Towards a Characterization of Stable Labelling Realizability – The Problem of Inherent Non-Determinism

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Abstract

We study realizability in terms of stable labellings for classical Dung frameworks. Although there is a close relationship between extension-based and labelling-based semantics, realizability with respect to labellings is significantly more challenging. This is due to the explicit treatment of non-accepted arguments (i.e., undecided and rejected ones), which fixes the argument set of any witnessing framework and rules out auxiliary arguments commonly used in constructions for extension-based semantics. Consequently, existing characterizations for extension-based semantics can only be reused to a limited extent. Towards a characterization we introduce so-called candidate sets, capturing sets of arguments that would become stable but should not. These sets constitute a central obstacle in our approach. In the absence of such sets, we provide a characterization theorem. For the general case, we identify a canonical over-approximating framework such that every stably realizable labelling set admits a witnessing argumentation framework obtainable as a subgraph of it. However, we observed an inherent non-determinism for our proposed method. Finally, we identified a connection between labelling-based realizability and the long-standing open problem of compact stable extension realizability.

Keywords

Formal Argumentation, Labelling Semantics, Realizability

1. Introduction

Realizability is a central question for any knowledge representation formalism [1, 2, 3]. In Dung's abstract argumentation theory [4, 5, 6], the question is whether, given a set $\mathbb{L}$ of desired extensions or labellings, there exists an argumentation framework F whose semantics exactly matches $\mathbb{L}$. A first systematic treatment of this question for extension-based semantics was given by Dunne et al. [7, 3], who provided a series of non-trivial characterization results. For labelling-based semantics, only conflict-free and naive labellings have been characterized so far [8, 9]. In this paper, we initiate the study of stable labelling realizability.

At first glance, one might expect realizability results for the extension-based setting to carry over to labelling-based semantics, given their close relationship [10, 11]. However, labellings encode strictly more information than their induced sets of extensions: they specify not only which arguments are accepted, but also which are not accepted, namely rejected or undecided [12]. As a consequence, the argument set of any witnessing framework is fixed a priori, leaving no freedom to introduce auxiliary self-looping dummy arguments to block unwanted extensions - a common technique in the extension-based setting [13, 7]. Therefore, existing characterizations for extension-based semantics can only be transferred in a very limited way. The following running example illustrates this difference in a concise manner.

Example 1. *Consider the following AF F and its associated set of stable labellings $\mathbb{L}$. (Intuitively, a stable labelling assigns each argument either* in *or* out*, with no argument left undecided, such that all rejected arguments are attacked by accepted ones)*

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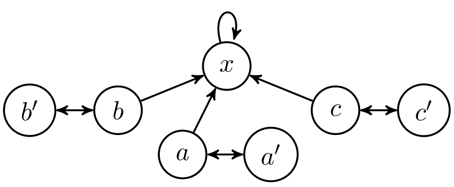
$F\colon$



$\mathbb{L}$	I	O	U
L_1	$\{a, b, c\}$	$\{a', b', c', x\}$	$\emptyset$
L_2	$\{a, b, c'\}$	$\{a', b', c, x\}$	$\emptyset$
L_3	$\{a, b', c\}$	$\{a', b, c', x\}$	$\emptyset$
L_4	$\{a', b, c\}$	$\{a, b', c', x\}$	$\emptyset$
L_5	$\{a, b', c'\}$	$\{a', b, c, x\}$	$\emptyset$
L_6	$\{a', b, c'\}$	$\{a, b', c, x\}$	$\emptyset$
L_7	$\{a', b', c\}$	$\{a, b, c', x\}$	$\emptyset$

Table 1: $\mathbb{L} = \{L_1, \ldots, L_7\}$

We observe that the argument x is rejected in any single labelling. Consider now the labelling-set $\mathbb{M}$ which results from $\mathbb{L}$ by removing x from any out-set. Obviously, the in-sets remain unchanged and thus are still extension-based realizable. In contrast, $\mathbb{M}$ is no longer labelling-based realizable. The reason is that the set $\{a', b', c'\}$ would be maximal conflict-free in any witnessing AF. In the original framework F, the argument x prevents $\{a', b', c'\}$ from being stable as it is not attacked by any of the arguments. Now, since x disappears, this structural witness is no longer possible and thus, no AF over the remaining arguments can block $\{a', b', c'\}$ from becoming stable. For a precise proof we refer the reader to [13, 7].

Sets like $\{a', b', c'\}$ are what we call *candidate sets*. They constitute a central object of study in this paper. We divide the analysis of stable realizability into two parts, namely the presence and the absence of such candidate sets.

In the absence of candidate sets, extension-based conditions (i.e., incomparability and tightness) are the essential ingredients for stable labelling realizability. We provide a general construction method for a witnessing framework such that stable and naive labellings coincide.

In the presence of candidate sets, however, a more detailed analysis is required. Each candidate set must be rendered non-stable by ensuring that it fails to attack at least one argument. We establish a construction of an over-approximating AF. A witnessing framework can then be obtained by removing a suitable set of attacks. Moreover, any witnessing AF can be transformed into a subgraph of this framework by performing semantically neutral modifications [14]. In other words, the generic framework captures the space of all witnessing AFs up to semantic redundancies. The question of which attacks need to be removed is non-trivial and, moreover, not uniquely determined. We refer to this as the *inherent non-determinism* of stable labelling realizability.

As a further contribution, we show that stable labelling realizability is directly connected to the open problem of compact stable extension realizability [13, 7]. Hence, any progress in its characterization yields corresponding results for this long-standing open problem.

2. Formal Preliminaries

2.1. Argumentation Frameworks and Semantics

We start with the necessary background on abstract argumentation. An *argumentation framework* (AF) is a pair $F = (A, R)$ where A, the set of arguments, is a finite subset of a fixed infinite background set $\mathcal{U}$, and $R \subseteq A \times A$. The set of all finite AFs is denoted by $\mathcal{F}$. We say a *attacks* b, or b is *defeated* by a in F whenever $(a, b) \in R$. If $F = (A, R)$, we use $A(F)$ as well as $R(F)$ to refer to the first or second component of F, i.e. A or R, respectively.

An *extension-based semantics* $\mathcal{E}_\sigma : \mathcal{F} \to 2^{2^{\mathcal{U}}}$ is a function which assigns to any AF $F = (A, R)$ a set of sets of arguments denoted by $\mathcal{E}_\sigma(F) \subseteq 2^A$. Each one of them, a so-called *σ-extension*, is considered to be acceptable with respect to F. The most basic criterion is *conflict-freeness* (cf), which guarantees no internal conflicts. Moreover, we consider further important refinements, namely *naive* and *stable* (abbr. as na, stb).

Definition 1. *Let $F = (A, R)$ be an AF and $E \subseteq A$. 1. $E \in \mathcal{E}_{cf}(F)$ iff there are no $a, b \in E$, s.t. $(a, b) \in R$, 2. $E \in \mathcal{E}_{na}(F)$ iff E is $\subseteq$-maximal in $\mathcal{E}_{cf}(F)$, 3. $E \in \mathcal{E}_{stb}(F)$ iff $E \in \mathcal{E}_{cf}(F)$ and $E^+ := \{b \mid (a, b) \in R, a \in E\} = A \setminus E$, i.e. E attacks all arguments not contained in E itself.*

A *labelling-based semantics* $\mathcal{L}_\sigma : \mathcal{F} \to 2^{\left(2^{\mathcal{U}}\right)^3}$ is a function which assigns to any AF $F = (A, R)$ a set of triples of sets of arguments denoted by $\mathcal{L}_\sigma(F) \subseteq \left(2^A\right)^3$. Each one of them, a so-called *σ-labelling* of F, is a triple $L = (I, O, U)$ indicating that arguments in I, O or U are considered to be *accepted (in)*, *rejected (out)* or *undecided* with respect to F. We further assume pairwise disjointness and covering, i.e. $I \cap O = I \cap U = O \cap U = \emptyset$ and $I \cup O \cup U = A$. We use L^{I} for the first component of the labelling L. Analogously for L^{O} and L^{U}. We proceed with the central notions of *conflict-free labellings* [15].

Definition 2. *A labelling L of $F = (A, R)$ is called conflict-free: 1. If $a, b \in L^{\mathrm{I}}$, then $(a, b) \notin R$ (no internal conflicts) and 2. if $a \in L^{\mathrm{O}}$, then there is a $b \in L^{\mathrm{I}}$ with $(b, a) \in R$ (reason for rejecting).*

We now introduce the labelling-based counterparts to the extension-based semantics.

Definition 3. *Let $F = (A, R)$ be an AF and $L \in \left(2^A\right)^3$ a labelling of F. 1. $L \in \mathcal{L}_{cf}(F)$ iff L is a conflict-free labelling of F, 2. $L \in \mathcal{L}_{na}(F)$ iff $L \in \mathcal{L}_{cf}(F)$ and L^{I} is $\subseteq$-maximal in $\{M^{\mathrm{I}} \mid M \in \mathcal{L}_{cf}(F)\}$, 3. $L \in \mathcal{L}_{stb}(F)$ iff $L \in \mathcal{L}_{cf}(F)$ and $L^{\mathrm{U}} = \emptyset$.*

In the following we list some well-known properties and relations between semantics which will be frequently used throughout the whole paper (confer [11]).

Proposition 1. *Given an AF $F = (A, R)$, a set $E \subseteq A$ and a semantics $\sigma \in \{cf, na, stb\}$. In the following we use $E^{\mathcal{L}}$ as shorthand for $(E, E^+, A \setminus (E \cup E^+))$. 1. $\mathcal{E}_{stb}(F) \subseteq \mathcal{E}_{na}(F) \subseteq \mathcal{E}_{cf}(F)$, 2. $\mathcal{L}_{stb}(F) \subseteq \mathcal{L}_{na}(F) \subseteq \mathcal{L}_{cf}(F)$, 3. If $E \in \mathcal{E}_\sigma(F)$, then $E^{\mathcal{L}} \in \mathcal{L}_\sigma(F)$, 4. If $L \in \mathcal{L}_\sigma(F)$, then $L^{\mathrm{I}} \in \mathcal{E}_\sigma(F)$, and 5. $(E^{\mathcal{L}})^I = E$. In case of stable semantics we even have the following additional relations: 6. For any $L, M \in \mathcal{L}_{stb}(F)$, $L^{\mathrm{I}} = M^{\mathrm{I}}$ iff $L = M$, 7. Given $L \in \mathcal{L}_{stb}(F)$, then $(L^{\mathrm{I}})^{\mathcal{L}} = L$, and 8. $|\mathcal{L}_{stb}(F)| = |\mathcal{E}_{stb}(F)|$.*

In the course of the paper we will make use of some known results about equivalence. We will give the definitions of the two main notions of equivalence: the *ordinary equivalence* and *strong equivalence* [16, 10].

Definition 4. *Given a semantics $X \in \{\mathcal{E}_\sigma, \mathcal{L}_\sigma\}$ and two AFs F and G. F and G are called ordinarily equivalent if $X(F) = X(G)$. We use $F \equiv^X G$ to indicate this relation.*

We mention one decisive difference between the two versions of semantics. In contrast to extension-based semantics, being ordinarily equivalent in the realm of labellings requires them to share the same arguments. This is due to the fact that each labelling assigns a label to any argument of the considered AF implying that labellings are necessarily different if the frameworks in question possess different arguments.

Definition 5. *Given a semantics $X \in \{\mathcal{E}_\sigma, \mathcal{L}_\sigma\}$ and two AFs F and G. F and G are called strongly equivalent if for each AF H, $X(F \sqcup H) = X(G \sqcup H)$. We use $F \equiv_s^X G$ to indicate this relation.*

It was shown that in order to decide strong equivalence for extension-based semantics as well as labelling semantics one may use a syntactical concept called *kernels* [16]. A kernel is again a directed graph obtained from the initial AF via adding or deleting attacks. In the following we introduce the *stable kernel*.

Definition 6. *Let $F = (A, R)$ be an AF. The stable kernel of F is defined as: $F^{sk} = (A^{sk}, R^{sk}) = (A, R \setminus \{(a, b) \in R \mid a \neq b, (a, a) \in R\})$.*

Proposition 2. *Given two AFs F and G and a semantics $X \in \{\mathcal{E}_{stb}, \mathcal{L}_{stb}\}$ we have: $F \equiv_s^X G \Leftrightarrow F^{sk} = G^{sk}$.*

2.2. Realizability in Abstract Argumentation

Let us start with the two central concepts, namely *realizability* as well as *signature*. In a nutshell, we say that a certain set $\mathbb{S}$ of sets, or labellings, respectively, is realizable under the semantics σ, if there is an AF F such that its set of σ-extensions/σ-labellings coincides with $\mathbb{S}$. Collecting all realizable sets defines the concept of a signature. Note that only $n = 1$ (extension-case) and $n = 3$ (labelling-case) will be relevant for this paper.

Definition 7. *Given a semantics $\sigma : \mathcal{F} \to 2^{\left(2^{\mathcal{U}}\right)^n}$. A set $\mathbb{S} \subseteq \left(2^{\mathcal{U}}\right)^n$ is σ-realizable if there is an AF $F \in \mathcal{F}$, s.t. $\sigma(F) = \mathbb{S}$. Moreover, the σ-signature is defined as $\Sigma_\sigma = \{\sigma(F) \mid F \in \mathcal{F}\}$.*

We proceed with further notation as well as the central notions of *downward-closedness* and *tightness* [7] used to formulate the characterization of realizability of extension-based semantics.

Definition 8. *A finite $\mathbb{S} \subseteq 2^{\mathcal{U}}$ is called an extension-set. We use $Args_{\mathbb{S}}$ to denote $\bigcup_{S \in \mathbb{S}} S$ and $|\mathbb{S}|$ for $|Args_{\mathbb{S}}|$. Furthermore, Pairs$_{\mathbb{S}}$ denotes $\{(a, b) \mid \exists S \in \mathbb{S} : \{a, b\} \subseteq S\}$ and*
dcl$(\mathbb{S})$ denotes the so-called downward-closure $\{S' \subseteq S \mid S \in \mathbb{S}\}$.

Definition 9. *Given an extension-set $\mathbb{S} \subseteq 2^{\mathcal{U}}$. We call $\mathbb{S}$ 1. downward-closed if $\mathbb{S} =$ dcl$(\mathbb{S})$, 2. incomparable if $\mathbb{S}$ is a $\subseteq$-antichain, and 3. tight if for all $S \in \mathbb{S}$ and $a \in Args_{\mathbb{S}}$ it holds that if $S \cup \{a\} \notin \mathbb{S}$ then there exists an $s \in S$ such that $(a, s) \notin$ Pairs$_{\mathbb{S}}$.*

Now, we present the central characterization theorems in case of extension-based semantics. Confer [7, 3].

Theorem 3. *Given a finite extension-set $\mathbb{S} \subseteq 2^{\mathcal{U}}$, then 1. $\mathbb{S} \in \Sigma_{\mathcal{E}_{cf}} \Leftrightarrow \mathbb{S}$ is a non-empty, downward-closed, and tight extension-set, 2. $\mathbb{S} \in \Sigma_{\mathcal{E}_{na}} \Leftrightarrow \mathbb{S}$ is a non-empty, incomparable extension-set and dcl$(\mathbb{S})$ is tight, 3. $\mathbb{S} \in \Sigma_{\mathcal{E}_{stb}} \Leftrightarrow \mathbb{S}$ is an incomparable and tight extension-set.*

Now let us turn to labelling-based semantics. Again, we start with some relevant notations and shorthands. For a detailed introduction see [8].

Definition 10. *Given a finite set $\mathbb{L} \subseteq \left(2^{\mathcal{U}}\right)^3$. We use $\mathbb{L}^I$, $\mathbb{L}^O$ and $\mathbb{L}^U$ to denote $\{L^I \mid L \in \mathbb{L}\}$, $\{L^O \mid L \in \mathbb{L}\}$ or $\{L^U \mid L \in \mathbb{L}\}$, respectively. Moreover, we say that $\mathbb{L}$ is a labelling-set if*

1. *$L_1^I \cup L_1^O \cup L_1^U = L_2^I \cup L_2^O \cup L_2^U$ for any $L_1, L_2 \in \mathbb{L}$ (same arguments) and*
2. *$L_1^I \cap L_1^O = L_1^I \cap L_1^U = L_1^O \cap L_1^U = \emptyset$ for each $L_1 \in \mathbb{L}$ (disjointness).*

Finally, for a fixed set of arguments $E \subseteq \mathcal{U}$ we use $\mathbb{L}_{I=E} = \{L \mid L \in \mathbb{L}, L^I = E\}$ (corresponding labellings) and $\mathbb{L}^O_{I=E} = \{L^O \mid L \in \mathbb{L}, L^I = E\}$ (corresponding out-labels). If $|\mathbb{L}^O_{I=E}| = 1$ for every $E \in \mathbb{L}^I$ then $L^O_{I=E}$ denotes the sole out-set belonging to the in-set E.

Let us illustrate the introduced concepts with the following example.

Example 2. *Consider $\mathbb{L} = \{(\{a, d\}, \{b\}, \{c\}), (\{b, d\}, \{a, c\}, \emptyset), (\{b, d\}, \{a\}, \{c\})\}$. First of all, we observe that $\mathbb{L}$ is indeed a labelling-set as all triples refer to the same arguments, namely a, b, c, d and moreover, for any triple we have that each argument occurs in one of the three sets only. We obtain the following sets: $\mathbb{L}^I = \{\{a, d\}, \{b, d\}\}$, $\mathbb{L}^O = \{\{b\}, \{a, c\}, \{a\}\}$ and $\mathbb{L}^U = \{\{c\}, \emptyset\}$ as well as $\mathbb{L}^O_{I=\{a,d\}} = \{\{b\}\}$ and $\mathbb{L}^O_{I=\{b,d\}} = \{\{a, c\}, \{a\}\}$. Furthermore, $\mathbb{L}_{I=\{a,d\}} = \{(\{a, d\}, \{b\}, \{c\})\}$ and $\mathbb{L}_{I=\{b,d\}} = \{(\{b, d\}, \{a, c\}, \emptyset), (\{b, d\}, \{a\}, \{c\})\}$.*

We now recall the properties relevant for characterizing conflict-free labellings and naive labellings. We will restrict ourselves to the concepts used in the course of this paper. For a detailed introduction to naive and conflict-free labelling-realizability see [8]. First we will recount two central definition extending the concept of tightness and downward-closure to the realm of labellings. While we will not repeat the characterization theorem of naive and conflict-free labellings here, we nevertheless want to mention that L-tightness is a key property required for both semantics.

Definition 11. *A labelling-set $\mathbb{L}$ is called L-tight, if for each $E \in \mathbb{L}^I$ we have: $\mathbb{L}^O_{I=E}$ possesses a $\subseteq$-greatest element. (Notation: We use $\overline{\mathbb{L}}^O_{I=E}$ for the $\subseteq$-greatest element and $\overline{\mathbb{L}}_{I=E}$ for the associated labelling.) Moreover, for all $E_1, E_2 \in \mathbb{L}^I$ it holds that $E_1 \cup E_2 \in \mathbb{L}^I \Leftrightarrow (\overline{\mathbb{L}}^O_{I=E_1} \cap E_2) \cup (\overline{\mathbb{L}}^O_{I=E_2} \cap E_1) = \emptyset$.*

Definition 12. *For a given labelling-set $\mathbb{L}$ we define a new labelling-set ${}^{\downarrow}\mathbb{L}$, the so-called downward-closure of $\mathbb{L}$, as follows: ${}^{\downarrow}\mathbb{L} := \{(E, O, Args_{\mathbb{L}} \setminus (E \cup O)) | E \subseteq L^I, L^I \in \mathbb{L}^I, O \subseteq O_E\}$ where $O_E = \bigcup_{a \in E} O_a$ with $O_a = \bigcap_{E' \in \mathbb{L}^I, \{a\} \subseteq E'} \bigcup \mathbb{L}^O_{I=E'}$. Please note that $\bigcup \mathbb{L}^O_{I=E'}$ can be replaced with $\overline{\mathbb{L}}^O_{I=E'}$ in case of L-tight labelling-sets.*

We will give a short example for the construction of a labelling downward-closure. Please take special note of the sets O_a which will be a recurring tool later on.

Example 3. *Consider $\mathbb{L} := \{(\{c\}, \{a\}, \{b, d\}), (\{c\}, \emptyset, \{a, b, d\}), (\{a, d\}, \{b, c\}, \emptyset),$ $(\{a, d\}, \{c\}, \{b\}), (\{a, d\}, \{b\}, \{c\}), (\{a, d\}, \emptyset, \{b, c\}), (\{b, d\}, \{c\}, \{a\}), (\{b, d\}, \emptyset, \{a, c\})\}$. We determine the associated downward-closure ${}^{\downarrow}\mathbb{L}$. In order to obtain the resulting labellings we first have to compute the set O_a for each single $a \in Args_{\mathbb{L}^I}$. Note that the considered labelling-set $\mathbb{L}$ is L-tight. Consequently, we may compute O_a with the more convenient $\bigcap\limits_{I' \in \mathbb{L}^I, \{a\} \subseteq I'} \overline{\mathbb{L}}^O_{I=I'}$.*
As an example we determine O_d in detail :

$$O_d = \bigcap_{I' \in \mathbb{L}^I, \{d\} \subseteq I'} \overline{\mathbb{L}}^O_{I=I'} = \overline{\mathbb{L}}^O_{I=\{a,d\}} \cap \overline{\mathbb{L}}^O_{I=\{b,d\}} = L^O_3 \cap L^O_7 = \{b, c\} \cap \{c\} = \{c\}.$$

In the same way we obtain: $O_a = \{b, c\}$, $O_b = \{c\}$ and $O_c = \{a\}$. The next step is to build O_E for each set $E \subseteq I \in \mathbb{L}^I$. This set is built via the union of any O_a with $a \in E$. As an example we show this for the in-set $\{a, d\} \in \mathbb{L}^I$. We have to consider the four subsets $E_1 = \{a, d\}$, $E_2 = \{a\}$, $E_3 = \{d\}$ and $E_4 = \emptyset$. We obtain: $O_{E_1} = \bigcup_{x \in E_1} O_x = O_a \cup O_d = \{b, c\} \cup \{c\} = \{b, c\}$, $O_{E_2} = O_a = \{b, c\}$, $O_{E_3} = O_d = \{c\}$ and $O_{E_4} = \emptyset$. In this way we determine the out-set for each subset of an in-set resulting in the new labelling-set ${}^{\downarrow}\mathbb{L}$.

As a last introductory step we will give a short overview on the standard construction for witnessing frameworks used for conflict-free and naive labelling-realizability. These will be central in certain cases of the characterization of stable labellings. The construction for a witnessing AF in case $\mathbb{L}$ is naive realizable is simply applying the conflict-free construction (Definition 13) to the downward-closure of the labelling-set in question.

Definition 13. *Given a labelling-set $\mathbb{L}$. We define $F^{cf}_{\mathbb{L}} = (A_{\mathbb{L}}, R_{\mathbb{L}})$ with $A_{\mathbb{L}} = Args_{\mathbb{L}}$ and 1. $\forall a \in A_{\mathbb{L}}$: $(a, a) \in R_{\mathbb{L}}$ if and only if $\{a\} \notin \mathbb{L}^I$, and 2. $\forall a, b \in A_{\mathbb{L}}$: If $a \neq b$, then $(a, b) \in R_{\mathbb{L}}$ iff $\{a\} \in \mathbb{L}^I$ and $b \in \bigcup \mathbb{L}^O_{I=\{a\}}$.*

Definition 14. *Given an L-tight labelling-set $\mathbb{L}$. We define $F^{cf}_{\downarrow\mathbb{L}} = (A_{\downarrow\mathbb{L}}, R_{\downarrow\mathbb{L}})$ with $A_{\downarrow\mathbb{L}} = Args_{\mathbb{L}}$ and 1. $\forall a \in A_{\downarrow\mathbb{L}}$: $(a, a) \in R_{\downarrow\mathbb{L}}$ if and only if $\{a\} \notin {}^{\downarrow}\mathbb{L}^I$, and 2. $\forall a, b \in A_{\downarrow\mathbb{L}}$: If $a \neq b$, then $(a, b) \in R_{\downarrow\mathbb{L}}$ iff $\{a\} \in {}^{\downarrow}\mathbb{L}^I$ and $b \in \bigcup {}^{\downarrow}\mathbb{L}^O_{I=\{a\}}$.*

Theorem 4. *For any labelling-set $\mathbb{L}$, it holds: $\mathbb{L} \in \Sigma_{\mathcal{L}_{cf}} \Leftrightarrow \mathbb{L} = \mathcal{L}_{cf}\left(F^{cf}_{\mathbb{L}}\right)$ and $\mathbb{L} \in \Sigma_{\mathcal{L}_{cf}} \Leftrightarrow \mathbb{L} = \mathcal{L}_{na}\left(F^{cf}_{\downarrow\mathbb{L}}\right)$.*

3. Realizability of Stable Labellings

3.1. First Reflections on Realizability

Finally, let us consider stable labellings. The following proposition strengthens a former result regarding realizability [3, Theorem 3.70]. If a set of labellings is stable realizable, then the set of in-labels is incomparable and tight. Moreover, no argument is labelled undecided.

Proposition 5. *Let $\mathbb{L} \subseteq \left(2^{\mathcal{U}}\right)^3$ be a labelling-set. If $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$, then $\bigcup \mathbb{L}^U = \emptyset$ and $\mathbb{L}^I$ is incomparable and tight.*

Proof. Let $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$. Hence, there is an AF F with $\mathcal{L}_{stb}(F) = \mathbb{L}$. We further deduce $\mathbb{L}^I = \mathcal{E}_{stb}(F)$ (Items 3,4 of Proposition 1). According to Theorem 3 we obtain $\mathbb{L}^I$ is incomparable and tight. Moreover, $\bigcup \mathbb{L}^U = \emptyset$ is fulfilled since Definition 3 requires $L^U = \emptyset$ for each $L \in \mathcal{L}_{stb}(F)$. □

We would like to mention that we initially thought that the given necessary condition is even sufficient for realizing stable labellings. However as the introductory Example 1 has shown, this was a misconception. A labelling-set may induce sets that must be actively prevented from becoming stable in any witnessing AF. We call these *candidate sets*. These sets are maximal conflict-free within the confines of the labelling set but are not supposed to be stable themselves. Therefore, in a witnessing AF for $\mathbb{L}$ for each of the candidate sets there has to be some argument that is not attacked by this set thus preventing stability. We will build our characterization of stable labelling realizability on this idea. The first step is to find and formalize these possible problematic argument-sets from a given labelling-set.

Definition 15. *Let $\mathbb{L}$ be a set of labellings. We define the following sets: $\mathbb{C}^+_{\mathbb{L}} := \max_{\subseteq}(\{E \mid \forall a, b \in E : (a, b) \in Pairs_{\mathbb{L}^I}\})$ and $\mathbb{C}_{\mathbb{L}} := (\max_{\subseteq}(\{E \mid \forall a, b \in E : (a, b) \in Pairs_{\mathbb{L}^I}\})) \setminus \mathbb{L}^I$. $\mathbb{C}_{\mathbb{L}}$ are called the candidate sets of $\mathbb{L}$.*

Intuitively the sets in $\mathbb{C}^+_{\mathbb{L}}$ are naive in-sets (wrt. a possible witnessing AF for $\mathbb{L}$) and $\mathbb{C}_{\mathbb{L}}$ naive in-sets that are not stable, subsequently we call the sets in $\mathbb{C}_{\mathbb{L}}$ candidate sets.
We will divide this chapter by the case $\mathbb{C}_{\mathbb{L}} = \emptyset$ and $\mathbb{C}_{\mathbb{L}} \neq \emptyset$.

Example 4. *We continue Example 1. As can be easily verified using the definition above we actually identify the problematic set correctly with $\mathbb{C}_{\mathbb{L}} = \{\{a', b', c'\}\}$. If we examine the graph of the AF given in the example, we can see that $\{a', b', c'\}$ is indeed a naive extension albeit not a stable one since x is not attacked.*

3.2. Towards a Characterization

In this section we aim to deepen the general discussion from the last section towards a characterization. As mentioned before we will divide our observation into two parts: First, there are no candidate sets, i.e $\mathbb{C}_{\mathbb{L}} = \emptyset$ and secondly, there are candidate sets, i.e. $\mathbb{C}_{\mathbb{L}} \neq \emptyset$. We start with the former case.

3.2.1. No additional candidate sets

The analysis of the first case $\mathbb{C}_{\mathbb{L}} = \emptyset$ is based on the naive labelling realizability. The underlying idea is: the set $\mathbb{C}_{\mathbb{L}}$ illustrates possible sets that are naive but not stable. However when that set is empty we can construct a witnessing AF F where the examined labelling-set matches the stables as well as the naive labellings. We will construct this AF by making use of the naive AF-construction (Definition 14). In that case we applied the AF-construction of the cf-realizability (Definition 12) and applied it to the labellings-downward-closure (Definition 13). In fact, we will show that under our circumstances the construction simplifies s.t. it will no longer be necessary to construct the labelling-downward-closure.

Furthermore, we will make use of a result about extensions-based realizability. In [13] a connection between the extension-based realizability and the set $\mathbb{C}_{\mathbb{L}}$ was already shown. We adopt this results to the realm of labellings. An alternative characterization of naive and stable extension-realizability is presented: a set of extensions or in our case a set of in-sets $\mathbb{L}^I$ is naive extension-realizable iff $\mathbb{L}^I$ matches $\mathbb{C}^+_{\mathbb{L}}$ and for stable realizability we at least can assume a subset relation (Proposition 6). However it is important to keep in mind that these statements only concern the extension-realizability of the in-set and the relation to the given out-sets has to be examined separately.

Proposition 6 ([13, Proposition 13]). *It holds $\Sigma_{\mathcal{E}_{na}} = \left\{\mathbb{L}^I \mid \mathbb{L} \text{ is a labelling-set}, \ \mathbb{L} \neq \emptyset, \ \mathbb{L}^I = \mathbb{C}^+_{\mathbb{L}}\right\}$ and $\Sigma_{\mathcal{E}_{stb}} = \left\{\mathbb{L}^I \mid \mathbb{L} \text{ is a labelling-set}, \ \mathbb{L}^I \subseteq \mathbb{C}^+_{\mathbb{L}}\right\}$*

As mentioned before, we will make use of the AF-construction already used in naive realizability. To use this construction, however, a key property of naive as well as conflict-free realizability is required, namely the L-tightness [8, Proposition 10]. In the naive case we always deal with multiple out-sets per in-set. L-tightness ensures the existence of a maximal out-set for each inset as well as fulfills a similar notion as tightness in the extensions case, i.e. there is always a reason for an argument to be out (see Definition 11). Therefore we note that, even while not explicitly mentioned, L-tightness will still be implicitly required regarding stable realizability of labellings. We will see that it suffices to assume the properties of extension realizability, namely $\mathbb{L}^I$ is incomparable and tight and the emptiness of all undec-sets. The following lemma shows that just by assuming this minimal set of properties, L-tightness is implied.

Lemma 7. *Let $\mathbb{L}$ be a labelling set where $\mathbb{L}^I$ is incomparable and $\bigcup \mathbb{L}^U = \emptyset$. If $\mathbb{L}^I$ is tight, then $\mathbb{L}$ is L-tight.*

Proof. Assume $\mathbb{L}^I$ is tight. For $\mathbb{L}$-tightness we have to prove two properties: the existence of a maximal out-set for each in-set and for all $E_1, E_2 \in \mathbb{L}^I$ we have: $E_1 \cup E_2 \in \mathbb{L}^I \Leftrightarrow (\overline{\mathbb{L}}^O_{I=E_1} \cap E_2) \cup (\overline{\mathbb{L}}^O_{I=E_2} \cap E_1) = \emptyset$. The first property is trivial for our case, since through the requirement of $\bigcup \mathbb{L}^U = \emptyset$ there is only one out-set per in-set, i.e. $|\overline{\mathbb{L}}^O_{I=E}| = 1$ for every $E \in \mathbb{L}^I$ (as a reminder: we denote this singular out-set with $L^O_{I=E}$ see Definition 10).
Now for the second property. Let $E_1, E_2 \in \mathbb{L}^I$ and $E_1 \cup E_2 \notin \mathbb{L}^I$. Thus, $E_1 \neq E_2$ and there exists (WLOG) an $a \in E_2$ s.t. $a \notin E_1$ and due to the incomparability $E_1 \cup \{a\} \notin \mathbb{L}^I$. Since we assumed tightness of $\mathbb{L}^I$ there exists an $b \in E_1$ s.t. $(a, b) \notin Pairs_{\mathbb{L}^I}$. Hence $b \notin E_2$ and so $b \in L^O_{I=E_2}$ since $L^U_{I=E_2} = \emptyset$. Consequently, $b \in E_1 \cap L^O_{I=E_2} \neq \emptyset$. Consider now $(E_1 \cap L^O_{I=E_2}) \cup (E_2 \cap L^O_{I=E_1}) \neq \emptyset$. Let WLOG $E_1 \cap L^O_{I=E_2} \neq \emptyset$. This means, there exists a $b \in E_1 \cap L^O_{I=E_2}$. Then $b \notin E_2$ and since the undecided sets are empty $b \in L^O_{I=E_1}$. Then the incomparability allows us to conclude $E_2 \cup \{b\} \notin \mathbb{L}^I$ and thus due to the tightness there exists $a \in E_2$ s.t. $(a, b) \notin Pairs_{\mathbb{L}^I}$. Hence $E_1 \cup E_2 \notin \mathbb{L}^I$ since otherwise $(a, b) \notin Pairs_{\mathbb{L}^I}$ would be contradicted. $\square$

The assumption that all undecided sets are empty yields a simpler version of the naive AF-construction (Definition 14) as described below. In particular, we do not have to consider the labelling-downward-closure.

Proposition 8. *Let $\mathbb{L}$ be a labelling-set where $\mathbb{L}^I$ is incomparable and tight as well as $\bigcup \mathbb{L}^U = \emptyset$ then $F^{cf}_{\downarrow\mathbb{L}} =: F^{na}_{\mathbb{L}} = (A^{na}, R^{na})$ with $A^{na} = Args_{\mathbb{L}}$ can be written as*

1'. $\forall a \in A^{na}$: $(a, a) \in R^{na}$ if and only if $a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$, and

2'. $\forall a, b \in A^{na}$: If $a \neq b$, then $(a, b) \in R^{na}$ iff $b \in O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E}$.

Proof. The AF $F^{cf}_{\downarrow\mathbb{L}}$ (Definition 14) is defined through $F^{cf}_{\downarrow\mathbb{L}} = (A_{\downarrow\mathbb{L}}, R_{\downarrow\mathbb{L}})$ with $A_{\downarrow\mathbb{L}} = Args_{\mathbb{L}}$ and 1. $\forall a \in A_{\downarrow\mathbb{L}}$: $(a, a) \in R_{\downarrow\mathbb{L}}$ if and only if $\{a\} \notin {}^{\downarrow}\mathbb{L}^I$, and 2. $\forall a, b \in A_{\downarrow\mathbb{L}}$: If $a \neq b$, then $(a, b) \in R_{\downarrow\mathbb{L}}$ iff $\{a\} \in {}^{\downarrow}\mathbb{L}^I$ and $b \in \bigcup {}^{\downarrow}\mathbb{L}^O_{I=\{a\}}$. We will show the transformation of each rule separately:
(1.) $\Rightarrow$ (1'.) Assume $\{a\} \notin {}^{\downarrow}\mathbb{L}^I$. From the definition of the downward-closure (Definition 12) we know $\{a\} \nsubseteq E$ for all $E \in \mathbb{L}^I$. Thus $a \notin Args_{\mathbb{L}^I}$ and since $\bigcup \mathbb{L}^U = \emptyset$ we get $a \notin Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$.
For the reverse we start by assuming $a \notin Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$. This means $a \notin E$ for all $E \in \mathbb{L}^I$ and thus again from Definition 12: $\{a\} \notin {}^{\downarrow}\mathbb{L}^I$
(2.) $\Rightarrow$ (2'.) Assume $\{a\} \in {}^{\downarrow}\mathbb{L}^I$ and $b \in \bigcup {}^{\downarrow}\mathbb{L}^O_{I=\{a\}}$. Through the definition of the downward-closure we can transform this to $\{a\} \in {}^{\downarrow}\mathbb{L}^I$ and

$$\begin{aligned} b \in & \bigcup \left({}^{\downarrow}\mathbb{L}_{I=\{a\}}\right)^O = \bigcup(\{(E, O, Args_{\mathbb{L}} \setminus (E \cup O)) | E \subseteq L^I, L^I \in \mathbb{L}^I, O \subseteq O_E\}_{I=\{a\}})^O \\ = & \bigcup(\{(\{a\}, O, Args_{\mathbb{L}} \setminus (\{a\} \cup O)) | O \subseteq O_a\})^O = \bigcup \{O \mid O \subseteq O_{\{a\}}\} \\ = & \bigcup \{O \mid O \subseteq \bigcap_{I' \in \mathbb{L}^I, \{a\} \subseteq I'} \bigcup \mathbb{L}^O_{I=I'}\} = \bigcup \{O \mid O \subseteq \bigcap_{I' \in \mathbb{L}^I, a \in I'} L^O_{I=I'}\} \end{aligned}$$

$$= \bigcap_{I' \in \mathbb{L}^I, a \in I'} L^O_{I=I'} = O_a$$

The reasoning for the last two steps is the following: In this circumstance we can simplify the definition of O_a since $\mathbb{L}^I$ is L-tight (Lemma 7) and thus the existence of a maximal out-set in $\mathbb{L}$ is assured, we can continue this thought, since through $\bigcup \mathbb{L}^U = \emptyset$ we also know that there is only one out-set per in-set in $\mathbb{L}$. Hence we end at $b \in O_a = \bigcap_{I' \in \mathbb{L}^I, a \in I'} L^O_{I=I'}$ as claimed in $(2'.)$ Now to the only-if direction. Assume $b \in O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E}$. Immediately: O_a only exists if there is an $E \in \mathbb{L}^I$ s.t. $\{a\} \subseteq E$, which is exactly $\{a\} \in {\downarrow}\mathbb{L}^I$. $b \in \bigcup {\downarrow}\mathbb{L}^O_{I=\{a\}}$ follows from $b \in O_a$ through the transformation shown above. $\square$

As mentioned before we will use the naive (and consequently the conflict-free) labelling realizability as a foundation or rather a point of reference for the characterization of stable realizability. In naive realizable labelling-set there exists a maximal out-set for each in-set and the set of all out-set for an in-set is downward-closed. To improve the clarity of the notation and proof and we use a special notation for the naive labellings of an AF, where we only observe the maximal out-set for each in-set.

Definition 16. *For an AF $F = (A, R)$ we define $\mathcal{L}_{\overline{na}}(F) = \left\{(E, \overline{\mathbb{L}}^O_{I=E}, U) \mid E \in \mathbb{L}^I\right\}$ where $\mathbb{L} = \mathcal{L}_{na}(F)$.*

Now we can present the first characterization theorem regarding stable realizability restricted to the case $\mathbb{C}_{\mathbb{L}} = \emptyset$.

Theorem 9. *Given a labelling-set $\mathbb{L}$ where $\mathbb{C}_{\mathbb{L}} = \emptyset$ we have: $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}} \Leftrightarrow$ 1. $\mathbb{L}^I$ is incomparable and tight and 2. $\bigcup \mathbb{L}^U = \emptyset$*

Proof. We will split the proof into if- and only-if-direction. The if-direction follows immediately from Proposition 5. It remains to show the only-if direction: From 1. and Theorem 3 we know immediately that $\mathbb{L}^I \in \Sigma_{\mathcal{E}_{stb}}$ and thus from Proposition 6 that $\mathbb{L}^I \subseteq \mathbb{C}^+_{\mathbb{L}}$. Together with $\mathbb{C}_{\mathbb{L}} = \emptyset$ this provides the information that $\emptyset = \mathbb{C}_{\mathbb{L}} = \mathbb{C}^+_{\mathbb{L}} \setminus \mathbb{L}^I$ and thus $\mathbb{C}^+_{\mathbb{L}} = \mathbb{L}^I$.
Now we use the AF-Construction from Proposition 8 on $\mathbb{L}$. Our goal is to show that $F^{na}_{\mathbb{L}}$ is a witnessing framework for $\mathbb{L}$, i.e. $\mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$:

- $\mathbb{L} \subseteq \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$: $\bigcup \mathbb{L}^U = \emptyset$ is provided by Item 2. Thus we only have to show that $\mathbb{L}$ is a conflict-free labelling-set of $F^{na}_{\mathbb{L}}$:
  1. No internal conflicts, i.e. if $a, b \in L^I$, then $(a, b) \notin R$. We use that $\mathbb{L}^I = \mathbb{C}^+_{\mathbb{L}}$ as shown above, note that for this direction even $\mathbb{L}^I \subseteq \mathbb{C}^+_{\mathbb{L}}$ is sufficient. This means for all $L \in \mathbb{L}$ and for all $a, b \in L^I$ with $a \neq b$ it holds that $a, b \in Pairs_{\mathbb{L}^I}$. So there exists an $L \in \mathbb{L}$ s.t. $\{a, b\} \subseteq L^I$ and thus $b \notin O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E}$ and $a \notin O_b$. Hence $(a, b), (b, a) \notin R^{na}$. The case $a = b$ can be ignored since $(a, a) \in R^{na}$ only iff $a \notin Args_{\mathbb{L}^I}$.
  2. Reason for rejecting, i.e $a \in L^O$ iff there is some $b \in L^I$ with $(b, a) \in R^{na}$ for all $L \in \mathbb{L}$.
  Let $L \in \mathbb{L}$ and $b \in L^O$. Then $b \notin L^I \in \mathbb{C}^+_{\mathbb{L}}$. Thus it exists $a \in L^I$ s.t. $(a, b) \notin Pairs_{\mathbb{L}^I}$ Consequently, by definition of $Pairs_{\mathbb{L}^I}$ we deduce for any $L' \in \mathbb{L}$: if $a \in L'^I$ then $b \notin L'^I$ meaning $b \in Args_{\mathbb{L}} \setminus L'^I = L'^O$ because of $L'^U = \emptyset$. However this is exactly the definition for $b \in O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E}$ and thus $(a, b) \in R^{na}$.

- $\mathcal{L}_{stb}(F^{na}_{\mathbb{L}}) \subseteq \mathbb{L}$: Let $L = (L^I, L^O, \emptyset) \in \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$. We show first that $L^I \in \mathbb{L}^I$: $L^I \in \left(\mathcal{L}_{stb}(F^{na}_{\mathbb{L}})\right)^I$ and thus has no internal conflicts i.e. if $a, b \in L^I$, then $(a, b) \notin R^{na}$. Thus $\forall a, b \in L^I : (a, b), (b, a) \notin R^{na}$. From the construction of R^{na} we know then that $b \notin O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E}$ and $a \notin O_b$. Intuitively, the definition of O_a (or O_b respectively) means that if a is in then b is out and vice versa, meaning they never co-occur in an in-set. In reverse if $a \notin O_b$, we know there has to be some in-set where a and b co-occur, since we assumed $\bigcup \mathbb{L}^U = \emptyset$. Formally this means for all $a, b \in L^I$ we know $(a, b) \in Pairs_{\mathbb{L}^I}$. Additionally we know that for any $c \in A^{na} \setminus L^I = Args_{\mathbb{L}} \setminus L^I$ that $c \in L^O$ (again because $L^U = \emptyset$). Thus there exists an

$a' \in L^I$ s.t. $(a', c) \in R^{na}$ (since $L \in \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$). Thus $c \in O_{a'}$, through the definition of R^{na}, meaning for all $E \in \mathbb{L}^I$ if $a' \in E$ then $c \in L^O_{I=E}$ and with this $(a', c) \notin Pairs_{\mathbb{L}^I}$. Together with the fact from before $(a, b) \in Pairs_{\mathbb{L}^I}$ for all $a, b \in L^I$ we see that $L^I \in \mathbb{C}^+_{\mathbb{L}}$. We combine this with our conclusion from the beginning of the proof $\mathbb{L}^I = \mathbb{C}^+_{\mathbb{L}}$ and $L^I \in \mathbb{L}^I$ finishing the first part of the proof.
Furthermore, by definition of $F^{na}_{\mathbb{L}}$ we obtain $Args_{F^{na}_{\mathbb{L}}} = Args_{\mathbb{L}}$. Moreover, for any $L \in \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$ the associated out-set can be given as $L^O = Args_{F^{na}_{\mathbb{L}}} \setminus L^I$ since $L^U = \emptyset$. Hence $L \in \mathbb{L}$. □

Corollary 10. *Let $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$, then $\mathbb{L} \subseteq \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$.*

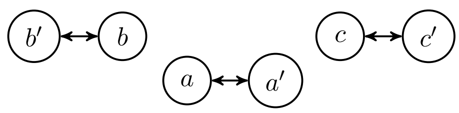
Proof. $\mathbb{L}$ is stable realizable, thus Proposition 6 provides $\mathbb{L}^I \subseteq \mathbb{C}^+_{\mathbb{L}}$. With this $\mathbb{L} \subseteq \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$ was already shown in the only-if direction in the proof of Theorem 9. Note that while Theorem 9 was under the assumption $\mathbb{C}_{\mathbb{L}} = \emptyset$, in that particular part of the proof it is not used. □

Example 5. *We use this opportunity to examine a slight modification of our running example set. We modify it exactly in the way we reflected in Example 1. We "delete" x and add the candidate set $\{a', b', c'\}$ (Example 4). We already know that the in-sets are incomparable and tight, the undecided sets empty and by the inclusion of the candidate set: $\mathbb{C}_{\mathbb{L}} = \emptyset$. We construct the witnessing AF: For each $e \in \mathbb{L}^I$ we build the set O_e: We show this in detail for the argument a.*

$$\begin{aligned} O_a = \bigcap_{E \in \mathbb{L}^I, a \in E} L^O_{I=E} &= L^O_{I=\{a,b,c\}} \cap L^O_{I=\{a,b,c'\}} \cap L^O_{I=\{a,b',c\}} \cap L^O_{I=\{a,b',c'\}} \\ &= \{a', b', c'\} \cap \{a', b', c\} \cap \{a', b, c'\} \cap \{a', b, c\} = \{a'\} \end{aligned}$$

In the same way we get the set $O_b = \{b'\}$, $O_c = \{c'\}$, $O_{a'} = \{a\}$, $O_{b'} = \{b\}$ and $O_{c'} = \{c\}$. Here we can easily see the attack $(e, e') \in F^{na}_{\mathbb{L}}$ iff $e' \in O_e$ and construct the AF.

$\mathbb{L}$	I	O	U
L_1	$\{a, b, c\}$	$\{a', b', c'\}$	$\{\}$
L_2	$\{a, b, c'\}$	$\{a', b', c\}$	$\{\}$
L_3	$\{a, b', c\}$	$\{a', b, c'\}$	$\{\}$
L_4	$\{a', b, c\}$	$\{a, b', c'\}$	$\{\}$
L_5	$\{a, b', c'\}$	$\{a', b, c\}$	$\{\}$
L_6	$\{a', b, c'\}$	$\{a, b', c\}$	$\{\}$
L_7	$\{a', b', c\}$	$\{a, b, c'\}$	$\{\}$
L_8	$\{a', b', c'\}$	$\{a, b, c\}$	$\{\}$

$F^{na}_{\mathbb{L}}$:


Besides a characterization for stable realizability in the case $\mathbb{C}_{\mathbb{L}} = \emptyset$ the proof of Theorem 9 also defines a construction for a witnessing AF and gives us a strong connection to the naive semantics. We summarize these results in the following corollary:

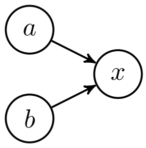
Proposition 11. *Let $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$. If $\mathbb{C}_{\mathbb{L}} = \emptyset$, then $\mathcal{L}_{stb}(F^{na}_{\mathbb{L}}) = \mathcal{L}_{\overline{na}}(F^{na}_{\mathbb{L}}) = \mathbb{L}$.*

Proof. From $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$ and Item 4 of Proposition 1 we know immediately that $\mathbb{L}^I \in \Sigma_{\mathcal{E}_{stb}}$ and thus from Proposition 6 that $\mathbb{L}^I \subseteq \mathbb{C}^+_{\mathbb{L}}$. Together with $\mathbb{C}_{\mathbb{L}} = \emptyset$ this provides the information that $\emptyset = \mathbb{C}_{\mathbb{L}} = \mathbb{C}^+_{\mathbb{L}} \setminus \mathbb{L}^I$ and thus $\mathbb{C}^+_{\mathbb{L}} = \mathbb{L}^I$. Under these circumstances we have seen in the proof of Theorem 9 that $\mathbb{L} = \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$. Additionally with Proposition 6 we can conclude $\mathbb{L}^I = \mathbb{C}^+_{\mathbb{L}} = \mathcal{E}_{na}(F^{na}_{\mathbb{L}})$ and thus since $\bigcup \mathbb{L}^U = \emptyset$ also $\mathbb{L} = \mathcal{L}_{\overline{na}}(F^{na}_{\mathbb{L}})$. □

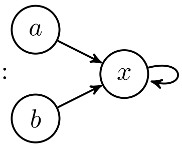
The following example shows that Proposition 11 cannot be generalized to arbitrary witnessing AFs. More precisely, there are AFs stable realizing a set $\mathbb{L}$ with different naive labellings, although no additional candidate sets exist.

Example 6. *Let $\mathbb{L} = \{(\{a, b\}, \{x\}, \{\})\}$. We deduce that $\mathbb{C}_{\mathbb{L}} = \emptyset$. The following AF F realizes the labelling-set $\mathbb{L}$ under stable semantics.*

F :



$F^{na}_{\mathbb{L}}$:



However, computing the naive labellings $\mathcal{L}_{\overline{na}}(F)$ of F reveals a further naive labelling, namely $(\{x\}, \emptyset, \{a, b\})$. This means, for the witnessing AF F we have: $\mathcal{L}_{\overline{na}}(F) \neq \mathcal{L}_{stb}(F)$. According to Proposition 11 such additional labellings are impossible in case of the witnessing $F^{na}_{\mathbb{L}}$. Indeed, $\mathcal{L}_{stb}(F^{na}_{\mathbb{L}}) = \mathcal{L}_{\overline{na}}(F^{na}_{\mathbb{L}}) = \mathbb{L}$.

3.2.2. The Existence of Candidate Sets

Now let us turn to the case where additional naive sets exist which should not turn stable. For this case we introduce the notion of a *critical argument*. Intuitively, for each $C \in \mathbb{C}_{\mathbb{L}}$ there has to be at least one argument that is not attacked by C, since otherwise C would be stable. This is exactly the job of a critical argument. However, identifying a critical argument is sometimes challenging, as we will see in the remainder of this section. In the following we introduce C*-critical candidates* which serve as preselected arguments that might play the role of a critical argument.

Definition 17. *Let $\mathbb{L}$ be a labelling-set and $C \in \mathbb{C}_{\mathbb{L}}$. We call $x \in Args_{\mathbb{L}}$ a C-critical candidate if $x \in Args_{\mathbb{L}} \setminus C$, and for each $E' \in \mathbb{L}^I$ with $x \notin E'$: $E' \cap S^C_x \neq \emptyset$, where $S^C_x = \{s \in Args_{\mathbb{L}} \setminus (C \cup \{x\}) \mid (s, x) \notin Pairs_{\mathbb{L}^I}\}$. We denote the set of all critical candidates for C as $Krit^C_{\mathbb{L}}$.*

Intuitively, for each candidate set $C \in \mathbb{C}_{\mathbb{L}}$ we identify critical candidate arguments. From these we choose a critical argument that in a witnessing AF is the witness for the non-stability of C, i.e. it is not included in C itself but also not attacked by C. Furthermore, if all arguments in C are not allowed to attack a C-critical candidate x, each in-set in $\mathbb{L}^I$ where the critical candidate x is not included still has to be able to reject said x to retain its stability. This means: we need at least one argument in each such in-set, that is not included in C and that does not co-occur with x in any in-set, i.e. an attack is theoretically possible without violation of conflict-freeness. In terms of the second condition of Definition 17: The set S^C_x collects all possible attackers of x outside of C and then we check if the intersection with each in-set where x is not included is non-empty, thus ensuring an attack on x is still possible.

Example 7. *We continue our running Example 1 and consider labelling-set $\mathbb{L}$ as depicted in Table 1. We already observed that $\mathbb{C}_{\mathbb{L}} = \{\{a', b', c'\}\}$. Now, we determine $Krit^{\{a',b',c'\}}_{\mathbb{L}}$. We therefore have to generate the sets $S^{\{a',b',c'\}}_{x'}$ for each single $x' \in Args_{\mathbb{L}} \setminus \{a', b', c'\} = \{a, b, c, x\}$. Consider the first case: $x' = a$.*

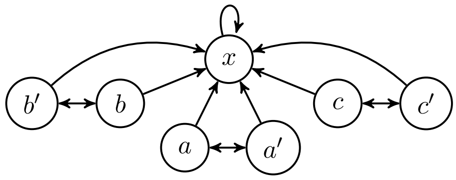
$$\begin{aligned} S^{\{a',b',c'\}}_a &= \{s \in Args_{\mathbb{L}} \setminus (\{a', b', c'\} \cup \{a\}) \mid (s, a) \notin Pairs_{\mathbb{L}^I}\} \\ &= \{s \in \{b, c, x\} \mid (s, a) \notin Pairs_{\mathbb{L}^I}\} = \{x\} \end{aligned}$$

By symmetry, we obtain $S^{\{a',b',c'\}}_b = \{x\}$ and $S^{\{a',b',c'\}}_c = \{x\}$. For these first three sets it is easy to see that $S^C_x \cap E = \emptyset$ for all $E \in \mathbb{L}^I$ since x is not included in any in-set. Thus neither a, b nor c are critical candidates. However, in case of $x' = x$ we get $S^{\{a',b',c'\}}_x = \{a, b, c\}$. We observe that the second item of Definition 17 is fulfilled by $S^{\{a',b',c'\}}_x$ only. More precisely, in every in-set $E' \in \mathbb{L}^I$ at least one of the arguments a, b or c is contained. In summary, we have $Krit^{\{a',b',c'\}}_{\mathbb{L}} = \{x\}$.

In order to realize labelling sets with additional candidate sets, we take our standard construction $F^{na}_{\mathbb{L}}$ as a starting point. By Corollary 10 we already know that $\mathbb{L} \subseteq \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$. Thus, we might have to get rid of additional stable labellings. In Theorem 13 we will formally show that this procedure is valid. Consider the following example.

Example 8 (Example 1 cont.). *Given the labelling-set $\mathbb{L}$ from our running example (Table 1). We already know $\mathbb{C}_{\mathbb{L}} = \{\{a', b', c'\}\}$. We use the standard construction $F^{na}_{\mathbb{L}}$ from Proposition 8.*

$F^{na}_{\mathbb{L}}$:



We observe $(\mathcal{L}_{stb}(F^{na}_{\mathbb{L}}))^I = \mathbb{L}^I \cup \mathbb{C}_{\mathbb{L}}$. The attacks (a', x), (b', x) and (c', x) turn $\{a', b', c'\}$ into an unwanted stable set. In fact, already one of them would have this effect. It is easy to verify that by removing these attacks we render $\{a', b', c'\}$ unstable while not violating the stability of the other desired stable in-sets.

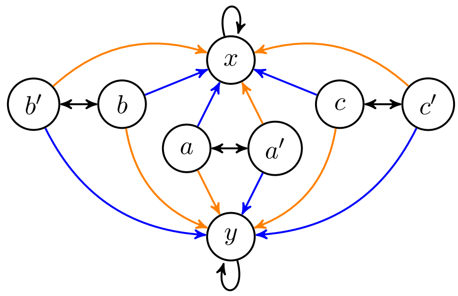
The example shows we cannot just use the construction of the case without candidate-sets as is. Although the construction does not yield an exact AF for the labellings set, we nevertheless obtain an AF where $\mathbb{L} \subset \mathcal{L}_{stb}(F^{na}_{\mathbb{L}})$ (with the addition of the candidate sets) and the witnessing AF for $\mathbb{L}$ is a subgraph of $F^{na}_{\mathbb{L}}$. We will show later on, that there always exists a witnessing AF that is a subgraph of $F^{na}_{\mathbb{L}}$: More precisely every witnessing AF can be transformed into a subgraph of $F^{na}_{\mathbb{L}}$ via semantically neutral modifications. Thus validating our method in general. This leads us to the question, which attacks can be removed from $F^{na}_{\mathbb{L}}$ without violating the stability of the desired stable labellings?

The solution of just removing all attacks from the candidate set towards its critical candidates does not work in general. The following example shows some of the difficulties that can occur and indeed why these arguments are only called critical candidates.

Example 9. *Given the following labelling-set $\mathbb{L}$.*

$\mathbb{L}$	I	O	U
L_2	$\{a, b, c'\}$	$\{a', b', c, x, y\}$	$\{\}$
L_3	$\{a, b', c\}$	$\{a', b, c', x, y\}$	$\{\}$
L_4	$\{a', b, c\}$	$\{a, b', c', x, y\}$	$\{\}$
L_5	$\{a, b', c'\}$	$\{a', b, c, x, y\}$	$\{\}$
L_6	$\{a', b, c'\}$	$\{a, b', c, x, y\}$	$\{\}$
L_7	$\{a', b', c\}$	$\{a, b, c', x, y\}$	$\{\}$

$F^{na}_{\mathbb{L}}$:



We will leave it to the reader to check that $\mathbb{C}_{\mathbb{L}} = \{C_1, C_2\} = \{\{a', b', c'\}, \{a, b, c\}\}$ and $Krit^{C_1}_{\mathbb{L}} = Krit^{C_2}_{\mathbb{L}} = \{x, y\}$. We again construct the AF $F^{na}_{\mathbb{L}}$ (as seen above). C_1 as well as C_2 have the critical candidates x and y. As we have already seen in the example before, just taking the construction as is, i.e. not removing any attacks does not work, since then both, C_1 and C_2 would also be stable. However, if we disallow all attacks on critical candidates from their respective C (we remove the orange and blue attacks), then neither x nor y get attacked at all, thus rendering all desired labellings unstable. Consequently, we have to make a decision: we remove the attacks from C_1 towards x and from C_2 towards y (remove all blue attacks) or vice versa (remove all orange attacks), but one of these attack-patterns is necessary to achieve the desired stable labellings.

The above example shows two decisive points on the way of finding witnessing frameworks.
1. We must balance removing attacks because: On the one hand, we have to remove enough attacks, s.t. candidate sets are rendered non-stable. On the other hand, we cannot remove too many attacks as critical candidates have to be rejected (see Lemma 12).
2. We encounter non-deterministic situations: There are cases where a decision becomes necessary which argument is actually critical for a candidate set. Not only that but these decisions can also be dependent on each other.

The following lemma shows indeed the necessity of rejecting critical candidates.

Lemma 12. *Let $\mathbb{L}$ be a labelling-set with $\bigcup \mathbb{L}^U = \emptyset$ and let $C \in \mathbb{C}_{\mathbb{L}}$. For any $x \in Krit^C_{\mathbb{L}}$ there exists an $O \in \mathbb{L}^O$, s.t. $x \in O$.*

Proof. Let $C \in \mathbb{C}_{\mathbb{L}}$ and $x \in Krit^C_{\mathbb{L}}$. We assume toward a contradiction: $x \notin O$ for all $O \in \mathbb{L}^O$. Since $\bigcup \mathbb{L}^U = \emptyset$, then $x \in E$ for all $E \in \mathbb{L}^I$. Then $(x, e) \in Pairs_{\mathbb{L}^I}$ for all $e \in Args_{\mathbb{L}^I}$. Per Definition 15 follows $x \in C'$ for all $C' \in \mathbb{C}_{\mathbb{L}} = \mathbb{C}^+_{\mathbb{L}} \setminus \mathbb{L}^I$, because of the $\subseteq$-maximality condition of $\mathbb{C}^+_{\mathbb{L}}$. Hence, in particular $x \in C$. Thus contradicting $x \in Krit^C_{\mathbb{L}}$ since Definition 17 demands $x \in Args_{\mathbb{L}} \setminus C$. □

We want to establish that our proposed method of removing attacks from the AF $F^{na}_{\mathbb{L}}$ can always be successful, if the labelling-set $\mathbb{L}$ is stable realizable:

Theorem 13. *Let $\mathbb{L} \in \Sigma_{\mathcal{L}_{stb}}$. There exists an AF F, s.t. $F \sqsubseteq F^{na}_{\mathbb{L}}$ and $\mathcal{L}_{stb}(F) = \mathbb{L}$.*

Proof. Since $\mathbb{L}$ is stable realizable there exists $F = (A, R)$, s.t. $\mathcal{L}_{stb}(F) = \mathbb{L}$. We will show that either directly $F \sqsubseteq F^{\mathbb{L}}_{na}$ or that there is a modification $F' = (A, R')$ of F, s.t. $F' \sqsubseteq F^{\mathbb{L}}_{na}$ and $\mathcal{L}_{stb}(F) = \mathbb{L}$. We immediately know that $A = Args_{\mathbb{L}} = A^{na}$. Let $(a, b) \in R$. To show the subgraph relationship with regards to the attack relation we distinguish the following cases:

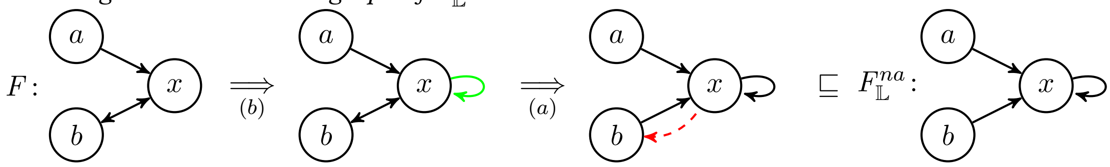
(i) $a \in Args_{\mathbb{L}^I}$, $a \neq b$. From the stability of $\mathbb{L}$ we know then that $(a, b) \notin Pairs_{\mathbb{L}^I}$. Furthermore since $\bigcup \mathbb{L}^U = \emptyset$ it follows: For all $L \in \mathbb{L}$, if $a \in L^I$, then $b \in L^O$. Thus $b \in O_a$ and the construction of R^{na}: $(a, b) \in R^{na}$.

(ii) $a = b$. Then $a \notin Args_{\mathbb{L}^I}$ since $\mathbb{L}$ is stable and thus conflict-free. Moreover $a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$ given the fact that $\bigcup \mathbb{L}^U = \emptyset$. Hence $(a, a) \in R^{na}$.

(iii) $a \notin Args_{\mathbb{L}^I}$, $a \neq b$. Then $a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$, since $\bigcup \mathbb{L}^U = \emptyset$. Hence $(a, a) \in R^{na}$. We now distinguish two subcases:
(a) $(a, a) \in R$. Then the AF $F' = (A, R')$ where $R' = R \setminus \{(a, b)\}$ has the same stable labellings as F i.e. $\mathcal{L}_{stb}(F) = \mathcal{L}_{stb}(F') = \mathbb{L}$ since $F^{sk} = (F')^{sk}$ which even guarantees strong equivalence. (Proposition 2)
(b) $(a, a) \notin R$. Now we have to show that adding this self-loop is a neutral modification regarding the stable labellings $\mathbb{L}$. Intuitively neither the conflict-freeness of $\mathbb{L}^I$, since $a \notin \mathbb{L}^I$ nor the attacks on all outlying arguments are influenced. We will show this formally. Let $F^a = (A, R^a)$ where $R^a = R \cup \{(a, a)\}$. We show both inclusion:
Therefore let $L \in \mathbb{L}$. We know $a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$, or, to be more explicit, for each $L' \in \mathbb{L}$ it holds that $a \in (L')^O$ since $\bigcup \mathbb{L}^U = \emptyset$. Especially $a \in L^O$. Thus the inner conflict-freeness is not violated by adding (a, a), meaning for all $b, b' \in L^I$ $(b, b') \notin R^a$. Additionally we know through the stability that $L^O = A \setminus L^I$, again because of the emptiness of the undecided sets, and for each $c \in L^O$ exists $b \in L^I$ s.t. $(b, c) \in R$, since we only add a self-loop and do not delete attacks also $(b, c) \in R^a$. Thus $L \in \mathcal{L}_{stb}(F^a)$.
The reverse inclusion is quite similar: Let $L \in \mathcal{L}_{stb}(F^a)$. Then for all $b, b' \in L^I$ we know $(b, b') \notin R^a$ because of the inner conflict-freeness. Additionally $(a, a) \in R^a$ and thus $a \notin L^I$, with the same reasoning as before $a \in A \setminus L^I = L^O$. Hence for all $b, b' \in L^I$ also $(b, b') \notin R$. Furthermore, again because of the stability of L, for all $c \in L^O$ exists $b \in L^I$ s.t. $(b, c) \in R^a$. Then $(b, c) \in R$ and thus $L \in \mathbb{L}$.

In summary we have shown the following: We can add a self-loop to all purely out-arguments without changing the stable labellings as well as delete all outgoing attacks from purely out-arguments (here we even have strong equivalence). Meaning for the modified AF $F^{mod} = (A, R^{mod})$ with $R^{mod} = (R \cup \{(a, a) \mid a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}\}) \setminus \{(a, b) \mid a \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}\}$ it holds that $\mathcal{L}_{stb}(F^{mod}) = \mathcal{L}_{stb}(F) = \mathbb{L}$ and that $F^{mod} \sqsubseteq F^{na}_{\mathbb{L}}$. □

Example 10. *Let $\mathbb{L} = \{(\{a, b\}, \{x\}, \{\})\}$. This set is realizable through the AF F. However, it is easy to see that F is not a subgraph of $F^{na}_{\mathbb{L}}$. In the proof of Theorem 13 we have shown that there are two neutral (in the sense that they do not change the stable labellings) syntactic actions I can take: (a) I can delete* *outgoing attacks from purely out-arguments, i.e. $x \in Args_{\mathbb{L}^O} \setminus Args_{\mathbb{L}^I}$ and (b) I can add a self-loop to a purely out-arguments. We modify F by applying these operations as often as possible: in our case we can apply (b) once - adding a self-loop to x and (a) once - deleting the outgoing attack from x towards b. The resulting AF is indeed a subgraph of $F^{na}_{\mathbb{L}}$:*



3.3. Relation to Compact Realizability

A noteworthy consequence concerns compact stable extension realizability, which has remained open since [13, 7]. In this context, *compact* means that the argument set of the witnessing AF coincides exactly with the union of all extensions, admitting no auxiliary arguments; equivalently, in the labelling setting, every argument appears in the in-set of at least one labelling. Hence each result on stable labelling realizability would simultaneously resolve compact stable extension realizability. The following proposition makes the connection precise.

Proposition 14. *Let $\mathbb{S}$ be an extension-set. $\mathbb{S} \in \Sigma^c_{\mathcal{E}_{stb}}$ if and only if the labelling-set $\mathbb{L} = \{(E, Args_{\mathbb{S}} \setminus E, \emptyset) \mid E \in \mathbb{S}\} \in \Sigma_{\mathcal{L}_{stb}}$.*

Proof. Let $\mathbb{S} \in \Sigma^c_{\mathcal{E}_{stb}}$. Then there exists a compact witnessing AF $F = (A, R)$ with $\mathcal{E}_{stb}(F) = \mathbb{S}$ and $A = Args_{\mathbb{S}}$. It follows that $\mathcal{L}_{stb}(F) = \mathbb{L}$ since $\mathbb{L}^I = \mathbb{S}$ and $\bigcup_{L \in \mathbb{L}} L^U = \emptyset$. For the reverse direction, assume $G = (A', R')$ with $\mathcal{L}_{stb}(G) = \mathbb{L}$. By Proposition 1 we get $\mathcal{E}_{stb}(G) = \mathbb{L}^I = \mathbb{S}$. Compactness follows since $A' = E \cup (Args_{\mathbb{S}} \setminus E) = Args_{\mathbb{S}}$ for any $L \in \mathbb{L}$, so G realizes $\mathbb{S}$ compactly. $\square$

4. Conclusion and Related Work

Expressibility is one central issue for knowledge representation formalisms [2]. Dung AFs are well-studied under extension-based semantics [7, 3]. A characterization for labelling-based semantics has recently been developed for conflict-free and naive labellings [8]. In this paper we took the next step and studied realizability for stable labellings.

The central observation is that although stable labellings and stable extensions are closely related (Proposition 1), the labelling level imposes strictly more structure. The argument set of any witnessing AF is determined by the labelling-set itself, which, on the one hand, constrains the search space for witnessing frameworks compared to the extension-based setting. On the other hand, however, it introduces an *inherent non-determinism* as in general there is no canonical choice of which attacks to include and which not. The latter problem is strongly related to so-called *candidate sets*, i.e. maximal conflict-free sets which should not turn stable.

In the absence of candidate sets we showed that the classical extension-based conditions incomparability and tightness of $\mathbb{L}^I$ together with $\bigcup \mathbb{L}^U = \emptyset$ turn out to be sufficient as well as necessary. Furthermore the construction $F^{na}_{\mathbb{L}}$ provides a witness. As a further result, $\mathbb{C}_{\mathbb{L}} = \emptyset$ characterizes exactly those labelling-sets admitting a witnessing AF whose stable and naive labellings coincide.

In the presence of candidate sets we introduced so-called *critical candidates*. These are arguments associated with a certain candidate set which might serve as a witness for non-stability. As Example 9 shows it is not possible to simply leave all of them unattacked. In some situations a decision must be made, which is the source of the non-determinism. We established $F^{na}_{\mathbb{L}}$ as an over-approximation for a witnessing AF. Any witnessing AF can be transformed, via two semantically neutral operations - adding self-loops to purely out-arguments and deleting their outgoing attacks - into a subgraph of $F^{na}_{\mathbb{L}}$ with identical stable labellings. The general realizability problem thus reduces to the question of which attacks from each candidate towards their critical candidates in $F^{na}_{\mathbb{L}}$ must be removed to obtain a valid witnessing AF. We leave the exact characterization of this for future work. It will further be interesting to examine to which extent the results carry over to other semantics such as stage semantics [11].

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Declaration on Generative AI

During the preparation of this work, the author(s) used GPT-5.3 in order to: Grammar and spelling check. After using these tool(s)/service(s), the author(s) reviewed and edited the content as needed and take(s) full responsibility for the publication's content.

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