A tableau for Many-Valued Multi-Modal Logic*

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Abstract

Melvin Fitting proposed in 1995 a tableau for many-valued modal logic, where the many-valued component was addressed leveraging Heyting algebras. However, the latter are not suitable to model common fuzzy logics different from Gödel logic, such as the widespread Łukasiewicz logic. We introduce a generalization of Fitting's many-valued modal logic based on the more general $\mathrm{FL_{ew}}$-algebras, encompassing both Heyting algebras and all popular fuzzy logics. Moreover, we propose a tableau system to address the satisfiability of formulae in the given framework, also allowing for the definition of more modalities, hence enabling for the treatment of common temporal and spatial modal logics, among others. The system has also been implemented in the Julia programming language as part of an open-source framework for symbolic learning and reasoning, namely Sole.jl.

Keywords

many-valued logic, multi-modal logic, satisfiability, automated theorem proving

1. Introduction

In the real world, information is often characterized by uncertainty and unclear boundaries.In the dimensionless case (i.e., at the pure propositional level), one classic approach to deal with these challenges is resorting to fuzzy logics, that come in a few well-known flavors. The most common examples include logics based on Gödel algebras (G) [1], MV-algebras (MV) [2], and product algebras (Π) [3]; in their typical formulation, the truth value domains are based on the interval $[0, 1]$ in $\mathbb{R}$, and are, therefore, chains. At the modal level, the most important contribution to date is probably Fitting's [4, 5, 6], who provided some interesting insights on how to combine modal logic with Heyting algebras (H) [7], aiming to model many-expert scenarios leveraging a partial order between truth degrees. Unfortunately, Heyting algebras are not suitable to cover other interesting many-valued approaches such as the aforementioned fuzzy logics, except for the Gödel case. In fact, to generalize both Fitting's and fuzzy logics approaches, one has to resort to a family of logics based on algebraic structures called $\mathrm{FL_{ew}}$-algebras.

Definition 1. *An algebraic structure*

$$\langle A, \cap, \cup, \cdot, \hookrightarrow, 0, 1\rangle,$$

where we define a binary relation $\preceq$ as $a \preceq b$ if and only if $a \cap b = a$, is called an $\mathrm{FL_{ew}}$-algebra *iff:* (i) $\langle A, \cap, \cup, 0, 1\rangle$ *is a bounded lattice with upper bound 1 and lower bound 0,* (ii) $\langle A, \cdot, 1\rangle$ *is a commutative*

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monoid, (iii) $\cdot$ *is monotone w.r.t.* $\preceq$*, i.e., if* $\gamma \preceq \alpha$*,* $\delta \preceq \beta$*, then* $\gamma \cdot \delta \preceq \alpha \cdot \beta$*, and* (iv) *for each* $a, b, c \in A$*,* $a \cdot b \preceq c$ *if and only if* $a \preceq b \hookrightarrow c$*.*

We refer, as it is customary, to $\cap$ as meet, $\cup$ as join, $\hookrightarrow$ as implication, and $\cdot$ as monoid. If the lattice order is complete (that is, each subset has infimum and supremum), we call the structure a complete $\mathrm{FL_{ew}}$-algebra. Given a subset $A' \subseteq A$, we denote the infimum and the supremum of A' respectively as $\bigcap A'$ and $\bigcup A'$. An $\mathrm{FL_{ew}}$-algebra may be linearly ordered, or a chain, if its lattice order is total; a standard algebra is an algebra whose lattice reduct is the real unit interval $[0, 1]$. We are also interested in finite algebras, that is, algebras with a finite number of elements. A many-valued logic may have its truth values in an $\mathrm{FL_{ew}}$-algebra, interpreting logical conjunction, disjunction, and implication as the monoid, join, and implication operations of the algebra, respectively.

Following Fitting, we proceed to give a general definition of many-valued modal logic parametrized to an $\mathrm{FL_{ew}}$-algebra. Differently from Fitting, we also consider modal logics encompassing more than one relation, aiming to provide a more flexible framework allowing for the treatment of many interesting scenarios such as temporal and spatial logics. Let us first recall the definition of Multi-Modal Logic [8].

Definition 2. *Let* $\mathcal{P}$ *be a countable set of propositional letters,* $\{\neg, \vee\}$ *the classical Boolean connectives, and* $\{\langle X_1\rangle, \ldots, \langle X_n\rangle\}$ *a finite set of existential modalities. Then, well-formed formulas of the* Multi-Modal Logic K_n *are obtained by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X_i\rangle\varphi,$$

for $1 \leq i \leq n$ *and* $p \in \mathcal{P}$*.*

The usual (classical) abbreviations apply, so that conjunction ($\wedge$), implication ($\rightarrow$), and one universal modality for each existential one ($[X_i]$) are considered shortcuts; in particular, $[X_i]\varphi \equiv \neg\langle X_i\rangle\neg\varphi$. Formulas of K_n are interpreted on Kripke structures, as follows.

Definition 3. *Given a set of propositional letters* $\mathcal{P}$ *and a non-empty set of* worlds W*, a* Kripke frame *is an object* $F = \langle W, R_1 \ldots, R_n\rangle$*, where each* $R_i \subseteq W \times W$ *is an* accessibility *relation. A* Kripke structure *is a Kripke frame enriched with a* valuation function $V\colon W \longrightarrow 2^{\mathcal{P}}$*, and it is denoted by* $M = \langle F, V\rangle$*. Finally, given a well-formed formula* φ*, we say that* φ *is* satisfied in M at w*, for some world* w*, and we denote it by* $M, w \Vdash \varphi$*, if and only if*

$$\begin{array}{lll}
M, w \Vdash p & \textit{iff} & p \in V(w), \\
M, w \Vdash \neg\psi & \textit{iff} & M, w \nVdash \psi, \\
M, w \Vdash \psi \vee \xi & \textit{iff} & M, w \Vdash \psi \textit{ or } M, w \Vdash \xi \\
M, w \Vdash \langle X_i\rangle\psi & \textit{iff} & \textit{there is an } s \textit{ s.t. } wR_is \textit{ and } M, s \Vdash \psi.
\end{array}$$

A formula φ *of* K_n *is* satisfied *at a world* w *in a Kripke structure* M *if and only if*

$$M, w \Vdash \varphi.$$

A formula φ *is* satisfiable *if and only if there exists a structure and a world in which it is satisfied, and* valid *if it is satisfied at every world in every structure.*

Definition 4. *Let* $\mathcal{P}$ *be a set of propositional letters and* $\boldsymbol{A}$ *a complete* $\mathrm{FL_{ew}}$*-algebra. The well-formed formulas of the* Multi-Modal Logic $\mathrm{FL_{ew}}$-K_n *are obtained by the following grammar:*

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X_i\rangle\varphi \mid [X_i]\varphi,$$

for $1 \leq i \leq n$*,* $p \in \mathcal{P}$*, and* $\alpha \in \boldsymbol{A}$*.*

The many-valued nature of $\mathrm{FL_{ew}}$-K_n requires an explicit inclusion of Boolean operators otherwise used as abbreviations, as well as the universal version of every modality. This is a consequence of the fact that while in $\mathrm{FL_{ew}}$-algebras, as in Heyting algebras, is still possible to define negation such as $\neg\varphi \equiv \varphi \rightarrow 0$, such a negation is not involutive in the general case (i.e, it does not satisfy the law of double negation). Formulas of $\mathrm{FL_{ew}}$-K_n are interpreted on $\mathrm{FL_{ew}}$-Kripke structures as follows.

Definition 5. *Given a set of propositional letters $\mathcal{P}$, a non-empty set of* worlds W*, and a complete* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$*, an* $\mathrm{FL_{ew}}$-Kripke frame *is an object* $\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle$*, where each* $\widetilde{R}_i : (W \times W) \longrightarrow \boldsymbol{A}$ *is an* accessibility *function. An* $\mathrm{FL_{ew}}$-Kripke structure *is an* $\mathrm{FL_{ew}}$*-Kripke frame enriched with a* valuation function $\widetilde{V} : (W \times \mathcal{P}) \longrightarrow \boldsymbol{A}$*, and it is denoted by* $\widetilde{M} = \langle \widetilde{F}, \widetilde{V} \rangle$*. Given a well-formed formula* φ*, we compute its* value in $\widetilde{M}$ at w*, for some* $w \in W$*, by extending* $\widetilde{V}$ *to formulas as follows:*

$$
\begin{array}{lcl}
\widetilde{V}(w, \alpha) & = & \alpha, \\
\widetilde{V}(w, \varphi \wedge \psi) & = & \widetilde{V}(w, \varphi) \cdot \widetilde{V}(w, \psi), \\
\widetilde{V}(w, \varphi \vee \psi) & = & \widetilde{V}(w, \varphi) \cup \widetilde{V}(w, \psi), \\
\widetilde{V}(w, \varphi \rightarrow \psi) & = & \widetilde{V}(w, \varphi) \hookrightarrow \widetilde{V}(w, \psi), \\
\widetilde{V}(w, \langle X_i \rangle \varphi) & = & \bigcup \{ \widetilde{R}_i(w, s) \cdot \widetilde{V}(s, \varphi) \}, \\
\widetilde{V}(w, [X_i] \varphi) & = & \bigcap \{ \widetilde{R}_i(w, s) \hookrightarrow \widetilde{V}(s, \varphi) \}.
\end{array}
$$

A formula φ *of* $\mathrm{FL_{ew}}$*-*K_n *is* α-satisfied *at world* w *in an* $\mathrm{FL_{ew}}$*-Kripke structure* $\widetilde{M}$ *if and only if*

$$\widetilde{V}(w, \varphi) \succeq \alpha.$$

A formula φ *is* α-satisfiable *if and only if there exists a structure and a world in which it is* α*-satisfied, and it is* satisfiable *if it is* α*-satisfiable for some* $\alpha \in \boldsymbol{A}$*,* $\alpha \succ 0$*; respectively, a formula is* α-valid *if it is* α*-satisfied at every world in every model, and* valid *if it is* 1*-valid.*

2. A tableau system for Many-Valued Multi-Modal Logic

Reasoning systems for fuzzy and many-valued logic exist mainly at the propositional, and sometimes first-order level [9, 10]; among the few attempts at proposing working reasoning systems for fuzzy modal logic, we mention [11]. However, none of those is currently available. Hence, we extend Fitting's work [6] introducing a tableau system for the Multi-Modal Logic $\mathrm{FL_{ew}}$-K_n, and we provide an implementation for it released open-source as part of a larger framework for learning and reasoning with many-valued multi-modal logic, namely Sole.jl [12, 13].

Definition 6. *Take an* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$*, an* $\mathrm{FL_{ew}}$*-*K_n*-formula* φ*, and a finite* $\mathrm{FL_{ew}}$*-Kripke frame* $\widetilde{F} = \langle W, R_1, \ldots, R_n \rangle$*. A* decoration *is an object of the type*

$$Q(\alpha \preceq \varphi, w, \widetilde{F}) \text{ or } Q(\varphi \preceq \alpha, w, \widetilde{F}),$$

where $\alpha \in \boldsymbol{A}$*,* $w \in W$*,* $\alpha \preceq \varphi$ *(resp.,* $\varphi \preceq \alpha$*) is an* assertion *on* w*, and* $Q \in \{\mathit{true}, \mathit{false}\}$ *is a* judgment*. The universe of all possible decorations is denoted by* $\mathcal{D}$*.*

Definition 7. *Given a finite* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$ *and an* $\mathrm{FL_{ew}}$*-*K_n*-formula* φ*, a* tableau τ *for* φ *and* $\alpha \in \boldsymbol{A}$ *is defined as a tuple*

$$\tau = \langle \mathcal{V}, \mathcal{E}, d, f, c \rangle,$$

where $\langle \mathcal{V}, \mathcal{E} \rangle$ *is a tree with nodes in* $\mathcal{V}$ *and edges in* $\mathcal{E}$*, and whose set of branches is denoted by* $\mathcal{B}$*. The vertices are partially ordered by a relation* $\lhd$*, which is induced by the edges. The function*

$$d : \mathcal{V} \longrightarrow \mathcal{D}$$

is a node labeling function *that assigns to each node* $\nu \in \mathcal{V}$ *a decoration of the form* $Q(\beta \preceq \psi, w, \widetilde{F})$ *or* $Q(\psi \preceq \beta, w, \widetilde{F})$*, where* $\beta \in \boldsymbol{A}$ *and* ψ *is a subformula of* φ*. The function*

$$f : \mathcal{V} \longrightarrow \{0, 1\}$$

is a node flag function *indicating whether nodes are expanded (1) or not expanded (0). The function*

$$c : \mathcal{B} \longrightarrow \mathfrak{F},$$

$$(true\ \wedge)\ \frac{true(\beta \preceq (\psi \wedge \xi), w, \widetilde{F})}{\begin{array}{c} true(\beta_1 \preceq \psi, w, c(B)) \mid \ldots \mid true(\beta_n \preceq \psi, w, c(B)) \\ true(\gamma_1 \preceq \xi, w, c(B)) \mid \ldots \mid true(\gamma_n \preceq \xi, w, c(B)) \end{array}}$$

where $\beta \succ 0$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \preceq \beta_j \cdot \gamma_j$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \preceq \beta'_j \cdot \gamma'_j$, $\beta'_j \preceq \beta_j$ and $\gamma'_j \preceq \gamma_j$

$$(false\ \wedge)\ \frac{false(\beta \preceq (\psi \wedge \xi), w, \widetilde{F})}{\begin{array}{c} true(\psi \preceq \beta_1, w, c(B)) \mid \ldots \mid true(\psi \preceq \beta_n, w, c(B)) \\ true(\xi \preceq \gamma_1, w, c(B)) \mid \ldots \mid true(\xi \preceq \gamma_n, w, c(B)) \end{array}}$$

where $\beta \succ 0$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \not\preceq \beta_j \cdot \gamma_j$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \not\preceq \beta'_j \cdot \gamma'_j$, $\beta_j \preceq \beta'_j$ and $\gamma_j \preceq \gamma'_j$

$$(true\ \vee)\ \frac{true((\psi \vee \xi) \preceq \beta, w, \widetilde{F})}{\begin{array}{c} true(\psi \preceq \beta_1, w, c(B)) \mid \ldots \mid true(\psi \preceq \beta_n, w, c(B)) \\ true(\xi \preceq \gamma_1, w, c(B)) \mid \ldots \mid true(\xi \preceq \gamma_n, w, c(B)) \end{array}}$$

where $\beta \prec 1$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta_j \cup \gamma_j \preceq \beta$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta'_j \cup \gamma'_j \preceq \beta$, $\beta_j \preceq \beta'_j$ and $\gamma_j \preceq \gamma'_j$

$$(false\ \vee)\ \frac{false((\psi \vee \xi) \preceq \beta, w, \widetilde{F})}{\begin{array}{c} true(\beta_1 \preceq \psi, w, c(B)) \mid \ldots \mid true(\beta_n \preceq \psi, w, c(B)) \\ true(\gamma_1 \preceq \xi, w, c(B)) \mid \ldots \mid true(\gamma_n \preceq \xi, w, c(B)) \end{array}}$$

where $\beta \prec 1$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta_j \cup \gamma_j \not\preceq \beta$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta'_j \cup \gamma'_j \not\preceq \beta$, $\beta'_j \preceq \beta_j$ and $\gamma'_j \preceq \gamma_j$

$$(true\ \rightarrow)\ \frac{true(\beta \preceq (\psi \rightarrow \xi), w, \widetilde{F})}{\begin{array}{c} true(\psi \preceq \beta_1, w, c(B)) \mid \ldots \mid true(\psi \preceq \beta_n, w, c(B)) \\ true(\gamma_1 \preceq \xi, w, c(B)) \mid \ldots \mid true(\gamma_n \preceq \xi, w, c(B)) \end{array}}$$

where $\beta \succ 0$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \preceq \beta_j \hookrightarrow \gamma_j$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \preceq \beta'_j \hookrightarrow \gamma'_j$, $\beta_j \preceq \beta'_j$ and $\gamma'_j \preceq \gamma_j$

$$(false\ \rightarrow)\ \frac{false(\beta \preceq (\psi \rightarrow \xi), w, \widetilde{F})}{\begin{array}{c} true(\beta_1 \preceq \psi, w, c(B)) \mid \ldots \mid true(\beta_n \preceq \psi, w, c(B)) \\ true(\xi \preceq \gamma_1, w, c(B)) \mid \ldots \mid true(\xi \preceq \gamma_n, w, c(B)) \end{array}}$$

where $\beta \succ 0$, $(\beta_j, \gamma_j) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \not\preceq \beta_j \hookrightarrow \gamma_j$ and there is no other $(\beta'_j, \gamma'_j) \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \not\preceq \beta'_j \hookrightarrow \gamma'_j$, $\beta'_j \preceq \beta_j$ and $\gamma_j \preceq \gamma'_j$

Figure 1: Propositional rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node ν which is being expanded.

where $\mathfrak{F}$ is the universe of all possible finite $\mathrm{FL_{ew}}$-K_n frames, is a branch labeling function *that associates to every branch the frame that belongs to the decoration of its leaf. The initial tableau τ_0 can be either*

$$\langle \{\nu_0\}, \emptyset, \{\nu_0 \mapsto true(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \ldots, \emptyset \rangle)\}, \{\nu_0 \mapsto 0\}, \{\{\nu_0\} \mapsto \langle \{w_0\}, \emptyset, \ldots, \emptyset \rangle\} \rangle$$

if considering α-satisfiability, or

$$\langle \{\nu_0\}, \emptyset, \{\nu_0 \mapsto false(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \ldots, \emptyset \rangle)\}, \{\nu_0 \mapsto 0\}, \{\{\nu_0\} \mapsto \langle \{w_0\}, \emptyset, \ldots, \emptyset \rangle\} \rangle$$

if considering α-validity, and it evolves by iteratively applying the branch expansion rules *(see Fig. 1 and 2) or, if not applicable, the* reverse rules *(see Fig. 3). These rules are applied to the node ν closest to the root*

$$(true\ \Box)\ \frac{true(\beta \preceq [X_i]\psi, w, \widetilde{F})}{\begin{array}{c} true((\beta \cdot \gamma_1) \preceq \psi, w_1, c(B)) \\ \dots \\ true((\beta \cdot \gamma_m) \preceq \psi, w_m, c(B)) \\ true(\beta \preceq [X_i]\psi, w, c(B)) \end{array}}$$

where $\gamma_j = \widetilde{R}_i(w, w_j)$, $w_j \in o(c(B))$, $\gamma_j \succ 0$, and $\beta \cdot \gamma_j \succ 0$

$$(true\ \Diamond)\ \frac{true(\langle X_i\rangle\psi \preceq \beta, w, \widetilde{F})}{\begin{array}{c} true(\psi \preceq (\gamma_1 \hookrightarrow \beta), w_1, c(B)) \\ \dots \\ true(\psi \preceq (\gamma_m \hookrightarrow \beta), w_m, c(B)) \\ true(\langle X_i\rangle\psi \preceq \beta, w, c(B)) \end{array}}$$

where $\gamma_j = \widetilde{R}_i(w, w_j)$, $w_j \in o(c(B))$, $\gamma_j \succ 0$, and $\gamma_j \hookrightarrow \beta \prec 1$

$$(false\ \Box)\ \frac{false(\beta \preceq [X_i]\psi, w, \widetilde{F})}{false((\beta \cdot \gamma_1) \preceq \psi, w_1, c(B')) \mid \dots \mid false((\beta \cdot \gamma_m) \preceq \psi, w_m, c(B'))}$$

where $\gamma_j = \widetilde{R}_i(w, w_j)$, $w_j \in o(c(B)) \cup n(c(B))$, $\gamma_j \succ 0$, and $\beta \cdot \gamma_j \succ 0$

$$(false\ \Diamond)\ \frac{false(\langle X_i\rangle\psi \preceq \beta, w, \widetilde{F})}{false(\psi \preceq (\gamma_1 \hookrightarrow \beta), w_1, c(B')) \mid \dots \mid false(\psi \preceq (\gamma_m \hookrightarrow \beta), w_m, c(B'))}$$

where $\gamma_j = \widetilde{R}_i(w, w_j)$, $w_j \in o(c(B)) \cup n(c(B))$, $\gamma_j \succ 0$, and $\gamma_j \hookrightarrow \beta \prec 1$

Figure 2: Modal rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node ν which is being expanded. When expanding a node of the type $false(\cdot, \cdot, \widetilde{F})$, a new branch B' is generated for each possible way a new world can be added to the frame $c(B)$. $o(c(B))$ is the set of all existing worlds in the frame $c(B)$, while $n(c(B))$ is the new world added to $c(B)$ to obtain $c(B')$.

$$(true\ \succeq)\ \frac{true(\beta \preceq \psi, w, \widetilde{F})}{false(\psi \preceq \gamma, w, c(B))}$$

where $\beta \succ 0$ and γ is any maximal element not above β, i.e., $\gamma \not\succeq \beta$

$$(false\ \succeq)\ \frac{false(\beta \preceq \psi, w, \widetilde{F})}{true(\psi \preceq \gamma_j, w, c(B)) \mid \dots \mid true(\psi \preceq \gamma_n, w, c(B))}$$

where $\beta \succ 0$ and $\gamma_1, \dots, \gamma_n$ are all maximal elements not above β, i.e., $\gamma_1, \dots, \gamma_n \not\succeq \beta$

$$(true\ \preceq)\ \frac{true(\psi \preceq \beta, w, \widetilde{F})}{false(\gamma \preceq \psi, w, c(B))}$$

where $\beta \prec 1$ and γ is any minimal element not below β, i.e., $\gamma \not\preceq \beta$

$$(false\ \preceq)\ \frac{false(\psi \preceq \beta, w, \widetilde{F})}{true(\gamma_j \preceq \psi, w, c(B)) \mid \dots \mid true(\gamma_j \preceq \psi, w, c(B))}$$

where $\beta \prec 1$ and $\gamma_1, \dots, \gamma_n$ are all minimal elements not below β, i.e., $\gamma_1, \dots, \gamma_n \not\preceq \beta$

Figure 3: Reverse rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node ν which is being expanded.

such that $f(\nu) = 0$, affecting every descendant leaf ν' for which $\nu \lhd \nu'$. An application of either rule to ν where $f(\nu) = 0$ will result in setting $f(\nu)$ to 1.

Consider, in particular, the rules *false* $\Box$ and *false* $\Diamond$ in Fig. 2. If we want to prove that $[X_i]\psi$ (resp., $\langle X_i\rangle\psi$) cannot take value greater than β when evaluated on the world w, we need to make sure that in some world w' which is $\widetilde{R}_i$-reachable from w with value γ, the value of ψ is so that $\beta \cdot \gamma$ (resp., $\gamma \hookrightarrow \beta$) is not greater than β. This must be predicated over all existing worlds in the frame at the leaf of the branch and over a possible new existing world.

Definition 8. *Given a finite* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$ *and an* $\mathrm{FL_{ew}}$*-*K_n*-formula* φ*, a tableau* τ *for* φ *and* $\alpha \in \boldsymbol{A}$ *is* closed *if the* branch closing rules *(see Fig. 4), can be applied to all of its branches. Conversely,* τ *is* open *if there exists at least one branch to which the branch closing rules cannot be applied.*

Given a formula φ and a value α, a tableau starting with $true(\alpha \preceq \varphi, w_0, \langle\{w_0\}, \emptyset, \dots, \emptyset\rangle)$ (resp., $false(\varphi \preceq \alpha, w_0, \langle\{w_0\}, \emptyset, \dots, \emptyset\rangle))$ is termed SAT-tableau (resp., VAL-tableau). An open SAT-tableau

$$(\text{✗}1)\ \frac{true(\beta \preceq \gamma, w, \widetilde{F})}{\text{✗}} \text{ where } \beta \not\preceq \gamma \qquad (\text{✗}2)\ \frac{false(\beta \preceq \gamma, w, \widetilde{F})}{\text{✗}} \text{ where } \beta \succ 0, \gamma \prec 1, \text{ and } \beta \preceq \gamma \qquad (\text{✗}3)\ \frac{false(0 \preceq \psi, w, \widetilde{F})}{\text{✗}}$$

$$(\text{✗}4)\ \frac{false(\psi \preceq 1, w, \widetilde{F})}{\text{✗}} \qquad (\text{✗}5)\ \frac{\begin{array}{c} true(\gamma \preceq \psi, w, \widetilde{F}) \\ false(\beta \preceq \psi, w, \widetilde{F}') \end{array}}{\text{✗}} \text{ where } \beta \preceq \gamma \qquad (\text{✗}6)\ \frac{Q(\cdot, \cdot, \widetilde{F})}{\text{✗}} \text{ where } \widetilde{F} \text{ is inconsistent}$$

Figure 4: Branch closing rules. Every branch $B \in \mathcal{B}$ containing the node ν which is being expanded must be closed.

(resp., closed VAL-tableau) for φ and α demonstrates that φ is α-satisfiable (resp., α-valid). Conversely, the reversed decorations $true(\varphi \preceq \alpha, w_0, \langle\{w_0\}, \emptyset, \ldots, \emptyset\rangle)$ and $false(\varphi \preceq \alpha, w_0, \langle\{w_0\}, \emptyset, \ldots, \emptyset\rangle)$ are generally not employed as starting decorations to prove substantial statements. For example, $true(\varphi \preceq \alpha, w_0, \langle\{w_0\}, \emptyset, \ldots, \emptyset\rangle)$ might suggest the possibility of constructing a structure where φ has a value less than α. In classical logic, setting $\alpha = 1$, this initial condition would imply that $\neg\varphi$ is valid, as φ would consistently take the value 0. However, in many-valued logic, such duality is absent (there is no guarantee of a formula that consistently attains the value 1 when φ is strictly less than 1).

Lemma 1. *The tableau system for many-valued multi-modal logic of finite* $\mathrm{FL_{ew}}$*-algebras is sound for proving* α*-satisfiability, that is, given a finite* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$*, an* $\mathrm{FL_{ew}}$*-*K_n*-formula* φ*, if* φ *is* α*-satisfiable then there exists an open SAT-tableau for* φ *and* α*.*

Lemma 2. *The tableau system for many-valued multi-modal logic of finite* $\mathrm{FL_{ew}}$*-algebras is complete for proving* α*-satisfiability, that is, given a finite* $\mathrm{FL_{ew}}$*-algebra* $\boldsymbol{A}$*, an* $\mathrm{FL_{ew}}$*-*K_n*-formula* φ*, and a value* $\alpha \in A$*, if there is an open SAT-tableau for* φ *and* α*, then* φ *is* α*-satisfiable.*

Theorem 3. *The tableau system for* $\mathrm{FL_{ew}}$*-*K_n *is sound and complete.*

3. Applications

Temporal and spatial data are undoubtedly two of the most prominent and resourceful sources of information for most real-world scenarios, ranging from electrical and audio signals, to medical imaging, to strategic planning, and many more. The literature on modal spatial and temporal logic is very wide, and temporal and spatial logic languages have been introduced in several forms [14, 15, 16, 17] for knowledge representation, although they are usually not examined together, taking points or intervals as primitive objects in the temporal case, and points or extended areas in the spatial case. The typical choices in terms of relations include directional relations (e.g., 'in the future', 'upwards'), as well as topological ones (e.g., 'during', 'inside'). More recently, some of these languages have also been leveraged in machine learning applications [18, 19], effectively opening new lines of research. However, certain spatial and temporal languages share a common multi-modal nature, and it is sometimes convenient to study them as particular cases of a generic multi-modal language, typically selecting worlds from a linear order or an Euclidean space and constraining the accessibility relations accordingly.

Introducing a many-valued version for such logic is fairly simple; in fact, one only has to define a many-valued interpretation for the linear order, and adapt the accessibility relations accordingly. There are several possible definitions of non-crisp linear orders. Zadeh [20] defines a similarity relation in a set, imposing that it is reflexive, symmetric, and transitive, as well as a notion of many-valued ordering, anti-symmetry, and totality. Similarly, Bodenhofer [21] advocates for the use of similarity-based many-valued orderings, in which the linearity is in a strong form; the same notion is also used in [22]. Ovchinnikov [23] proposes a notion of many-valued ordering with a non-strict ordering relation.

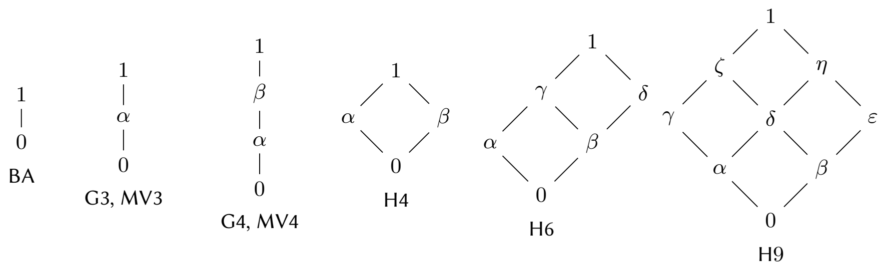

Figure 5: Algebras used in our experiments. G3 and MV3 (resp. G4 and MV4) only differ for the monoid.

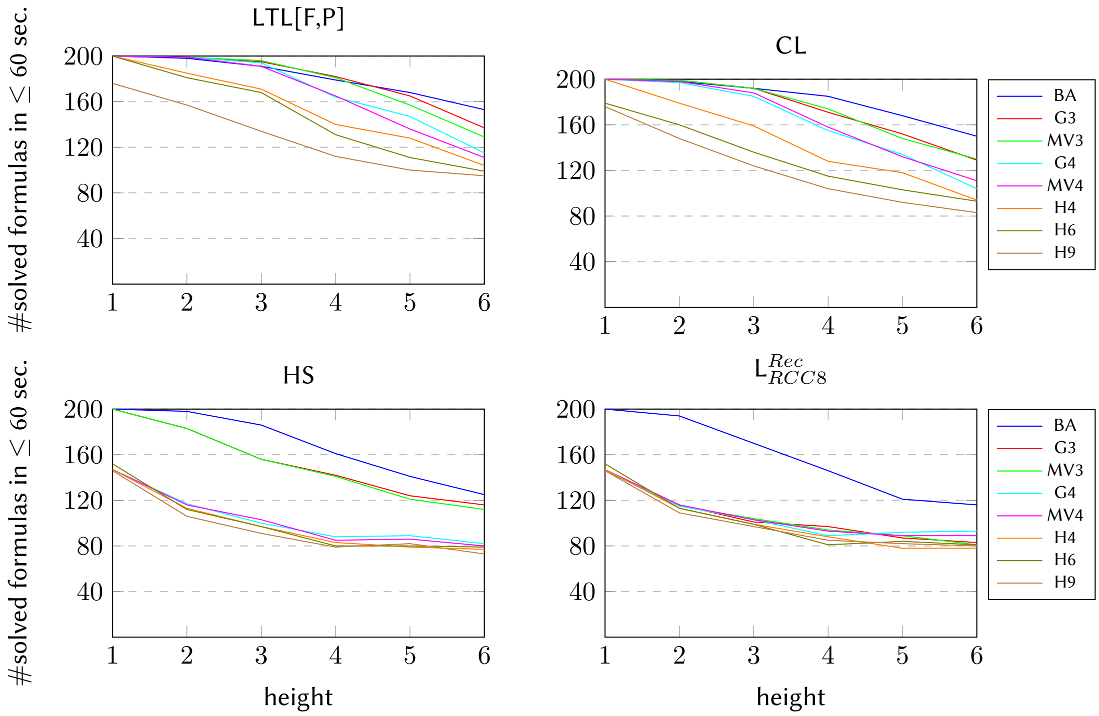

Figure 6: Estimation of how the computation time for various many-valued temporal and spatial logics is influenced by the height of the formula and the type of algebra, specifically, how many formulas can be computed within a 60-second timeout over 200 for each eight from 1 to 6 for algebras in Fig. 5.

A common denominator to all such proposals is the definition of a very weak version of the transitivity property, which allows one to obtain very general definitions.

The tableau system proposed in the former section already supports many-valued temporal and spatial logics out of the box. An open-source implementation for common many-valued temporal spatial and logics is also provided through the Sole.jl framework [13], namely Pnueli's Linear Time Logic with Future and Past (LTL[F,P]) [14], Venema's Compass Logic (CL) [15], Halpern and Shoham's Modal Logic of Time Intervals (HS) [24], and Lutz and Wolter's Modal Logic of Topological Relations [17], specifically in the case of rectangular areas aligned with the axes (L^{Rec}_{RCC8}). Fig. 6 depicts some performance results of running the tableau to check α-satisfiability in the aforementioned logics for algebras in Fig. 5.

4. Conclusions

In this work, we proposed a possible generalization for the tableau system for Many-Valued Modal Logic proposed by Melvin Fitting in 1995 suitable to treat other many-valued approaches, such as fuzzy logics, and easily adaptable to temporal and spatial logics. We also provided some experimental results for an open-source implementation of the system, emphasizing performance over different algebras.

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Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

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