be transformed into a normal form.

Lemma 3. *Every $\sigma \in \mathfrak{T}^*$ is equivalent to a sequence of the form $\sigma_{\mathsf{inf}} \triangle_1 \triangledown_1 \dots \triangle_k [\triangledown_k]$, where*

- *$\triangle$ denotes an operator type ($\Diamond$ or $\Box$) and $\triangledown$ the opposite type,*
- *$\triangledown_k$ is optional,*
- *σ_{inf} is one of the following sequences of classical operators: $\triangledownplus$, $\triangledown$,$\triangledownpm$, $\triangleplus\ \triangledownplus$, or $\triangle\ \triangledown$,*
- *for all $i \in \{1, \dots, k\}$, $\triangle_i = \trianglec_{j_i}$ with $j_1 > \dots > j_k > 0$, and*
- *for all $i \in \{1, \dots, k\}$, $\triangledown_i = \triangledownc_{\ell_i}$ with $\ell_1 > \dots > \ell_k > 0$ (or $\ell_1 > \dots > \ell_{k-1} > 0$ if $\triangledown_k$ is absent).*

Proof. Lemma 1(h) and Corollary 1(h) show that the composition of any two operators of the same type is equivalent to another operator of that type, and thus we can merge any two consecutive operators of the same type in σ. Furthermore, the result of applying any operator from $\mathfrak{D}_{\mathsf{inf}} \cup \mathfrak{B}_{\mathsf{inf}}$ to any $M \subseteq \mathbb{Z}$ cannot contain any finite gaps or peaks, and thus applying convex operators after (to the left of) classical operators has no effect. For sequences σ_{inf} of classical operators, the following equivalences hold (for all $\triangle \in \mathfrak{D}_{\mathsf{inf}} \cup \mathfrak{B}_{\mathsf{inf}}$):

$$\begin{aligned} &\triangle \diamondpm \equiv \diamondpm, \quad \triangle \boxpm \equiv \boxpm, \quad \boxminus \diamondplus \equiv \diamondplus, \quad \diamondminus \boxplus \equiv \boxplus, \quad \boxplus \diamondminus \equiv \diamondminus, \quad \diamondplus \boxminus \equiv \boxminus, \\ &\boxplus \diamondplus \equiv \boxpm \diamondplus, \quad \diamondplus \boxplus \equiv \diamondpm \boxplus, \quad \boxminus \diamondminus \equiv \boxpm \diamondminus, \quad \diamondminus \boxminus \equiv \diamondpm \boxminus . \end{aligned}$$

The first two equivalences hold because $\boxpm M$ and $\diamondpm M$ are always either $\varnothing$ or $\mathbb{Z}$ and by Lemma 1(f) and Corollary 1(f). Similarly, $\diamondplus M$ is always of the form $\varnothing$, $\mathbb{Z}$, or $(-\infty, \max M]$, on which applying $\boxminus$ has no effect, and $\boxpm$ has the same effect as $\boxplus$. The other equivalences follow by analogous arguments. Thus, there are essentially only four combinations of two classical operators that are not equivalent to a single operator: "always eventually" and "eventually always", in both directions.

It remains to show that every sequence of convex operators is equivalent to a sequence of the form $\triangle_1 \triangledown_1 \dots \triangle_k [\triangledown_k]$ as above. To see this, observe that $\boxc_k \diamondc_\ell \boxc_m \equiv \diamondc_\ell \boxc_m$ whenever $k \leq m$, since $\boxc_m M$ cannot contain any peaks of length $\leq m$ and $\diamondc_\ell$ can only increase the length of peaks, and thus the subsequent $\boxc_k$ has no effect. The dual equivalence $\diamondc_k \boxc_\ell \diamondc_m \equiv \boxc_\ell \diamondc_m$ follows by Lemma 2. This means that we can remove any operators from σ that do not have a larger index than any previous (to the right) operators of the same kind. Finally, note that all operators $\trianglec_0$ and $\triangledownc_0$ can be removed, since they are equivalent to ϵ by Lemma 1(a) and Corollary 1(a). $\square$

By applying the equivalences used in this proof, we can transform every sequence into normal form in polynomial time, since each step removes at least one operator (or transforms $\trianglepm$ into $\triangleplus$ or $\triangle$). In the following, we consider only sequences of *convex* operators in normal form (i.e., without σ_{inf}), and want to characterize subsumption between such sequences.

Example 1. Consider the finite subset $\{\boxc_4, \boxc_3, \diamondc_2, \diamondc_5\} \subseteq \mathfrak{T}$. Figure 2 depicts the $\sqsubseteq$-order between all possible sequences over these four operators in normal form.

To determine whether two sequences are subsumed by each other, we can first compare the right-most operator (which is applied first). Indeed, in the figure we can see that a sequence ending in $\boxc_4$ cannot subsume a sequence ending in $\boxc_3$, etc. (see the colored layers in the diagram). However, checking subsumption is not as straightforward as comparing the operators in each sequence one by one in

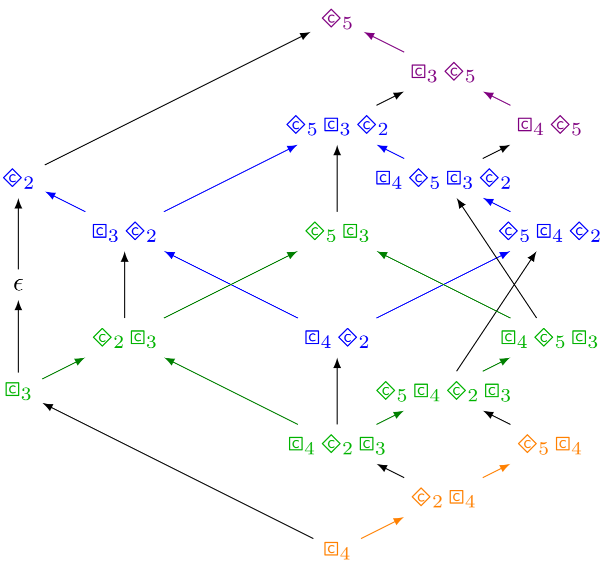


Figure 2: The order diagram of $\subseteq$ on a subset of $\mathfrak{T}^*$. Sequences ending in the same operator have the same color.

right-to-left order. For example, it holds that $\diamonddot_2 \boxdot_4 \subseteq \boxdot_4 \diamonddot_2 \boxdot_3$, and indeed $\boxdot_4 \subseteq \boxdot_3$ and $\diamonddot_2 \subseteq \diamonddot_2$, but the remaining $\boxdot_4$ does not have a clear corresponding operator. However, recall from the proof of Lemma 3 that $\diamonddot_2 \boxdot_4 \equiv \boxdot_4 \diamonddot_2 \boxdot_4$, and hence we can also match the last operator and conclude that $\diamonddot_2 \boxdot_4 \equiv \boxdot_4 \diamonddot_2 \boxdot_4 \subseteq \boxdot_4 \diamonddot_2 \boxdot_3$.

Likewise, we can add a $\boxdot_0 \equiv \diamonddot_0 \equiv \epsilon$ as the rightmost element of any sequence, e.g., to easily see that $\diamonddot_5 \boxdot_3 (\diamonddot_0) \subseteq \diamonddot_5 \boxdot_3 \diamonddot_2$. On the other hand, consider the pair $\diamonddot_2 \boxdot_4 \nsubseteq \boxdot_3$. Although it holds that $\boxdot_4 \subseteq \boxdot_3$, there is no equivalent extension to the second sequence that would allow us to match the remaining $\diamonddot_2$ operator.

For a more complex example, consider $\diamonddot_5 \boxdot_4 \subseteq \diamonddot_5 \boxdot_4 \diamonddot_2 \boxdot_3$. While $\boxdot_4 \subseteq \boxdot_3$ holds, the next operator $\diamonddot_5$ can only be matched by the last operator in the second sequence. However, we can consider the expansion $\diamonddot_5 \boxdot_4 \equiv \diamonddot_5 (\boxdot_4 \diamonddot_0) \boxdot_4 \subseteq \diamonddot_5 \boxdot_4 \diamonddot_2 \boxdot_3$ to align the sequences. This demonstrates that we can sometimes skip parts of sequences, if the indices of the surrounding operators allow this (in the example, it worked because the first operator was $\boxdot_4$).

3.1. Finite-State Transducers Comparing Sequences of Convex Operators

We will use a special model of finite-state transducer that implements the intuitions described in the example. We only consider a finite subset $\mathfrak{S}_{\mathsf{fin}} \subseteq \mathfrak{T}_{\mathsf{fin}} := \mathfrak{D}_{\mathsf{fin}} \cup \mathfrak{B}_{\mathsf{fin}}$ here, e.g., consisting of the convex operators occurring in the input sequences that we want to compare. The treatment of the infinite operators $\mathfrak{T}_{\mathsf{inf}}$ is left for future work, since their behavior is mostly independent of the finite operators and there are only finitely many combinations to consider (see Lemma 3). In the following, let $\sigma, \tau \in \mathfrak{S}^*_{\mathsf{fin}}$ be two sequences in normal form according to Lemma 3.

As seen in Example 1, the natural way to compare these sequences is from right to left, which is why we use a non-standard model of finite-state transducer that works in this direction.

Definition 1. A *(right-to-left) finite-state transducer* is a tuple $T = (Q, \Sigma, \Gamma, I, \Delta, F)$, where Q is a finite set of *states*, Σ is a finite *input alphabet*, Γ is a finite *output alphabet*, $I \subseteq Q$ is the set of *initial states*, $\Delta \subseteq Q \times \Sigma^* \times \Gamma^* \times Q$ is the *transition relation*, and $F \subseteq Q$ is the set of *final states*.

A *run* of T on two words $\sigma \in \Sigma^*$ and $\tau \in \Sigma^*$ is a sequence $q_n, (\sigma_n, \tau_n), \dots, q_1, (\sigma_1, \tau_1), q_0$, $n \geq 0$, alternating between states q_i and pairs (σ_i, τ_i) of input-output words, such that $\sigma_n \dots \sigma_1 = \sigma$, $\tau_n \dots \tau_1 = \tau$, and $(q_i, \sigma_i, \tau_i, q_{i-1}) \in \Delta$ for all $i \in \{1, \dots, n\}$. We write $q_n \in \Delta(\sigma, \tau, q_0)$ if there is such a run. The *language* accepted by T is $L(T) := \{(\sigma, \tau) \in \Sigma^* \times \Gamma^* \mid \exists q_0 \in I, q_n \in F \colon q_n \in \Delta(\sigma, \tau, q_0)\}$.

Table 2
The transition relation Δ_{fin} of T_{fin}

successor state	input word	output word	current state			
$(q_{0,\ell,\boxdot},$	$\epsilon,$	$\diamonddot_\ell,$	q_0	$)$		(T1)
$(q_{k,0,\diamonddot},$	$\boxdot_k,$	$\epsilon,$	q_0	$)$		(T2)
$(q_{0,0,\boxdot},$	$\epsilon,$	$\epsilon,$	q_0	$)$		(T3)
$(q_{0,0,\diamonddot},$	$\epsilon,$	$\epsilon,$	q_0	$)$		(T4)
$(q_{k,j,\diamonddot},$	$\boxdot_k,$	$\boxdot_\ell,$	$q_{i,j,\boxdot}$	$)$	if $k \geq \ell$	(T5)
$(q_{i,\ell,\boxdot},$	$\diamonddot_k,$	$\diamonddot_\ell,$	$q_{i,j,\diamonddot}$	$)$	if $k \leq \ell$	(T6)
$(q_{i,\ell,\boxdot},$	$\epsilon,$	$\diamonddot_\ell \boxdot_k,$	$q_{i,j,\boxdot}$	$)$	if $k \leq i$	(T7)
$(q_{k,j,\boxdot},$	$\diamonddot_\ell \boxdot_k,$	$\epsilon,$	$q_{i,j,\boxdot}$	$)$	if $\ell \leq j$	(T8)
$(q_{\ell,j,\diamonddot},$	$\boxdot_\ell \diamonddot_k,$	$\epsilon,$	$q_{i,j,\diamonddot}$	$)$	if $k \leq j$	(T9)
$(q_{i,k,\diamonddot},$	$\epsilon,$	$\boxdot_\ell \diamonddot_k,$	$q_{i,j,\diamonddot}$	$)$	if $\ell \leq i$	(T10)
$(q_f,$	$\epsilon,$	$\boxdot_k,$	$q_{i,j,\boxdot}$	$)$	if $k \leq i$	(T11)
$(q_f,$	$\boxdot_k,$	$\epsilon,$	$q_{i,j,\boxdot}$	$)$		(T12)
$(q_f,$	$\diamonddot_k,$	$\epsilon,$	$q_{i,j,\diamonddot}$	$)$	if $k \leq j$	(T13)
$(q_f,$	ϵ	$\diamonddot_k,$	$q_{i,j,\diamonddot}$	$)$		(T14)
$(q_f,$	$\epsilon,$	$\epsilon,$	$q_{i,j,\boxdot}$	$)$		(T15)
$(q_f,$	$\epsilon,$	$\epsilon,$	$q_{i,j,\diamonddot}$	$)$		(T16)

We now define the transducer $T_{\mathsf{fin}} := (Q_{\mathsf{fin}}, \mathfrak{S}_{\mathsf{fin}}, \mathfrak{S}_{\mathsf{fin}}, \{q_0\}, \Delta_{\mathsf{fin}}, \{q_f\})$ that expresses the semantic relation $\sigma \subseteq \tau$ between two sequences $\sigma, \tau \in \mathfrak{S}_{\mathsf{fin}}^*$. The states in $Q_{\mathcal{T}} := \{q_0, q_f\} \cup \{q_{i,j,\triangle} \mid \boxdot_i, \diamonddot_j \in \mathfrak{S}_{\mathsf{fin}}, \triangle \in \{\boxdot, \diamonddot\}\}$ track the bound i of the latest $\boxdot$ operator encountered in σ, the bound j of the latest $\diamonddot$ operator in τ, and whether the next operators to be compared are of the form $\boxdot$ or $\diamonddot$. Recall that we assume σ and τ to be in normal form, and hence they are alternating sequences of $\boxdot$ and $\diamonddot$ operators, and the bounds of each operator type are increasing (when reading from right to left). However, the two sequences do not necessarily start or end with the same type of operator. To process such sequences, Δ_{fin} contains the transitions listed in Table 2, for all $\boxdot_i, \boxdot_k, \boxdot_\ell, \diamonddot_j, \diamonddot_k, \diamonddot_\ell \in \mathfrak{S}_{\mathsf{fin}}$.[1] Transitions (T1)–(T2) match the first $\diamonddot_\ell$ or $\boxdot_k$ to an imaginary $\diamonddot_0 \subseteq \diamonddot_\ell$ or $\boxdot_0 \supseteq \boxdot_k$ in the other sequence, in order to align the sequences to the same operator type. The next two transitions (T3)–(T4) are used if the sequences are already aligned. The transitions (T5)–(T6) take care of matching $\diamonddot$ and $\boxdot$ operators in an alternating fashion, obeying the order $\subseteq$ on $\mathfrak{S}_{\mathsf{fin}}$. The next transition (T7) allows skipping over any $\diamonddot_\ell$ (and the intermediate $\boxdot_k$) in the output sequence τ, as long as $\boxdot_k$ subsumes the last $\boxdot_i$ read from the input sequence σ. Then, transition (T8) allows the same in σ, but only if $\diamonddot_\ell$ is subsumed by the last $\diamonddot_j$ seen in τ. The subsequent two transitions (T9)–(T10) encode the analogous behavior when the two operator types are swapped. Finally, at the end of the sequences, it is possible to introduce one more imaginary $\diamonddot_j$ or $\boxdot_i$ (or $\diamonddot_0$ or $\boxdot_0$) operator to match the last remaining operator in the other sequence ((T11)–(T14)), or simply accept if both sequences have been processed completely ((T15)–(T16)).

Lemma 4. *For all $\sigma, \tau \in \mathfrak{S}_{\mathsf{fin}}^*$ in normal form, we have $(\sigma, \tau) \in L(T_{\mathsf{fin}})$ iff $\sigma \subseteq \tau$.*

To prove this result, we introduce some auxiliary notations. We say that σ and τ are $\triangle$-*compatible* if either both sequences start with the same kind of operator $\triangle \in \{\boxdot, \diamonddot\}$ or one (or both) are empty. Moreover, $\sigma \subseteq_{i,j} \tau$ denotes the fact that $\sigma M \subseteq \tau M$ holds for all $M \subseteq \mathbb{Z}$ with $\boxdot_i M = M$ and $\diamonddot_j M = M$, i.e., M is closed under applications of $\boxdot_k$, $k \leq i$, and $\diamonddot_\ell$, $\ell \leq j$, which means M does not have peaks of length $\leq i$ or gaps of length $\leq j$. Intuitively, in a state $q_{i,j,\triangle}$ we have already applied $\boxdot_i$ and $\diamonddot_j$, and thus can assume that the set of time points is closed under these operators. Moreover, we know that all

[1] We write the successor state first and the current state last in order to be consistent with the direction in which the words are processed. In contrast, we write the input word before the output word to stay consistent with the direction of the subsumption $\sigma \subseteq \tau$.

remaining $\boxdot_k$ operators in σ must be such that $k > i$, and similarly for $\diamonddot_\ell$ operators in τ. Note that $\subseteq_{0,0}$ is the same as $\subseteq$, since $\boxdot_0 M = M = \diamonddot_0 M$ by Lemmas 1(a) and 2.

Lemma 5. *If σ and τ are $\triangledot$-compatible, σ contains no $\boxdot_k$ with $k \le i$, and τ contains no $\diamonddot_\ell$ with $\ell \le j$, then $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau, q_{i,j,\triangledot})$ iff $\sigma \subseteq_{i,j} \tau$.*

Proof. We consider only the case of $\triangledot = \boxdot$ here, since the other case can be handled by dual arguments.

"$\Rightarrow$": We show $\sigma \subseteq_{i,j} \tau$ by induction on the length of the run of T_{fin} to q_f from $q_{i,j,\triangledot}$. We consider any $M \subseteq \mathbb{Z}$ such that $\boxdot_i M = M = \diamonddot_j M$, and show that $\sigma M \subseteq \tau M$. For the induction base, the run consists of exactly one transition leading to q_f:

- If the last transition is $(q_f, \epsilon, \boxdot_k, q_{i,j,\boxdot})$ (T11), then $\sigma = \epsilon$ and $\tau = \boxdot_k$ with $k \le i$. Thus, we have $\sigma M = M = \boxdot_i M \subseteq \boxdot_k M = \tau M$ by Corollary 1(b).
- For $(q_f, \boxdot_k, \epsilon, q_{i,j,\boxdot})$ (T12), we have $\sigma = \boxdot_k$ and $\tau = \epsilon$, and thus $\sigma M = \boxdot_k M \subseteq \boxdot_0 M = M = \tau M$.
- For $(q_f, \epsilon, \epsilon, q_{i,j,\boxdot})$ (T15), we trivially have $\sigma = \epsilon = \tau$.

For the induction step, we assume that the claim holds for all shorter runs ending in q_f, and make a case distinction on the shape of the first transition starting from $q_{i,j,\boxdot}$:

- $(q_{k,j,\diamonddot}, \boxdot_k, \boxdot_\ell, q_{i,j,\boxdot})$ with $k \ge \ell$ (T5) and $q_f \in \Delta_{\mathsf{fin}}(\sigma', \tau', q_{k,j,\diamonddot})$ with $\sigma = \sigma' \boxdot_k$ and $\tau = \tau' \boxdot_\ell$: By the induction hypothesis, we have $\sigma' \subseteq_{k,j} \tau'$, and have to show that $\sigma' \boxdot_k M \subseteq \tau' \boxdot_\ell M$. For $M' := \boxdot_k M$, we have $\boxdot_k M' = \boxdot_k \boxdot_k M = \boxdot_k M = M'$ by Corollary 1(h). Moreover, $\diamonddot_j M' = \diamonddot_j \boxdot_k M = \diamonddot_j \boxdot_k \diamonddot_j M = \boxdot_k \diamonddot_j M = \boxdot_k M = M'$ by Lemma 3. Thus, $\sigma' \boxdot_k M = \sigma' M' \subseteq \tau' M' = \tau' \boxdot_k M \subseteq \tau' \boxdot_\ell M$ by the fact that $\sigma' \subseteq_{k,j} \tau'$ and Lemma 1(e) and Corollary 1(b) and (e).
- $(q_{i,\ell,\boxdot}, \epsilon, \diamonddot_\ell \boxdot_k, q_{i,j,\boxdot})$ with $k \le i$ (T7) and $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau', q_{i,\ell,\boxdot})$ with $\tau = \tau' \diamonddot_\ell \boxdot_k$: By the induction hypothesis, we have $\sigma \subseteq_{i,\ell} \tau'$, and have to show $\sigma M \subseteq \tau' \diamonddot_\ell \boxdot_k M$. For $M' := \diamonddot_\ell M$, we have $\boxdot_i M' = \boxdot_i \diamonddot_\ell M = \boxdot_i \diamonddot_\ell \boxdot_i M = \diamonddot_\ell \boxdot_i M = \diamonddot_\ell M = M'$ and $\diamonddot_\ell M' = \diamonddot_\ell \diamonddot_\ell M = \diamonddot_\ell M = M'$ by Corollary 1(h). Thus, $\sigma M \subseteq \sigma \diamonddot_\ell M = \sigma M' \subseteq \tau' M' = \tau' \diamonddot_\ell M = \tau' \diamonddot_\ell \boxdot_i M \subseteq \tau' \diamonddot_\ell \boxdot_k M$ by the fact that $\sigma \subseteq_{i,\ell} \tau'$ and Lemma 1(d) and (e) and Corollary 1(b) and (e).
- $(q_{k,j,\boxdot}, \diamonddot_\ell \boxdot_k, \epsilon, q_{i,j,\boxdot})$ with $\ell \le j$ (T8) and $q_f \in \Delta_{\mathsf{fin}}(\sigma', \tau, q_{k,j,\boxdot})$ with $\sigma = \sigma' \diamonddot_\ell \boxdot_k$: By the induction hypothesis, we have $\sigma' \subseteq_{k,j} \tau$, and have to show $\sigma' \diamonddot_\ell \boxdot_k M \subseteq \tau M$. For $M' := \boxdot_k M$, we obtain $\boxdot_k M' = M' = \diamonddot_j M'$ as above. Thus, $\sigma' \diamonddot_\ell \boxdot_k M = \sigma' \diamonddot_\ell M' \subseteq \sigma' \diamonddot_j M' = \sigma' M' \subseteq \tau M' = \tau \boxdot_k M \subseteq \tau M$ by the fact that $\sigma' \subseteq_{k,j} \tau$ and Lemma 1(b) and (e), and Corollary 1(d) and (e).

"$\Leftarrow$": We show that $\sigma \subseteq_{i,j} \tau$ implies $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau, q_{i,j,\boxdot})$ by induction on $|\sigma| + |\tau|$. For the induction base, we assume that $|\sigma| + |\tau| \le 1$, and distinguish three cases:

- If $\sigma = \tau = \epsilon$, then $q_f \in \Delta_{\mathsf{fin}}(\epsilon, \epsilon, q_{i,j,\boxdot})$ holds by (T15).
- If $\sigma = \boxdot_k$ and $\tau = \epsilon$, then $q_f \in \Delta_{\mathsf{fin}}(\boxdot_k, \epsilon, q_{i,j,\boxdot})$ by (T12).
- If $\sigma = \epsilon$ and $\tau = \boxdot_k$, then $\sigma \subseteq_{i,j} \tau$ implies that $k \le i$ (otherwise the interval $M := [-i, 0]$ of length $i + 1$ would be a counterexample: $\boxdot_i M = M = \diamonddot_j M$ and $\sigma M = M \nsubseteq \emptyset = \boxdot_k M = \tau M$), and thus again $q_f \in \Delta_{\mathsf{fin}}(\epsilon, \boxdot_k, q_{i,j,\boxdot})$ by (T11).

For the induction step, we assume that $|\sigma| + |\tau| > 1$ and that the claim holds for all pairs of sequences with a smaller combined length. Assuming that $\sigma \subseteq_{i,j} \tau$, σ contains no $\boxdot_k$ with $k \le i$, and τ contains no $\diamonddot_\ell$ with $\ell \le j$, we again consider several cases:

- If $\sigma = \sigma' \boxdot_k$ and $\tau = \tau' \boxdot_\ell$ with $k \ge \ell$, then $|\sigma'| + |\tau'| < |\sigma| + |\tau|$, σ' and τ' are $\diamonddot$-compatible, and we can show that $\sigma' \subseteq_{k,j} \tau'$. For this, consider any $M \subseteq \mathbb{Z}$ with $\boxdot_k M = M = \diamonddot_j M$. Since $0 \le i < k$, we have $M \subseteq \boxdot_k M \subseteq \boxdot_i M \subseteq M$ by Corollary 1(d) and (e), i.e., $\boxdot_i M = M$. Thus, by the assumption $\sigma \subseteq_{i,j} \tau$, we obtain $\sigma' M = \sigma' \boxdot_k M \subseteq \tau' \boxdot_\ell M \subseteq \tau' M$ by Lemma 1(e) and Corollary 1(d) and (e), which shows that $\sigma' \subseteq_{k,j} \tau'$. Finally, since σ is in normal form (see Lemma 3), σ' contains no $\boxdot_{k'}$ with $k' \le k$, and thus we can apply the induction hypothesis to obtain that $q_f \in \Delta_{\mathsf{fin}}(\sigma', \tau', q_{k,j,\diamonddot})$. Together with the transition $(q_{k,j,\diamonddot}, \boxdot_k, \boxdot_\ell, q_{i,j,\boxdot})$ with $k \ge \ell$ (T5), we conclude that $q_f \in \Delta_{\mathcal{T}}(\sigma, \tau, q_{i,j,\boxdot})$.

- If $\sigma = \sigma' \diamonddot_m \boxdot_k$ with $m \le j$, then we show that σ' and τ satisfy the preconditions of the induction hypothesis w.r.t. $\subseteq_{k,j}$. It is clear that $|\sigma'| + |\tau| < |\sigma| + |\tau|$, σ' and τ are $\boxdot$-compatible, and σ' contains no $\boxdot_{k'}$ with $k' \le k$, and thus it remains to show that $\sigma' \subseteq_{k,j} \tau$. Consider any $M \subseteq \mathbb{Z}$ with $\boxdot_k M = M = \diamonddot_j M$. As in the previous case, we also know that $i < k$, and thus $\boxdot_i M = M$. By the assumption $\sigma \subseteq_{i,j} \tau$ and Lemma 1(d), we obtain $\sigma' M \subseteq \sigma' \diamonddot_m M = \sigma' \diamonddot_m \boxdot_k M = \sigma M \subseteq \tau M$, which shows that $\sigma' \subseteq_{k,j} \tau$ holds. By the induction hypothesis, we know that $q_f \in \Delta_{\mathsf{fin}}(\sigma', \tau, q_{k,j,\boxdot})$. Together with the transition $(q_{k,j,\boxdot}, \diamonddot_m \boxdot_k, \epsilon, q_{i,j,\boxdot})$ with $m \le j$ (T8), this shows that $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau, q_{i,j,\boxdot})$.

- If neither $k \ge \ell$ nor $m \le j$ hold (or the corresponding operators do not exist, e.g., if σ or τ are empty), then we can show that τ must have at least length 2. Suppose for a contradiction that $|\tau| \le 1$ (but recall that $|\sigma| + |\tau| > 1$).
  - If $\tau = \epsilon$, then σ must contain at least two operators, i.e., $\sigma = \sigma' \diamonddot_m \boxdot_k$, with $m > j$ and $k > i$. Consider the set $M := [-k, 0] \cup [j+2, \infty)$ (or $(-\infty, 0] \cup [j+2, \infty)$ if $k = \infty$) that has a peak of size $k+1$ followed by a gap of size $j+1$. Then $\boxdot_k M = \boxdot_i M = M = \diamonddot_j M$, but $\sigma M = \sigma' \diamonddot_m \boxdot_k M = \sigma' \diamonddot_m M = \sigma'[-k, \infty) = [-k, \infty) \nsubseteq M = \tau M$ (or correspondingly with $\mathbb{Z}$ instead of $[-k, \infty)$), which contradicts the assumption that $\sigma \subseteq_{i,j} \tau$.
  - If $\tau = \boxdot_\ell$, then σ contains at least one operator, i.e., $\sigma = \sigma' \boxdot_k$, with $i < k < \ell$. If $\sigma' = \epsilon$, then $M := [-k, 0]$ is a counterexample for $\boxdot_k = \sigma \subseteq_{i,j} \tau = \boxdot_\ell$ since $k < \infty$. Thus, $\sigma = \sigma'' \diamonddot_m \boxdot_k$ and $M := [-k, 0] \cup [j+2, \infty)$ or $M := (-\infty, k] \cup [j+2, \infty)$ from above is again a counterexample, because $\tau M = \boxdot_\ell M \subseteq M$ by Corollary 1(d).

  Thus, we must have $\tau = \tau' \diamonddot_p \boxdot_\ell$, and we show that $\ell \le i$. Assume to the contrary that $\ell > i$. Then σ cannot consist of a single $\boxdot_k$ (with $k < \ell$), since then again $M := [-k, 0]$ would be a counterexample to $\boxdot_k = \sigma \subseteq_{i,j} \tau = \tau' \diamonddot_p \boxdot_\ell$. The same holds for the case where $\sigma = \epsilon$. Thus, we must have $\sigma = \sigma' \diamonddot_m \boxdot_k$ with $i < k < \ell$ and $j < m$. Then, $M := [-k, 0] \cup [j+2, \infty)$ is again a counterexample to $\sigma \subseteq_{i,j} \tau$, since $\sigma M = \sigma' \diamonddot_m \boxdot_k M = \sigma'[-k, \infty] = [-k, \infty) \nsubseteq [j+2, \infty) = \tau' \diamonddot_p [j+2, \infty) = \tau' \diamonddot_p \boxdot_\ell M = \tau M$.

  Thus, we know that $\ell \le i$. We show that σ and τ' satisfy the preconditions of the induction hypothesis w.r.t. $\subseteq_{i,p}$. Since τ is in normal form (see Lemma 3), we have $|\sigma| + |\tau'| < |\sigma| + |\tau|$, σ and τ' are $\boxdot$-compatible, and τ' contains no $\diamonddot_{p'}$ with $p' \le p$. Consider now a set $M \subseteq \mathbb{Z}$ with $\boxdot_i M = M = \diamonddot_p M$. Since $j \le p$, we also have $\diamonddot_j M = M$, and thus by $\sigma \subseteq_{i,j} \tau$ we know that $\sigma M \subseteq \tau M = \tau' \diamonddot_p \boxdot_\ell M \subseteq \tau' \diamonddot_p M = \tau' M$ by Lemma 1(e) and Corollary 1(d) and (e). By the induction hypothesis, we obtain that $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau', q_{i,p,\boxdot})$, and the claim follows by the transition $(q_{i,p,\boxdot}, \epsilon, \diamonddot_p \boxdot_\ell, q_{i,j,\boxdot})$ with $\ell \le i$ (T7). $\square$

With this auxiliary result, we can now prove Lemma 4.

Proof of Lemma 4. "$\Rightarrow$": Let $(\sigma, \tau) \in L(T_{\mathsf{fin}})$, i.e., there is a run of T_{fin} on σ and τ starting in q_0 and ending in q_f. We consider the form of the first transition in this run:

- $(q_{0,\ell,\boxdot}, \epsilon, \diamonddot_\ell, q_0)$ (T1) and $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau', q_{0,\ell,\boxdot})$ with $\tau = \tau' \diamonddot_\ell$: By inspecting all transitions starting from $q_{0,\ell,\boxdot}$, we observe that σ must be empty or start with a $\boxdot$ operator, and thus σ and τ' are $\boxdot$-compatible. Moreover, τ' contains no $\diamonddot_{\ell'}$ with $\ell' \le \ell$. By Lemma 5, we obtain $\sigma \subseteq_{0,\ell} \tau'$. To show $\sigma \subseteq \tau$, consider an arbitrary set $M \subseteq \mathbb{Z}$. For $M' := \diamonddot_\ell M$, we have $\boxdot_0 M' = M' = \diamonddot_\ell M'$ by Lemma 1(h) and Corollary 1(a), and thus $\sigma M \subseteq \sigma \diamonddot_\ell M = \sigma M' \subseteq \tau' M' = \tau' \diamonddot_\ell M = \tau M$ by Lemma 1(d) and (e) and Corollary 1(e).

- $(q_{k,0,\Diamond}, \boxdot_k, \epsilon, q_0)$ (T2) and $q_f \in \Delta_{\mathsf{fin}}(\sigma', \tau, q_{k,0,\Diamond})$ with $\sigma = \sigma' \boxdot_k$: Due to the transitions of $q_{k,0,\Diamond}$, we know that τ must be empty or start with a $\Diamond$ operator. Since σ' contains no $\boxdot_{k'}$ with $k' \leq k$, by Lemma 5, we have $\sigma' \subseteq_{k,0} \tau$. For any $M \subseteq \mathbb{Z}$, we thus have $\sigma M = \sigma' \boxdot_k M \subseteq \tau \boxdot_k M \subseteq \tau M$ by Lemma 1(e) and Corollary 1(d) and (e).

- $(q_{0,0,\triangle}, \epsilon, \epsilon, q_0)$ (T3)–(T4) and $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau, q_{0,0,\triangle})$: Due to the transitions in Δ_{fin}, σ and τ must be $\triangle$-compatible. Thus, by Lemma 5 we get $\sigma \subseteq \tau$.

"$\Leftarrow$": Assume now that $\sigma \subseteq \tau$. If the two sequences are $\triangle$-compatible, then we can apply Lemma 5 to $\sigma \subseteq_{0,0} \tau$ to infer $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau, q_{0,0,\triangle})$, and thus the transition $(q_{0,0,\triangle}, \epsilon, \epsilon, q_0)$ (T3)–(T4) yields $(\sigma, \tau) \in L(T_{\mathsf{fin}})$. If σ and τ are not $\triangle$-compatible, then we consider two cases:

- If $\sigma = \sigma' \Diamond_m$, $\tau = \tau' \boxdot_\ell$, then we have $m > 0$ and $\ell > 0$ since the sequences are in normal form (see Lemma 3). Consider the set $M = \{0\} \cup [2, \infty)$ that consists of a peak of length 1 followed by a gap of length 1. Then, $\sigma M = \sigma' \Diamond_m M = \sigma'[0, \infty) \nsubseteq [2, \infty) = \tau'[2, \infty) = \tau' \boxdot_\ell M = \tau M$, which contradicts the assumption that $\sigma \subseteq \tau$.

- If $\sigma = \sigma' \boxdot_k$ and $\tau = \tau' \Diamond_\ell$, we show that the pair σ, τ' satisfies the preconditions of Lemma 5 w.r.t. $\subseteq_{0,\ell}$. By Lemma 3, they are $\boxdot$-compatible, σ contains no $\boxdot_{k'}$ with $k' \leq 0$, and τ' contains no $\Diamond_{\ell'}$ with $\ell' \leq \ell$. To show $\sigma \subseteq_{0,\ell} \tau'$, consider any $M \subseteq \mathbb{Z}$ with $\Diamond_\ell M = M$. Then, $\sigma M \subseteq \tau M = \tau' \Diamond_\ell M = \tau' M$. By Lemma 5, we obtain $q_f \in \Delta_{\mathsf{fin}}(\sigma, \tau', q_{0,\ell,\boxdot})$, and thus the transition $(q_{0,\ell,\boxdot}, \epsilon, \Diamond_\ell, q_0)$ (T1) shows that $(\sigma, \tau) \in L(T_{\mathsf{fin}})$. ☐

This yields the following complexity for checking subsumption between two sequences of convex temporal operators.

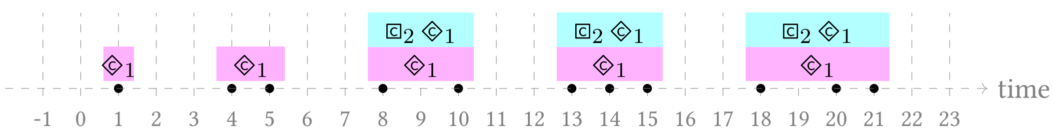
Theorem 1. *It can be decided in polynomial time whether $\sigma \subseteq \tau$ holds for $\sigma, \tau \in \mathfrak{T}^*_{\mathsf{fin}}$.*

Proof. Using Lemma 3, we can transform σ, τ into normal forms σ', τ' in polynomial time. Then, we can construct the transducer T_{fin} over the set $\mathfrak{S}_{\mathsf{fin}}$ of all operators occurring in σ', τ' and check whether $(\sigma', \tau') \in L(T_{\mathsf{fin}})$. The latter check can be done in polynomial time via dynamic programming, by iteratively constructing the set of states reachable from q_0 after reading any pair of suffixes σ'', τ'' of σ', τ'. Since there are only quadratically many pairs of suffixes, and Q_{fin} is of quadratic size, we can determine after polynomial time whether q_f is reachable from q_0 via the full sequences σ', τ', which is equivalent to checking $\sigma \equiv \sigma' \subseteq \tau' \equiv \tau$ by Lemma 4. ☐

4. Subsumption between Formulas over Bounded Operators

We now consider subsumption between complex formulas in $(\mathfrak{T}_{\mathsf{fin}} \setminus \{\Diamond_\infty, \boxdot_\infty\})^\times$, which is defined as for $\mathfrak{T}^\times$ above.

Example 2. Consider the effect of the operator $\boxdot_2 \Diamond_1$ that closes gaps of length 1 and subsequently removes intervals of length at most 2 on $M = \{2, 5, 6, 9, 11, 14, 15, 16, 19, 21, 22\}$:

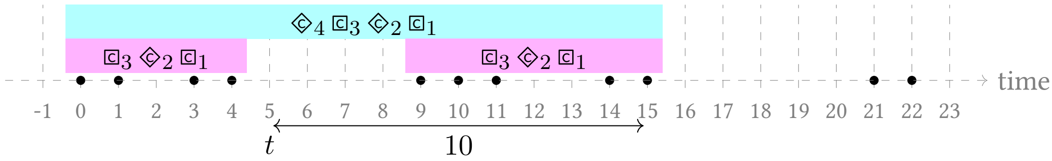


Visually, it is easy to notice that the difference between $\boxdot_2 \Diamond_1$ and $\Diamond_1$ lies in the identity operator, i.e., $\epsilon \cup \boxdot_2 \Diamond_1$ is equivalent to $\Diamond_1$.

In general, to check whether $\phi \subseteq \psi$ holds for $\phi, \psi \in \mathfrak{T}^\times_{\mathsf{fin}}$ that do not use $\Diamond_\infty$ or $\boxdot_\infty$, we can show that the behavior of these complex formulas is *local*, in the sense that, to check whether t is included in ϕM, we only need to consider a local neighborhood of time points $M \cap [t-n, t+n]$ near t. For a formula $\phi \in (\mathfrak{T}_{\mathsf{fin}} \setminus \{\Diamond_\infty, \boxdot_\infty\})^\times$, we recursively define its *reach* as:

$$\begin{aligned}
\mathsf{reach}(\triangle_n) &:= n \\
\mathsf{reach}(\phi \circ \psi) &:= \mathsf{reach}(\phi) + \mathsf{reach}(\psi) \\
\mathsf{reach}(\phi \cup \psi) &= \max\{\mathsf{reach}(\phi), \mathsf{reach}(\psi)\} \\
\mathsf{reach}(\phi \cap \psi) &= \max\{\mathsf{reach}(\phi), \mathsf{reach}(\psi)\}
\end{aligned}$$

Example 3. For $\Diamond_4 \boxdot_3 \Diamond_2 \boxdot_1$, we have $\mathsf{reach}(\Diamond_4 \boxdot_3 \Diamond_2 \boxdot_1) = 4+3+2+1 = 10$, and thus to determine whether $t \in \Diamond_4 \boxdot_3 \Diamond_2 \boxdot_1 M$ holds, we only need to look at time points of distance at most 10 from t:



Lemma 6. *For any $\phi \in (\mathfrak{T}_{\mathsf{fin}} \setminus \{\Diamond_\infty, \boxdot_\infty\})^\times$ with $\mathsf{reach}(\phi) = n$, any $M \subseteq \mathbb{Z}$, and any time point $i \in \mathbb{Z}$, we have $i \in \phi M$ iff $i \in \phi(M \cap [i-n, i+n])$.*

Proof. We show this by induction on the structure of ϕ. For $\phi = \Diamond_n$, we have $i \in \phi M$ iff $i \in M$ or there are $j, k \in M$ with $j < i < k$ and $k - j \leq n+1$, and thus one can equivalently check the existence of $j, k \in M \cap [i-n, i+n]$ with $j \leq i \leq k$ and $k - j \leq n+1$, since the distances of j and k from i are at least 1 and at most n. For $\phi = \boxdot_n$, we have $i \in \phi M$ iff there is an interval $[j,k] \subseteq M$ with $j \leq i \leq k$ and $k - j \geq n$ (i.e., the interval contains at least $n+1$ time points) iff there is an interval $I \subseteq M \cap [i-n, i+n]$ of at least $n+1$ time points that contains i.

For $\phi = \phi_1 \circ \phi_2$, let $n_1 = \mathsf{reach}(\phi_1)$ and $n_2 = \mathsf{reach}(\phi_2)$ and $M' := \phi_2 M$. By induction, we have $i \in \phi_1 M'$ iff $i \in \phi_1(M' \cap [i-n_1, i+n_1])$ and, for any $t \in [i-n_1, i+n_1]$, it holds that $t \in \phi_2 M$ iff $t \in \phi_2(M \cap [t-n_2, t+n_2])$ iff $t \in \phi_2(M \cap [i-n_2-n_1, i+n_2+n_1])$. We obtain that $i \in \phi_1\phi_2 M$ iff $i \in \phi_1(\phi_2 M \cap [i-n_1, i+n_1])$ iff $i \in \phi_1((\phi_2(M \cap [i-n_2-n_1, i+n_2+n_1])) \cap [i-n_1, i+n_1])$ iff $i \in \phi_1\phi_2(M \cap [i-n_2-n_1, i+n_2+n_1])$.

For $\phi = \phi_1 \cup \phi_2$, we similarly have $i \in \phi_j M$ iff $i \in \phi_j(M \cap [i-n_j, i+n_j])$, which is equivalent to $i \in \phi_j(M \cap [i - \max\{n_1, n_2\}, i + \max\{n_1, n_2\}])$, for $j \in \{1, 2\}$. Thus, $i \in \phi M$ iff $i \in \phi_1 M$ or $i \in \phi_2 M$ iff $i \in \phi(M \cap [i - \max\{n_1, n_2\}, i + \max\{n_1, n_2\}])$. The argument for $\phi = \phi_1 \cap \phi_2$ is similar. $\square$

This implies that, to solve the question $\phi \subseteq \psi$, it suffices to consider all $M \subseteq [-n, n]$, where n is the maximal reach of ϕ and ψ.

Theorem 2. *For any $\phi, \psi \in (\mathfrak{T}_{\mathsf{fin}} \setminus \{\Diamond_\infty, \boxdot_\infty\})^\times$, we can check whether $\phi \subseteq \psi$ holds in double exponential time if operator subscripts are encoded in binary, and in exponential time for unary encoding.*

Proof. If $\phi M \nsubseteq \psi M$, then there is $i \in \phi M$ with $i \notin \psi M$. By considering the set $M_{-j} := \{i - j \mid i \in M\}$, we obtain that $0 \in \phi M_{-j}$ and $0 \notin M_{-j}$, i.e., it suffices to check whether 0 is a counterexample for $\phi \subseteq \psi$ under any $M \subseteq \mathbb{Z}$. By Lemma 6, 0 is a counterexample in M iff it is a counterexample in $M \cap [-n, n]$, where $n := \max\{\mathsf{reach}(\phi), \mathsf{reach}(\psi)\}$. Thus, to check $\phi \subseteq \psi$, we can enumerate all (double) exponentially many subsets $M \subseteq [-n, n]$, compute ϕM and ψM in (double-)exponential time, and check whether $0 \in \phi M$ and $0 \notin \psi M$. $\square$

5. Conclusions

In this paper, we have presented first results on characterizing combinations of $\boxdot_n$ and $\Diamond_n$ operators, including their composition, point-wise intersection, and union. We have established that subsumption between such complex operators can be decided in double exponential time in the general case. For sequences of compositions, we have proposed a transducer-based procedure that solves the subsumption problem in polynomial time.

In future work, we will extend the developed characterizations of subsumption to the classical infinite operators $\boxplus$, $\diamond\!\!\!\!+$, etc., for which there are only finitely many cases to consider (see Lemma 3). More importantly, we want to determine tight complexity bounds for subsumption between arbitrary temporal formulas from $\mathfrak{T}^{\times}$. We conjecture that this problem can be decided in polynomial time, based on observations indicating that non-subsumption is always witnessed by a simple kind of counterexamples, of which there are only polynomially many. Based on this, our aim is to characterize which axioms of the form $\phi A \sqsubseteq B$ can be entailed by a temporal $\mathcal{TELH}_{\bot}^{\diamond\!\!\!\!\text{c},\mathsf{lhs}}$ ontology extended with $\boxed{\text{c}}$ operators, as a step towards developing a classification algorithm for this logic.

Acknowledgments

This work was supported by Deutsche Forschungsgemeinschaft (DFG) grant 389792660 as part of TRR 248 – CPEC, see https://perspicuous-computing.science.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

References

[1] A. Artale, E. Franconi, A survey of temporal extensions of description logics, Ann. Math. Artif. Intell. 30 (2000) 171–210. doi:10.1023/A:1016636131405.

[2] C. Lutz, F. Wolter, M. Zakharyaschev, Temporal description logics: A survey, in: S. Demri, C. S. Jensen (Eds.), 15th International Symposium on Temporal Representation and Reasoning (TIME), IEEE Computer Society, 2008, pp. 3–14. doi:10.1109/TIME.2008.14.

[3] A. Artale, R. Kontchakov, V. Ryzhikov, M. Zakharyaschev, A cookbook for temporal conceptual data modelling with description logics, ACM Trans. Comput. Log. 15 (2014) 25:1–25:50. doi:10.1145/2629565.

[4] A. Artale, R. Kontchakov, A. Kovtunova, V. Ryzhikov, F. Wolter, M. Zakharyaschev, Ontology-mediated query answering over temporal data: A survey (invited talk), in: S. Schewe, T. Schneider, J. Wijsen (Eds.), 24th International Symposium on Temporal Representation and Reasoning (TIME), LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017, pp. 1:1–1:37. doi:10.4230/LIPICS.TIME.2017.1.

[5] V. Gutiérrez-Basulto, J. C. Jung, T. Schneider, Lightweight temporal description logics with rigid roles and restricted TBoxes, in: Q. Yang, M. J. Wooldridge (Eds.), Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence (IJCAI), AAAI Press, 2015, pp. 3015–3021. URL: http://ijcai.org/Abstract/15/426.

[6] V. Gutiérrez-Basulto, J. C. Jung, R. Kontchakov, Temporalized EL ontologies for accessing temporal data: Complexity of atomic queries, in: S. Kambhampati (Ed.), Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence (IJCAI), IJCAI/AAAI Press, 2016, pp. 1102–1108. URL: http://www.ijcai.org/Abstract/16/160.

[7] S. Borgwardt, W. Forkel, A. Kovtunova, Finding new diamonds: Temporal minimal-world query answering over sparse ABoxes, in: P. Fodor, M. Montali, D. Calvanese, D. Roman (Eds.), Rules and Reasoning - Third International Joint Conference, RuleML+RR 2019, Proceedings, volume 11784 of *Lecture Notes in Computer Science*, Springer, 2019, pp. 3–18. doi:10.1007/978-3-030-31095-0_1.

[8] S. Borgwardt, W. Forkel, A. Kovtunova, Temporal minimal-world query answering over sparse aboxes, Theory Pract. Log. Program. 22 (2022) 193–228. doi:10.1017/S1471068421000119.

[9] A. Artale, R. Kontchakov, C. Lutz, F. Wolter, M. Zakharyaschev, Temporalising tractable description logics, in: 14th International Symposium on Temporal Representation and Reasoning (TIME), IEEE Computer Society, 2007, pp. 11–22. doi:10.1109/TIME.2007.62.

[10] F. Baader, S. Brandt, C. Lutz, Pushing the EL envelope, in: L. P. Kaelbling, A. Saffiotti (Eds.), IJCAI-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, Professional Book Center, 2005, pp. 364–369. URL: http://ijcai.org/Proceedings/05/Papers/0372.pdf.

[11] C. Lutz, D. Toman, F. Wolter, Conjunctive query answering in the description logic EL using a relational database system, in: C. Boutilier (Ed.), IJCAI 2009, Proceedings of the 21st International Joint Conference on Artificial Intelligence, 2009, pp. 2070–2075. URL: http://ijcai.org/Proceedings/09/Papers/341.pdf.

[12] V. Gutiérrez-Basulto, J. C. Jung, A. Ozaki, On metric temporal description logics, in: G. A. Kaminka, M. Fox, P. Bouquet, E. Hüllermeier, V. Dignum, F. Dignum, F. van Harmelen (Eds.), ECAI 2016 - 22nd European Conference on Artificial Intelligence, Frontiers in Artificial Intelligence and Applications, IOS Press, 2016, pp. 837–845. doi:10.3233/978-1-61499-672-9-837.

[13] F. Baader, S. Borgwardt, P. Koopmann, A. Ozaki, V. Thost, Metric temporal description logics with interval-rigid names, ACM Trans. Comput. Log. 21 (2020) 30:1–30:46. doi:10.1145/3399443.