Understanding Zhang's Partial Expansion

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Abstract

Partial expansion operators were presented in Zhang (2018) as intermediate operators that, when combined with choice contraction, lead to choice revision operators. We propose analyzing partial expansion operators in their own right, generalizing the original proposal, and studying different alternatives for this family. In doing so, we also point out a flaw in the original definition where some postulates are incompatible, and propose how to amend it by considering new postulates that maintain the original purpose of these operators.

In belief change theory, multiple change operators are presented by Fuhrmann and Hansson in a series of works [1, 2, 3]. There, the authors propose a generalization of classical AGM contraction and revision operators, where the new knowledge is represented by a set of formulae M instead of a simple formula. They distinguished two different objectives when dealing with a belief base K and a set of formulas M: the *package* case, where the whole M is involved, and the *choice* case, where a non-empty subset of M is considered for the change. However, no distinction was made for the expansion operator, which inherently depicts a package behavior: the new knowledge includes all the formulas of M. By doing so, the expansion operator retains its traditional role as a deterministic operation corresponding to closure under logical consequence, Cn. From our perspective, this assigns a rather limited role to expansion within belief change theory. In this paper, we argue that expansion should be placed on the same footing as contraction and revision by allowing it to be nondeterministic and thus developing a corresponding theory of choice expansion.

In [4], Zhang conceived a choice case operator for expansion, called *partial expansion*, but only as an ad-hoc operator to represent choice revision. In this work, we aim to show that partial expansion operators have a value of their own as a generalization of expansion that covers a more cautious expansion phenomenon. We also believe a better understanding of its behavior leads us to a better comprehension of how it shapes choice revision.

As in revision, a typical situation for partial expansion is when an inconsistency between the knowledge and the set of information arises. But in this approach, partial expansion allows us to capture a subset of the information compatible with the original knowledge. Another interesting scenario is one of internally inconsistent new information, where, nonetheless, the agent may identify and accept a consistent subset with its knowledge. Additionally, if this is not the case, the non-prioritized behavior of the operator preserves the consistency of the original knowledge. In any case, partial expansion models an active selection process by taking a specific subset of valid options, as in the following example.

Example 1. *It is common knowledge that, in some towns with only two football teams, namely Team P and Team Q, you have to support one, but you cannot support both. Thus, every agent in this town has to decide between "supporting team P" (p) or "supporting team Q" (q), knowing that $p \to \neg q$. Formally, $\{p \to \neg q\}$ is the knowledge of an agent of this town regarding this topic, while the new information is $\{p, q\}$. Therefore, the expected change is to expand her knowledge by adding either p or q.*

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This selection is part of a belief change behavior, where the new information proposes which formulas are available to be added, and the agent decides which of them will belong to the new knowledge. This scenario lies outside classical frameworks such as revision, contraction, or expansion, but is covered by partial expansion.

This article is organized as follows. In Section 1, we provide some formal preliminaries. In Section 2, we present the family of partial expansion operators based on the one given in [4]. While providing axiomatic definitions, constructions, and representation theorems, we compare our proposal with Zhang's, analyzing incompatibilities between postulates appearing in the original work. This also allows us to develop a family tree of different kinds of partial expansion operators. In Section 3, we explore the relations between partial expansion and external package revision. In Section 4, we analyze a non-monotonic frame based on partial expansion and the connection made with external package revision. Finally, in Section 5, we summarize our results and present directions for future research.

1. Preliminaries

We start by providing definitions and remarks needed to formally construct partial expansions. Although many of these are adaptations from [4], the changes we considered aim to develop a thorough analysis of these operators. Our framework is classical propositional logic (CPL), i.e., a supraclassical, compact, and Tarskian operator Cn over a propositional language $\mathcal{L}$ defined from a set of countable propositional variables $\{p_0, p_1, \dots, p_n, \dots\}$, truth-functional operations $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, constant $\top$ for tautologies, $\bot$ for contradiction, and lower-case Greek letters for formulas. We work in a belief base context, where K, $A \in \mathcal{P}(\mathcal{L})$ represent a knowledge base and a set of observations, respectively. In some cases, the sets are assumed to be finite, which we represent by belonging to $\mathcal{P}_f(\mathcal{L})$.

We start by considering the set of minimal partial sum sets, a construction based on AGM remainders.

Definition 2. The function $\bowtie\colon \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{L}))$ is called minimal partial sum if every $X \in K \bowtie A$ satisfy:

($\bowtie$ 1) $X \subseteq K \cup A$;

($\bowtie$ 2) $K \subseteq X$;

($\bowtie$ 3) $X \cap A \neq \emptyset$.

($\bowtie$ 4) if $K \subseteq X' \subsetneq X$ then $X' \cap (A \setminus K) = \emptyset$.

If it satisfies only **($\bowtie$ 1)**~**($\bowtie$ 3)**, we call it partial sum.

We add the **($\bowtie$ 4)** as a minimality condition to the definition given by Zhang of partial sum, not only to avoid redundancies while selecting sets from $K \bowtie A$, but also to provide a simpler characterization of which sets are available in $K \bowtie A$.

$$K \bowtie A = \{K \cup \{a\} \mid a \in A\} \tag{1}$$

In particular, we have that $K \bowtie A = \emptyset$ iff $A = \emptyset$, and $K \in K \bowtie A$ iff $K \cap A \neq \emptyset$. Thus, we have:

$$\{K\} \cup (K \bowtie A) = \{K\} \cup (K \bowtie B) \text{ iff } A \setminus K = B \setminus K \tag{2}$$

The following lemma formalizes the idea that minimal partial sum sets are enough to represent all the partial sum sets.

Lemma 3. *Let $\bowtie_m$ be the minimal partial sum set, i.e., the function satisfying **($\bowtie$ 1)**~**($\bowtie$ 4)**, and $\bowtie$ be the partial sum set, i.e., the one satisfying **($\bowtie$ 1)**~**($\bowtie$ 3)**. If $X \in K \bowtie A$, then $X = \bigcup S_X$ where $S_X = \{Y \in K \bowtie_m A \,|\, Y \subseteq X\}$.*

Continuing the adaptation of AGM and Zhang's constructions, we consider a weaker version of the two-place selection function defined by Hansson [2].

Definition 4. We call $\gamma : \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{P}(\mathcal{L})) \to \mathcal{P}(\mathcal{P}(\mathcal{L}))$ a general selection function if given $K \subseteq \mathcal{L}$, then either $\emptyset \neq \gamma(K, \mathcal{Y}) \subseteq \mathcal{Y}$ or $\gamma(K, \mathcal{Y}) = \{K\}$. Also, we say γ is:

non-empty if $\mathcal{Y} \neq \emptyset$ implies $\gamma(K, \mathcal{Y}) \subseteq \mathcal{Y}$.

$\bigcup$-consistency-preserving if $\bigcup \gamma(K, \mathcal{Y}) \not\models \bot$ whenever there is a set $S \in \mathcal{Y}$ such that $S \not\models \bot$.

Non-empty functions are the ones considered by Hansson in [2], while $\bigcup$-consistency-preserving functions are Zhang's proposal for consistent partial expansions in [4][1]. In the next section, we analyze the postulates of Zhang's partial expansion representation theorem and, motivated by this analysis, we propose alternatives to his proposal.

2. Partial Expansion

In this section, we formally define partial expansion operators based on minimal partial sum sets and the general selection function. But our proposal differs from Zhang's in terms of postulates. The following postulates are the ones considered in [4] for a representation theorem.

Definition 5. Given $\dot{+} : \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ a belief base operator, and $K, A \subseteq \mathcal{L}$, consider the following postulates:

($\dot{+}$1) $K \dot{+} A \subseteq K \cup A$ ($\dot{+}$ inclusion);

($\dot{+}$2) $K \subseteq K \dot{+} A$ ($\dot{+}$ preservation);

($\dot{+}$3) if $K \cap A \neq \emptyset$ and $A \subseteq B \subseteq K \cup A$,
then $K \dot{+} A = K \dot{+} B$ ($\dot{+}$ coincidence);

($\dot{+}$4) if $A \neq \emptyset$, then $A \cap (K \dot{+} A) \neq \emptyset$ ($\dot{+}$ success);

($\dot{+}$5) if there is an $X \subseteq \mathcal{L}$ such that $K \subseteq X \subseteq K \cup A$ and $X \not\models \bot$, then $K \dot{+} A \not\models \bot$ ($\dot{+}$ consistency);

In [4], the author affirms that an operator satisfying postulates ($\dot{+}$1)∼($\dot{+}$5) is equivalent to an operator constructively defined by minimal partial sum (due to Lemma 3) and a non-empty, $\bigcup$-consistency-preserving selection function:

$$K \dot{+} A = \bigcup \gamma(K, K \bowtie A)$$

However, postulates ($\dot{+}$2), ($\dot{+}$4) and ($\dot{+}$5) are incompatible, as the following example shows:

Example 6. *Consider $K = \{a\}$ and $A = \{\neg a\}$. Postulates ($\dot{+}$2) and ($\dot{+}$4) force $K \dot{+} A = \{a, \neg a\}$ an inconsistent set, contradicting ($\dot{+}$5). Meanwhile, ($\dot{+}$4) and ($\dot{+}$5) imply $K \dot{+} A = \{\neg a\} = A$, thus not an expansion of K. Lastly, ($\dot{+}$2) and ($\dot{+}$5) push towards $K \dot{+} A = \{a\} = K$ failing ($\dot{+}$4). If we consider ($\dot{+}$2) a core postulate for partial expansion, then postulates ($\dot{+}$4) or ($\dot{+}$5) should be adapted.*

Note that the extra properties given to the general selection function, namely non-empty and $\bigcup$-consistency-preserving, are compatible in this case: the set of minimal partial sum is $K \bowtie A = \{\{a, \neg a\}\}$. Since $\bigcup$-consistency-preserving is trivially satisfied, the construction prioritizes success over consistency, i.e., $K \dot{+} A = \{a, \neg a\}$. Thus, it is acceptable to assume the incompatibility only arises on the postulate side.

Having this example as reference, we decided to start with a general version of partial expansion where neither ($\dot{+}$4) nor ($\dot{+}$5) necessarily holds.

[1] Zhang's functions are only defined in terms of partial sum sets, so we generalize them to any given set of sets.

Definition 7. An operator $\dot{+} : \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a non-prioritized (NP) partial expansion if there is a general selection function γ such that for every $K, A \subseteq \mathcal{L}$:

$$K \dot{+} A = \bigcup \gamma(K, K \bowtie A)$$

Moreover, from [4], we call $\dot{+}$:

- partial expansion if γ is non-empty;
- consistency-preserved (CP) partial expansion if γ is also $\bigcup$-consistency-preserving.

The following representation theorem is the reason why we started with a weaker version of the selection function γ, where it is possible to select $\{K\}$ in non-empty cases.

Theorem 8. *An operator $\dot{+}$ is a non-prioritised (NP) partial expansion if and only if it satisfies ($\dot{+}$1)~($\dot{+}$3).*

Proof. **From Construction to Postulates:**

($\dot{+}$1) Trivial, since given $X \in \gamma(K, K \bowtie A)$, $X \subseteq K \cup A$.

($\dot{+}$2) Trivial by construction.

($\dot{+}$3) Assume $K \cap A \neq \emptyset$ and $A \subseteq B \subseteq A \cup K$. Then, we have $A \setminus K = B \setminus K$. Therefore, by equivalence (2), $\{K\} \cup (K \bowtie A) = \{K\} \cup (K \bowtie B)$ Since $K \cap A \neq \emptyset$, then $K \in K \bowtie A$ and $K \in K \bowtie B$. We conclude $K \bowtie A = K \bowtie B$.

From Postulates to Construction: Given K, $A \subseteq \mathcal{L}$ and $\mathcal{Y} \subseteq \mathcal{P}(\mathcal{L})$, consider the subset of $\mathcal{Y}$ defined as $\mathcal{Y}_{K \dot{+} A} = \{X \in \mathcal{Y} | X \subseteq K \dot{+} A\}$. Given γ' an arbitrary general selection function (such as $\gamma'(K, \mathcal{Y}) = \mathcal{Y}$), define γ such that:

$$\gamma(K, \mathcal{Y}) = \begin{cases} \{K\} & \text{if } \exists A \,|\, \mathcal{Y} = K \bowtie A \text{ and } \mathcal{Y}_{K \dot{+} A} = \emptyset \\ \mathcal{Y}_{K \dot{+} A} & \text{if } \exists A \,|\, \mathcal{Y} = K \bowtie A \text{ and } \mathcal{Y}_{K \dot{+} A} \neq \emptyset \\ \gamma'(K, \mathcal{Y}) & \text{otherwise} \end{cases}$$

Note that γ is well-defined: if there are A, A' sets such that $\mathcal{Y} = K \bowtie A = K \bowtie A'$, by equivalence (2) we have $A \setminus K = A' \setminus K$. If $K \notin \mathcal{Y}$, then $A = A'$; and if $K \in \mathcal{Y}$, then $K \dot{+} A = K \dot{+} A'$ by ($\dot{+}$3). In any case, $\mathcal{Y}_{K \dot{+} A} = \mathcal{Y}_{K \dot{+} A'}$.

Since γ is a general selection function by construction, if we check $K \dot{+} A = \bigcup \gamma(K, K \bowtie A)$, we finish the proof. By definition of γ, and in particular $\mathcal{Y}_{K \dot{+} A}$, we know $\bigcup \gamma(K, K \bowtie A) \subseteq K \dot{+} A$. Note that if $A \cap (K \dot{+} A) = \emptyset$, then $K \dot{+} A = K$ and $K \cap A = \emptyset$ by ($\dot{+}$1) and ($\dot{+}$2). In particular, by taking $\mathcal{Y} = K \bowtie A$, we have $\mathcal{Y}_{K \dot{+} A} = (K \bowtie A)_{K \dot{+} A} = \emptyset$. Therefore $\bigcup \gamma(K, K \bowtie A) = K = K \dot{+} A$. If not, take an arbitrary $\alpha \in A \cap (K \dot{+} A)$. Then $K \cup \{\alpha\} \subseteq K \dot{+} A$ by ($\dot{+}$2). Since $K \cup \{\alpha\} \in K \bowtie A$, then $K \cup \{\alpha\} \in (K \bowtie A)_{K \dot{+} A}$, hence $K \cup \{\alpha\} \subseteq \bigcup \gamma(K, K \bowtie A)$ by definition of γ. We deduce $K \cup (A \cap (K \dot{+} A)) \subseteq \bigcup \gamma(K, K \bowtie A)$. But by ($\dot{+}$1) we have $K \cup (A \cap (K \dot{+} A)) = K \dot{+} A$. Therefore, we conclude $K \dot{+} A \subseteq \bigcup \gamma(K, K \bowtie A)$, and the double inclusion holds. □

This representation theorem depicts postulates ($\dot{+}$1)~($\dot{+}$3) as structural for NP partial expansion, allowing us to analyze subfamilies based on this general setup. Moreover, we aim to develop a one-to-one correspondence between other postulates for $\dot{+}$ and properties of γ. Let us start by connecting the properties from Definition 4 with either known or new postulates.

Theorem 9. *Let $\dot{+}$ be an NP partial expansion constructed by a general selection function γ. Then:*

1. *γ is non-empty iff $\dot{+}$ satisfies ($\dot{+}$4);*
2. *γ is $\bigcup$-consistency-preserving iff $\dot{+}$ satisfies:*

   ($\dot{+}$6) *if there is an $X \subseteq \mathcal{L}$ such that $K \subseteq X \subseteq K \cup A$, $X \cap A \neq \emptyset$ and $X \not\models \bot$, then $K \dot{+} A \not\models \bot$ ($\dot{+}$ weak consistency);*

Proof.

1. Assume γ is non-empty and take $A \neq \emptyset$. This means $\emptyset \neq \gamma(K, K \bowtie A) \subseteq K \bowtie A$ since $K \bowtie A \neq \emptyset$ iff $A \neq \emptyset$ by (1). Hence, there is $X \in \gamma(K, K \bowtie A)$ such that $X \cap A \neq \emptyset$. By construction, $X \subseteq K \dot{+} A$, therefore $A \cap (K \dot{+} A) \neq \emptyset$. Thus $\dot{+}$ satisfies ($\dot{+}$4).
   Consider now $\dot{+}$ satisfies ($\dot{+}$4). Take the construction of γ in Theorem 8. By considering γ' a non-empty general selection function (such as $\gamma'(K, \mathcal{Y}) = \mathcal{Y}$ when $\mathcal{Y} \neq \emptyset$), we only need to check that if $K \bowtie A \neq \emptyset$ then $\emptyset \neq \gamma(K, K \bowtie A) \subseteq K \bowtie A$. Recall from (1), $K \bowtie A \neq \emptyset$ iff $A \neq \emptyset$. Hence, if $K \bowtie A \neq \emptyset$, then $A \cap (K \dot{+} A) \neq \emptyset$ by ($\dot{+}$4). Therefore, by the construction of $\dot{+}$, we deduce $\gamma(K, K \bowtie A) \subseteq K \bowtie A$ by (1).
2. Consider γ $\bigcup$-consistency-preserving and suppose there is $X \subseteq \mathcal{L}$ such that $K \subseteq X \subseteq (K \cup A)$, $X \cap A \neq \emptyset$ and $X \not\models \bot$. In particular, there is some $\alpha \in X \cap A$ such that $K \cup \{\alpha\} \not\models \bot$. By $\bigcup$-consistency-preserving, with $S = K \cup \{\alpha\}$, we deduce $\bigcup \gamma(K, K \bowtie A) \not\models \bot$. Thus $K \dot{+} A \not\models \bot$ by construction.
   Assume now $\dot{+}$ satisfies ($\dot{+}$6) and consider the construction of γ in Theorem 8. Take γ' satisfying $\bigcup$-consistency-preserving (such as the one that selects the consistent set S given by definition). Thus, now we can only focus on the first two cases. We prove it by contrapositive. Consider $\bigcup \gamma(K, K \bowtie A) \models \bot$. Then, by construction, $K \dot{+} A \models \bot$. Hence, by ($\dot{+}$6), it means that if $K \subseteq X \subseteq K \cup A$, then $X \models \bot$. In particular, we deduce that every element of $K \bowtie A$ is inconsistent. We conclude γ is $\bigcup$-consistency-preserved by contrarreciprocal.

□

By adding $X \cap A \neq \emptyset$ to the premise of ($\dot{+}$5), postulate ($\dot{+}$6) allows us to have a weaker version of consistency compatible with success. Note that forcing X to have at least one element of A (the $X \cap A \neq \emptyset$) differs from excluding K as an option for X (using $K \subsetneq X$), since in the latter case we would be excluding it even for the case where $K \cap A \neq \emptyset$. By considering $X \cap A \neq \emptyset$, we recover Zhang's representation theorem for partial expansion.

Corollary 10. *Let $\dot{+}$ be a non-prioritized (NP) partial expansion, i.e., it satisfies ($\dot{+}$1)~($\dot{+}$3). Then:*

1. *$\dot{+}$ is a partial expansion iff it also satisfies ($\dot{+}$4);*
2. *$\dot{+}$ is a Consistency-preserved (CP) partial expansion iff it also satisfies ($\dot{+}$4), ($\dot{+}$6).*

After Theorem 9, two questions remain regarding postulate ($\dot{+}$5). Specifically:

- Which property of γ relates to postulate ($\dot{+}$5)?
- Is there a weaker version of ($\dot{+}$4) compatible with ($\dot{+}$5)? If so, which property of γ is associated?

To answer them, first note that any operator satisfying ($\dot{+}$5) equivalently satisfies classical consistency:

($\dot{+}$5') If $K \not\models \bot$, then $K \dot{+} A \not\models \bot$.

Therefore, from now on, we are using ($\dot{+}$5') for this analysis.

From postulate ($\dot{+}$6), we can interpret that its premise is a critical case where both success, represented by $A \cap (K \dot{+} A) \neq \emptyset$, and consistency, represented by $K \dot{+} A \not\models \bot$, may collide. Following this idea, we suggest a weaker version of success compatible with ($\dot{+}$5'):

($\dot{+}$7) If there is an $X \subseteq \mathcal{L}$ such that $K \subseteq X \subseteq K \cup A$, $X \cap A \neq \emptyset$ and $X \not\models \bot$, then $A \cap (K \dot{+} A) \neq \emptyset$ ($\dot{+}$ consistent success).

The following theorem connects postulates ($\dot{+}$5') and ($\dot{+}$7) with new properties for the general selection function.

Theorem 11. *Let $\dot{+}$ be an NP partial expansion constructed by a selection function γ. Then:*

1. *$\dot{+}$ satisfies ($\dot{+}$5') iff γ is **$\bigcup$-consistent**: if $K \not\models \bot$ then $\bigcup \gamma(K, \mathcal{Y}) \not\models \bot$.*
2. *$\dot{+}$ satisfies ($\dot{+}$7) iff γ is **success-preserving**: if there is a set $S \in \mathcal{Y}$, $S \not\models \bot$, then $\gamma(K, \mathcal{Y}) \subseteq \mathcal{Y}$.*

Proof.

1. Consider $K \not\models \bot$. If γ is $\bigcup$-consistent, then $\dot{+}$ directly satisfies **($\dot{+}$5')** by construction. If $\dot{+}$ satisfies **($\dot{+}$5')**, then γ is $\bigcup$-consistent by considering $\gamma'(K, \mathcal{Y}) = \{K\}$.
2. Consider γ consistent-success and suppose there is $X \subseteq \mathcal{L}$ such that $K \subseteq X \subseteq (K \cup A)$, $X \cap A \neq \emptyset$ and $X \not\models \bot$. In particular, there is some $\alpha \in X \cap A$ and $K \cup \{\alpha\} \subseteq X$. Thus, $K \cup \{\alpha\} \in K \bowtie A$ and $K \cup \{\alpha\} \not\models \bot$, therefore $\gamma(K, K \bowtie A) \subseteq K \bowtie A$ by hypothesis. Since $\gamma(K, K \bowtie A) \neq \emptyset$, we conclude $\emptyset \neq A \cap (\bigcup \gamma(K, K \bowtie A))$, or equivalently, $A \cap (K \dot{+} A) \neq \emptyset$. Assume now $\dot{+}$ satisfies **($\dot{+}$8)** and consider the construction of γ in Theorem 8. Take γ' satisfying $\bigcup$-consistent-success (such as the one that selects the consistent set S given by definition). Thus, we can now focus only on the first two cases. We prove it by contrapositive. Suppose $\gamma(K, K \bowtie A) \not\subseteq K \bowtie A$. Then, by construction, $K \dot{+} A = K$ and $K \cap A = \emptyset$. Hence, $A \cap (K \dot{+} A) = \emptyset$, and by **($\dot{+}$6)**, if $K \subseteq X \subseteq K \cup A$ then either $X \cap A = \emptyset$ or $X \models \bot$. In particular, we deduce that every element of $K \bowtie A$ is inconsistent. We conclude γ is consistent-success by contrarreciprocal.

□

Note $\bigcup$-consistent guarantees consistency of the result when the original knowledge is consistent, while success-preserving ensures success only when there is at least a consistent option. It is directly deduced that **($\dot{+}$5')** implies **($\dot{+}$6)**, and **($\dot{+}$4)** implies **($\dot{+}$7)**.

Lastly, we analyze some properties for the general selection functions related to the classical base expansion operator and other postulates considered in [5].

Definition 12. We say a general selection function γ, where the first parameter is noted as $K \subseteq \mathcal{L}$ and the second parameter as $\mathcal{Y} \subseteq \mathcal{P}(\mathcal{L})$, is:

non-empty on consistency if $\mathcal{Y} \neq \emptyset$ and $\bigcup \mathcal{Y} \setminus K \not\models \bot$ implies $\gamma(K, \mathcal{Y}) \subseteq \mathcal{Y}$;

vacuity-full if $\emptyset \neq \mathcal{Y}$ and either $K \models \bot$ or $K \cup \bigcup \mathcal{Y} \not\models \bot$ implies $\gamma(K, \mathcal{Y}) = \mathcal{Y}$;

optimal if $\bigcup \gamma(K, \mathcal{Y}) \not\models \bot$ implies $\gamma(K, \mathcal{Y}) \in \mathcal{M} = \max\{S \subseteq \mathcal{Y} \mid S \neq \emptyset, \bigcup S \not\models \bot\}$;

choice-full ([4]) if $\mathcal{Y} \neq \emptyset$ implies $\gamma(K, \mathcal{Y}) = \mathcal{Y}$.

Choice-full property links NP partial expansion directly to expansion. Meanwhile, vacuity-full only forces expansion behaviour when $K \cup A \not\models \bot$ or $K \models \bot$, while optimal aims to maximise the resulting set of consistent formulas. Lastly, non-empty on consistency, a weaker version of non-empty, forces the selection of partial sum sets when A is both non-empty and consistent. Let us formalise this axiomatically.

Theorem 13. *Let $\dot{+}$ be an NP partial expansion constructed by a general selection function γ. Then:*

1. *(also in [5]) γ is non-empty on consistency iff $\dot{+}$ satisfies:*

   ($\dot{+}$8) *If $A \neq \emptyset$ and $A \not\models \bot$, then $A \cap (K \dot{+} A) \neq \emptyset$ ($\dot{+}$ weak success);*
2. *γ is vacuity-full iff $\dot{+}$ satisfies:*

   ($\dot{+}$9) *If $K \cup A \not\models \bot$ or $K \models \bot$ then $A \subseteq K \dot{+} A$ ($\dot{+}$ vacuity[2]);*
3. *γ is optimal iff $\dot{+}$ satisfies:*

   ($\dot{+}$10) *If $K \dot{+} A \subsetneq X \subseteq K \cup A$, then $X \models \bot$ ($\dot{+}$ optimal expansion);*
4. *γ is choice-full iff $\dot{+}$ satisfies:*

   ($\dot{+}$11) *$A \subseteq K \dot{+} A$ ($\dot{+}$ expansion inclusion).*

[2]A similar postulate called maximal expansion is considered in [5], which characterizes maximal partial expansion operators.

Proof.

1. Suppose γ is non-empty on consistency, and that $A \cap (K \dot{+} A) = \emptyset$. By construction, this means $\gamma(K, K \bowtie A) = \{K\}$ and that $K \cap A = \emptyset$. Then, either $A = \emptyset$ or $(\bigcup K \bowtie A) \setminus K \models \bot$. Since $K \cap A = \emptyset$, then either $A = \emptyset$ or $A \models \bot$. Therefore, $\dot{+}$ satisfies **($\dot{+}$8)**.
   If $\dot{+}$ satisfies **($\dot{+}$8)**, consider the construction of γ in Theorem 8. Take γ' satisfying non-empty on consistency (such as the one that selects one element of $\mathcal{Y}$ when $\mathcal{Y} \neq \emptyset$ and $\bigcup \mathcal{Y} \setminus K \not\models \bot$). Thus, we can now focus only on the first two cases. We prove it by contrapositive. Suppose $\gamma(K, K \bowtie A) \not\subseteq K \bowtie A$. Then, by construction, $K \dot{+} A = K$ and $K \cap A = \emptyset$. By **($\dot{+}$8)**, we deduce that either $A = \emptyset$ or $A \models \bot$. Since $A = (\bigcup K \bowtie A) \setminus K$ in case $A \neq \emptyset$, we have that either $K \bowtie A = \emptyset$ or $(\bigcup K \bowtie A) \setminus K \models \bot$.
2. Assume γ is vacuity-full, and that $K \cup A \not\models \bot$. Then $\bigcup K \bowtie A = K \cup A \not\models \bot$. Hence, either $K \bowtie A = \emptyset$, i.e. $A = \emptyset$, and **($\dot{+}$9)** is trivially satisfied, or $K \bowtie A \neq \emptyset$, and $\gamma(K, K \bowtie A) = K \bowtie A$ by hypothesis. In such a case, $A \subseteq \bigcup \gamma(K, K \bowtie A)$. Therefore $A \subseteq K \dot{+} A$, i.e. $\dot{+}$ satisfies **($\dot{+}$9)**.
   If $\dot{+}$ satisfies **($\dot{+}$9)**, suppose $\bigcup K \bowtie A = K \cup A \not\models \bot$. Then, in such a case, $A \subseteq K \dot{+} A$ by hypothesis. This means $A \subseteq \bigcup \gamma(K, K \bowtie A)$. If $A \neq \emptyset$, then $K \bowtie A \neq \emptyset$, and we can deduce $\gamma(K, K \bowtie A) = K \bowtie A$ by (1. If $A = \emptyset$, then $K \bowtie A = \emptyset$. In any case, γ is vacuity-full when using $K \bowtie A$ in the second parameter. The third case trivially satisfies vacuity-full if $\gamma'(K, \mathcal{Y}) = \mathcal{Y}$ is considered.
3. Assuming γ is optimal, consider $K \dot{+} A \subsetneq X \subseteq K \cup A$. If $K \dot{+} A \models \bot$, then $X \models \bot$. Otherwise, the strict inclusion implies $K \bowtie A \neq \emptyset$, and that there is some $a \in X \setminus (K \dot{+} A)$ such that $K \cup \{a\} \in K \bowtie A$ and $K \cup \{a\} \notin \gamma(K, K \bowtie A)$. Therefore, $K \cup \{a\} \cup \bigcup \gamma(K, K \bowtie A) \models \bot$ by γ being optimal. Since $K \cup \{a\} \cup \bigcup \gamma(K, K \bowtie A) \subseteq X$, we conclude $X \models \bot$.
   If $\dot{+}$ satisfies **($\dot{+}$10)**, consider the construction of γ in Theorem 8. Take γ' satisfying optimal (such as the one that satisfies $\gamma'(K, \mathcal{Y}) \in \mathcal{M}$ whenever $\mathcal{M} \neq \emptyset$, and $\gamma'(K, \mathcal{Y}) = \{K\}$ otherwise). Then, we can now focus only on the first two cases. If $K \dot{+} A = K$, then $X \models \bot$ for every $K \neq X \in K \bowtie A$ due to **($\dot{+}$10)**. Hence, either $\mathcal{M} = \emptyset$ (when $K \cap A = \emptyset$) or $\mathcal{M} = \{\{K \cup \{a\} \mid a \in A \cap K\}\}$ (when $K \cap A \neq \emptyset$). Therefore, either $\mathcal{M} = \emptyset$, or $\gamma(K, \mathcal{Y}) \in \mathcal{M}$ if $K \not\models \bot$ by construction, or $\gamma(K, \mathcal{Y}) \models \bot$. If $K \dot{+} A \neq K$, then $\emptyset \neq \gamma(K, \mathcal{Y}) \subseteq \mathcal{Y}$ by construction, and either $\gamma(K, \mathcal{Y}) \models \bot$ or $\gamma(K, \mathcal{Y}) \not\models \bot$. Assuming $\gamma(K, \mathcal{Y}) \not\models \bot$, there is some $S \in \mathcal{M}$ such that $\gamma(K, \mathcal{Y}) \subseteq S$. But if there is some $X \in S \setminus \gamma(K, \mathcal{Y})$, then $X \cup \bigcup \gamma(K, \mathcal{Y}) \models \bot$ by **($\dot{+}$10)**. This implies $\bigcup S \models \bot$, leading to a contradiction. Thus, we conclude $\gamma(K, \mathcal{Y}) \in \mathcal{M}$.
4. Consider γ choice-full. This means $\gamma(K, \mathcal{Y}) = \mathcal{Y}$ if $\mathcal{Y} \neq \emptyset$ and $\gamma(K, \emptyset) = \{K\}$. Therefore, $K \dot{+} A = \bigcup K \bowtie A = K \cup A$ if $A \neq \emptyset$ or $K \dot{+} A = K$ if $A = \emptyset$. In any case, $A \subseteq K \dot{+} A$, i.e. $\dot{+}$ satisfies **($\dot{+}$11)**.
   Suppose now $\dot{+}$ satisfies **($\dot{+}$11)**. Then $A \subseteq \bigcup \gamma(K, K \bowtie A)$. Hence, $\gamma(K, K \bowtie A) = K \bowtie A$ by (1). The third case trivially satisfies if $\gamma'(K, \mathcal{Y}) = \mathcal{Y}$ is considered. In any case, if $\mathcal{Y} \neq \emptyset$, then $\gamma(K, \mathcal{Y}) = \mathcal{Y}$, i.e. γ is choice-full.

□

Postulate **($\dot{+}$8)** characterises how to deal with inconsistent sets of information. Although it is a weaker version of postulate **($\dot{+}$4)**, the incompatibility exhibited in Example 6 persists. Postulates **($\dot{+}$9)** and **($\dot{+}$10)** characterise two ideas of minimal change. While **($\dot{+}$9)** forces the partial expansion to behave like an expansion in the extreme situations, **($\dot{+}$10)** implies that partial expansion should behave as closely as possible to an expansion whenever it is consistently possible. This postulate is therefore compatible with any other postulate presented here. Meanwhile, postulate **($\dot{+}$11)** expansion inclusion forces partial expansion to behave like expansion. In this case, all the postulates, except **($\dot{+}$5)** and **($\dot{+}$6)**, which are the ones related to consistency, are trivially satisfied.

The practical one-to-one correspondence between postulates and properties for the general selection function helps us understand and organise the different subfamilies based on NP partial expansion.

Definition 14. Let $\dot{+}$ be an NP partial expansion and γ its general selection function. We say $\dot{+}$ is:

- NP consistent partial expansion if γ is $\bigcup$-consistent;
- consistent partial expansion if γ is both $\bigcup$-consistent and success-preserving;
- weak partial expansion [5] if γ is non-empty on consistency;
- CP weak partial expansion [5] if γ is both non-empty on consistency and $\bigcup$ −consistency-preserving;
- optimal partial expansion if γ is both optimal and $\bigcup$-consistent;
- expansion if γ is choice-full.

We also add the prefix normal when γ is vacuity-full.

Example 15 (Continuation of Example 1)**.** *A tourist visits the town with her partner, expecting to attend the match between Teams P and Q, mainly for the supporters' experience. Even though she wants to remain neutral, she has to decide which side to be on during the match. Then, when buying the ticket, she decides on Team Q. Unfortunately, her partner, who rejects the idea of supporting either side, expected her to find 'neutral' tickets.*

Formally speaking, previous to the match, we have that $\{p \to \neg q\} \dot{+} \{p, q\} = \{p \to \neg q\}$ since she wanted to remain neutral. But when the moment to buy the ticket came $\{p \to \neg q\} \dot{+}_t \{p, q\} = \{p \to \neg q, q\}$. Lastly, her partner has another belief base: $K' = \{p \to \neg q, \neg p \wedge \neg q\}$, which is inconsistent with either selecting p or q, i.e. $K' \dot{+}' \{p, q\} = K'$. The first case is an example of NP partial expansion; the second case prioritizes success, thus a partial expansion; while the third prioritizes consistency, i.e., a consistent partial expansion. Also, you can see the last two as different outputs of a CP weak partial expansion, or even an optimal partial expansion.

The rest of the text is devoted to linking these families with other known structures.

3. Relation with External Package Revision

Package revision is defined in [1] as a generalisation of classical revision operators. A belief base adaptation was analysed in [2], where internal and external package revisions are axiomatically characterised. Due to belief base framework, operators typically take only finite subsets of $\mathcal{L}$. In terms of postulates, the key connection with partial expansion is given by pre-expansion, a characteristic postulate of external package revision operators.

Definition 16. Given $*_p : \mathcal{P}_f(\mathcal{L}) \times \mathcal{P}_f(\mathcal{L}) \to \mathcal{P}_f(\mathcal{L})$ a belief base operator, and $M, B \subseteq \mathcal{L}$, we say $*_p$ is an *external* package revision if it satisfies the following postulates:

(P∗1) $B \subseteq M *_p B$ ($*$ package success).

(P∗2) $M *_p B \subseteq M \cup B$ ($*$ inclusion).

(P∗3) If $B \not\models \perp$ then $M *_p B \not\models \perp$ ($*$ consistency).

(P∗4) If $\mu \in M \setminus (M *_p B)$, then there is some M' such that $M *_p B \subseteq M' \subseteq M \cup B$, $M' \not\models \perp$ and $M' \cup \{\mu\} \models \perp$ ($*$ package relevance).

(P∗5) If, $B, B' \subseteq M$ and for every $M' \subseteq M$, $M' \cup B \not\models \perp$ iff $M' \cup B' \not\models \perp$, then $M *_p B = M *_p B'$ ($*$ weak package uniformity).

(P∗6) $M *_p B = (M \cup B) *_p B$ ($*$ pre-expansion);

We say an external package revision is *maxichoice* if it also satisfies:

(P∗7) Either $\mu \in M *_p B$ or $M *_p B \models \neg\mu$ for every $\mu \in M$ ($*$ tenacity).

Interestingly, some of these postulates can be reinterpreted as partial expansion postulates by swapping the parameters of the operator, i.e., $K \dot{+} A = A *_p K$.

Lemma 17. *Let $\dotplus$, $*_p$ be belief base operators such that $K \dotplus A = A *_p K$ for every $K, A \in \mathcal{P}_f(\mathcal{L})$ then:*

1. *$*_p$ satisfies **(P$*$2)** iff $\dotplus$ satisfies **($\dotplus$1)**;*
2. *$*_p$ satisfies **(P$*$1)** iff $\dotplus$ satisfies **($\dotplus$2)**;*
3. *if $*_p$ satisfies **(P$*$6)** then $\dotplus$ satisfies **($\dotplus$3)**, due to **($\dotplus$3)** is equivalent to:*

   ($\dotplus$3') *If $K \cap A \neq \emptyset$ then $K \dotplus A = K \dotplus (K \cup A)$;*
4. *$*_p$ satisfies **(P$*$3)** iff $\dotplus$ satisfies **($\dotplus$5')**;*
5. *if $*_p$ satisfies **(P$*$1)** and **(P$*$4)** then $\dotplus$ satisfies **($\dotplus$9)**;*
6. *if $*_p$ satisfies **(P$*$1)**, then $*_p$ satisfies **(P$*$7)** iff $\dotplus$ satisfies **($\dotplus$10)**.*

Proof. By $K \dotplus A = A *_p K$, equivalencies **(P$*$3)** and **($\dotplus$5')**, **(P$*$2)** and **($\dotplus$1)**, **(P$*$1)** and **($\dotplus$2)**, **(P$*$5)** and **($\dotplus$13)** are trivially satisfied. Analogously, **(P$*$6)** implies **($\dotplus$3')**.

Assume now $*_p$ satisfies **(P$*$1)** and **(P$*$4)**, and consider $K \cup A \not\models \bot$ or $K \models \bot$. If $A \setminus (A *_p K) \neq \emptyset$, consider $a \in A \setminus (A *_p K)$, then by **(P$*$4)** there is some A' such that $A *_p K \subseteq A' \subseteq A \cup K$, $A' \not\models \bot$ and $A' \cup \{\alpha\} \models \bot$. But if $K \cup A \not\models \bot$, then in particular $A' \cup \{\alpha\} \not\models \bot$. Therefore $K \models \bot$. But due to **(P$*$1)**, $K \subseteq A$, which means $A' \models \bot$, leading to a contradiction. Thus, we conclude $A \subseteq A *_p K$. Hence $A \subseteq K \dotplus A$, i.e. $\dotplus$ satisfies **($\dotplus$9)**.

Let us check whether **($\dotplus$3)** and **($\dotplus$3')** are equivalent. Assume **($\dotplus$3)** holds. If $K \cap A \neq \emptyset$, take $A' = K \cup A$, then $K \dotplus A = K \dotplus (K \cup A)$. Thus **($\dotplus$3')** holds. Consider now $K \cap A \neq \emptyset$, $A \subseteq A' \subseteq K \cup A$, and suppose $\dotplus$ satisfies **($\dotplus$3')**. Then $K \cap A' \neq \emptyset$, and $K \cup A = K \cup A'$. Therefore, $K \dotplus A = K \dotplus (K \cup A) = K \dotplus (K \cup A') = K \dotplus A'$.

Lastly, suppose $\dotplus$ satisfies **($\dotplus$10)**. If $A \subseteq A *_p K$, then **(P$*$7)** is trivially satisfied. Otherwise, consider $\alpha \in A \setminus (A *_p K)$. Then, we have $X = (A *_p K) \cup \{\alpha\} \models \bot$. Equivalently, $(A *_p K) \models \neg\alpha$, which means $*_p$ satisfies **(P$*$7)**. If $*_p$ satisfies **(P$*$7)** and $K \dotplus A \subsetneq X \subseteq K \cup A$, then there is $a \in X \cap A \setminus (K \dotplus A)$ due to **(P$*$1)**. Therefore, $(K \dotplus A) \cup \{\alpha\} \models \bot$ by **(P$*$7)**. Since $(K \dotplus A) \cup \{\alpha\} \subseteq X$, this means $X \models \bot$. Hence, $\dotplus$ satisfies **($\dotplus$10)**. □

Thus, external package revision operators can be seen as NP consistent normal partial expansion operators but with their parameters swapped.

Corollary 18. *Let $*_p$ and $\dotplus$ be a belief base operators, where $K \dotplus A = A *_p K$ for every $K, A \in \mathcal{P}_f(\mathcal{L})$. Then, $*_p$ is an external package revision iff $\dotplus$ is a NP consistent normal partial expansion that also satisfies:*

($\dotplus$12) *If $\alpha \in A \setminus (K \dotplus A)$, then there is some A' such that $K \dotplus A \subseteq A' \subseteq K \cup A$, $A' \not\models \bot$ and $A' \cup \{\alpha\} \models \bot$ ($\dotplus$ relevance).*

($\dotplus$13) *If $K, K' \subseteq A$ and for every $A' \subseteq A$, $A' \cup K \not\models \bot$ iff $A' \cup K' \not\models \bot$, then $K \dotplus A = K' \dotplus A$ ($\dotplus$ weak uniformity);*

($\dotplus$14) *$K \dotplus A = K \dotplus (K \cup A)$ ($\dotplus$ strong coincidence);*

The postulates presented in Corollary 18 are the partial expansion versions of **(P$*$4)**, **(P$*$5)**, and **(P$*$6)**, where the parameters are swapped w.r.t. the ones for external package revision. Due to its simple relation, there is a one-to-one correspondence between external package revision and this subfamily of partial expansion operators. In the next section, we analyse how this connection is interpreted when the Ramsey Test is applied.

4. Non-monotonic Reasoning and Partial Expansion

In the previous section, we established a one-to-one correspondence between external package revision and a subfamily of partial expansion operators obtained by swapping the roles of the parameters. From a technical point of view, this means that both constructions are equivalent up to this reparameterization. However, this leads to a distinct conceptual reading, since it exchanges the roles of the knowledge base and the set of observations. In light of this, we now consider the Ramsey Test, the standard bridge between belief revision and non-monotonic consequence, from the perspective of partial expansion.

Definition 19. Given $\dot{+}$ an NP partial expansion operator and a set of observations A, we define the following non-monotonic entailment $K \mid\!\approx \alpha$ if and only if $K \dot{+} A \models \alpha$. Equivalently, we define the consequence operator $C_{\dot{+}} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ such that $C_{\dot{+}}(K) = Cn(K \dot{+} A)$.

In the Ramsey Test based on the external package revision, i.e., $M \mid\!\approx \alpha$ if and only if $B *_p M \models \alpha$, the implicit background is the knowledge base B. This is typically interpreted as: "given background knowledge B, if M then non-monotonically α". However, in the partial expansion version, swapping roles means that the implicit background is now the set of observations, which can be read as: "given background observations A, the state K entails non-monotonically α".

By following the ideas of [6], we investigate which properties of non-monotonic reasoning are preserved in our new setting. The main difference between our approach and theirs lies in the type of multiple revisions that are employed. Our notion of non-monotonic inference is implicitly based on an external package revision, whereas their framework relies on an internal multiple revision of belief sets.

Definition 20 ([6, 7])**.** Given $C : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ we consider the following conditions:

(N$\dot{+}$1) $K \subseteq C(K)$ (inclusion);

(N$\dot{+}$2) If $K \subseteq K' \subseteq C(K)$ then $C(K') \subseteq C(K)$ (cut);

(N$\dot{+}$3) If $K \subseteq K' \subseteq C(K)$ then $C(K) \subseteq C(K')$ (cautious monotony);

(N$\dot{+}$4) $Cn(K) \subseteq C(K)$ (supraclassicality);

(N$\dot{+}$5) if $K \not\models \bot$ then $C(K) \neq \mathcal{L}$ (consistency preservation);

(N$\dot{+}$6) $Cn(C(K)) \subseteq C(K)$ (left absorption or closure);

(N$\dot{+}$7) $C(K \cup K') \subseteq Cn(K \cup C(K'))$ (infinite conditionalization);

(N$\dot{+}$8) If $C(K) \cup K' \not\models \bot$, then $C(K) \subseteq C(K \cup K')$ (rational monotony).

We say C is a cumulative consequence operator if it satisfies **(N$\dot{+}$1)**~**(N$\dot{+}$3)**, and we say it is a (consistent) rational consequence operator if it satisfies **(N$\dot{+}$4)**~**(N$\dot{+}$8)**. From [6], we know every rational consequence operator is cumulative.

Since the only postulates associated solely with the first parameter are **($\dot{+}$2)** and **($\dot{+}$5')**, only these can be directly translated into conditions for $C_{\dot{+}}$, while the rest involve both parameters and do not directly constrain the induced consequence operator. Thus, we need to consider extra postulates focusing on the first parameter and also in interaction with the Cn operator.

Definition 21. Given $\dot{+}$ an NP partial expansion operator, we consider the following postulates:

($\dot{+}$15) $A \cap (K \dot{+} A) = A \cap Cn(K \dot{+} A)$ ($\dot{+}$ relative closure);

($\dot{+}$16) If $K \subseteq K' \subseteq Cn(K \dot{+} A)$ then $A \cap (K' \dot{+} A) \subseteq A \cap (K \dot{+} A)$ ($\dot{+}$ weak success reduction)

($\dot{+}$17) If $K \subseteq K' \subseteq Cn(K \dot{+} A)$ then $A \cap (K \dot{+} A) \subseteq A \cap (K' \dot{+} A)$ ($\dot{+}$ weak success propagation)

($\dot{+}$18) If $K \subseteq K'$ then $Cn(K' \dot{+} A) \subseteq Cn(K' \cup (K \dot{+} A))$ ($\dot{+}$ success reduction)

($\dot{+}$19) If $K \subseteq K'$ and $K' \cup (K \dot{+} A) \not\models \bot$ then $A \cap (K \dot{+} A) \subseteq A \cap (K' \dot{+} A)$ ($\dot{+}$ success propagation)

Postulate **($\dot{+}$15)** is the relative closure version for partial expansion, based on the one from [2]. It says that every element of A that is deduced by the expansion is already in the expansion. Postulates **($\dot{+}$16)**, **($\dot{+}$18)**, **($\dot{+}$17)** and **($\dot{+}$19)** are inspired in the ones appearing in [8] for non-consistent revision operators. On one side, **($\dot{+}$16)** and **($\dot{+}$18)** force states that entail more information to choose elements of A that were already considered for weaker states. In particular, if any weaker state fails to expand with A, then the state must fail too. On the other side, **($\dot{+}$19)** induces a state to preserve the selection made for weaker states as long as the partial expansion is consistent with the state. In particular, if any weaker state successfully expands with any element of A, then the state must expand too.

The next theorem connects the properties of $C_{\dot{+}}$ with the postulates considered.

Theorem 22. *If $\dot{+}$ is a NP partial expansion and $C_{\dot{+}}$ its associated consequence operator, then $C_{\dot{+}}$ satisfies supraclassicality (thus inclusion) and closure, and $C_{\dot{+}}$ satisfies* **(N$\dot{+}$5)** *if and only if $\dot{+}$ satisfies* **($\dot{+}$5')**. *Moreover, if $\dot{+}$ satisfies* **($\dot{+}$15)**, *then:*

1. *$C_{\dot{+}}$ satisfies* **(N$\dot{+}$2)** *if and only if $\dot{+}$ satisfies* **($\dot{+}$16)**
2. *$C_{\dot{+}}$ satisfies* **(N$\dot{+}$3)** *if and only if $\dot{+}$ satisfies* **($\dot{+}$17)**
3. *$C_{\dot{+}}$ satisfies* **(N$\dot{+}$7)** *if and only if $\dot{+}$ satisfies* **($\dot{+}$18)**
4. *$C_{\dot{+}}$ satisfies* **(N$\dot{+}$8)** *if and only if $\dot{+}$ satisfies* **($\dot{+}$19)**

Proof. Supraclassicality of $C_{\dot{+}}$ is given by definition, **($\dot{+}$2)**, and the inclusion property of Cn, while closure is directly deduced by definition and the idempotence property of Cn. Also, the double implication of **(N$\dot{+}$5)** and **($\dot{+}$5')** is directly deduced by the definition of $C_{\dot{+}}$. Continuing with the items:

1. $C_{\dot{+}}$ satisfies **(N$\dot{+}$2)** means that $Cn(K'\dot{+}A) \subseteq Cn(K\dot{+}A)$ whenever $K \subseteq K' \subseteq Cn(K\dot{+}A)$. Due to **($\dot{+}$15)**, this is equivalent to $A \cap (K'\dot{+}A) \subseteq A \cap (K\dot{+}A)$ whenever $K \subseteq K' \subseteq Cn(K\dot{+}A)$, i.e., $\dot{+}$ satisfying **($\dot{+}$16)**.
2. This proof is analogous to the previous one.
3. Note that $K' \cup Cn(K\dot{+}A) \models \psi$ iff $K' \cup (K\dot{+}A) \models \psi$ by property of $\models$. In terms of Cn, we have $Cn(K' \cup Cn(K\dot{+}A)) = Cn(K' \cup (K\dot{+}A))$. Hence, when **(N$\dot{+}$7)** holds, if $K \subseteq K'$ then:

$$Cn(K\dot{+}A) = C_{\dot{+}}(K') = C_{\dot{+}}(K \cup K') \subseteq Cn(K' \cup C(K)) = Cn(K' \cup (K\dot{+}A))$$

   Therefore, $\dot{+}$ satisfies **($\dot{+}$18)**. Analogously, given $K,\ K' \subseteq \mathcal{L}$, when **($\dot{+}$18)** holds we deduce $Cn((K \cup K')\dot{+}A) \subseteq Cn(K' \cup K \cup (K'\dot{+}A))$ since $K' \subseteq K \cup K'$. Recalling that $K' \subseteq K'\dot{+}A$ by **($\dot{+}$2)**, this means $C_{\dot{+}}(K \cup K') \subseteq Cn(K \cup C_{\dot{+}}(K'))$. Thus $C_{\dot{+}}$ satisfies **(N$\dot{+}$7)**.
4. Assume $\dot{+}$ satisfies **($\dot{+}$19)**, and $C_{\dot{+}}(K) \cup K' \not\models \bot$. In particular, $K \subseteq K \cup K'$ and $C_{\dot{+}}(K) \cup (K \cup K') \not\models \bot$ since $C_{\dot{+}}(K) \cup K = C_{\dot{+}}(K)$ by **($\dot{+}$2)**. Therefore, $A \cap (K\dot{+}A) \subseteq A \cap ((K \cup K')\dot{+}A)$ by hypothesis. Then, once again by adding $K \cup K'$ with a union and applying Cn both sides, we conclude $C_{\dot{+}}(K) \subseteq C_{\dot{+}}(K \cup K')$.
   Consider now that **(N$\dot{+}$8)** holds, and take $K \subseteq K'$ such that $K' \cup (K\dot{+}A) \not\models \bot$. Then, $K' \cup C_{\dot{+}}(K) \not\models \bot$ by property of Cn. We deduce by hypothesis that $C_{\dot{+}}(K) \subseteq C_{\dot{+}}(K \cup K') = C_{\dot{+}}(K')$. Therefore, when intersecting with A and applying **($\dot{+}$15)**, we conclude $A \cap (K\dot{+}A) \subseteq A \cap (K'\dot{+}A)$. Which means $\dot{+}$ satisfies **($\dot{+}$19)**.

□

In this way, we have established a first connection between general patterns of non-monotonic reasoning and NP partial expansions. We leave a deeper investigation of this relationship for future work.

5. Conclusions and Future Work

In this work, we have studied partial expansion as an independent belief change operator, rather than as an auxiliary tool for representing choice revision. We provided an axiomatic framework for partial expansion, together with constructions and representation results, and showed how different families of operators arise depending on the interaction between their postulates (see Figure 1). This allowed us to clarify and further develop the proposal originally introduced by Zhang [4]. Thus, one natural continuation of this work is to study how the diversity of partial expansion families can help us to analyze choice revision operators. This also holds for the selective revision operators linked to choice revision [5].

We also analyzed the relation between partial expansion and external package revision, establishing a one-to-one correspondence between both frameworks via a reparameterization of their arguments. While this correspondence shows a technical equivalence between the two approaches, it also highlights a difference in their conceptual interpretation, namely, the role assigned to the belief base and to the information. Further work is needed to better understand the role of partial expansion in the broader landscape of multiple belief change, in particular its interaction with other choice-based operators.

The idea of swapping parameters could also be related to [8, 9], where operators are defined using two revision operators, one of which swaps the position of knowledge and observation. A detailed analysis of this connection is left for future work, since these constructions are formulated for belief sets, whereas the present work is developed in terms of (finite) belief bases.

Finally, we studied the non-monotonic consequence relation induced by partial expansion, using the Ramsey Test as a bridge between belief change and inference. This allowed us to present another perspective on partial expansion by introducing new postulates and linking them to the behavior of the associated consequence operator. According to [10], we believe that $\mid\!\not\approx$ has connections with Default-Assumption Consequence. Another natural direction for future work is to further investigate the rest of the postulates that involve comparing the outcomes of the operator under different base parameters and to analyze how these translate into properties relating the corresponding consequence operators.

Family Tree of Partial Expansion Operators

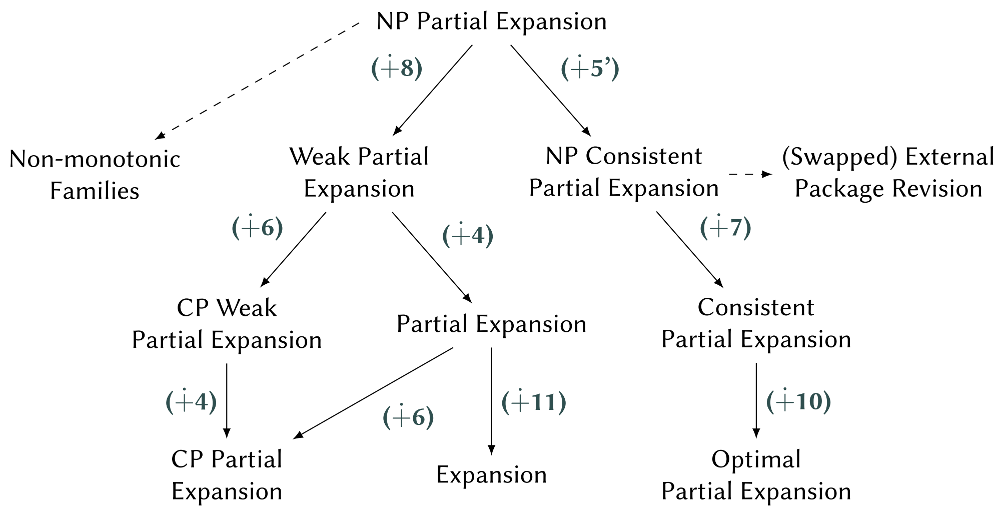

Figure 1: Family Tree of Partial Expansion Operators, based on the postulates they satisfy. Postulate $(\dotplus 9)$ can be added to any of these families.

Declaration on Generative AI

During the preparation of this work, the author(s) used ChatGPT, DeepSeek, and Gramarly in order to: improve writing style, paraphrase and reword, grammar and spelling check. After using these tools and services, the authors reviewed and edited the content as needed and take full responsibility for the publication's content.

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