Recursive Self-Identification at the Σ_1/Π_1 Boundary

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Abstract

We analyze recursively enumerable arithmetical predicates intended to represent epistemic knowledge. We isolate three structural conditions — recursive presentability, factivity, and fragment-level truth alignment — whose conjunction is called *Recursive Self-Identification* (RSI).

The central result is a definability-theoretic classification: for every syntactically decidable fragment Γ, RSI(Γ) holds if and only if the set of true Γ-sentences in the standard model is recursively enumerable. In particular, RSI(Σ_1) is consistent, whereas RSI(Π_n) for $n \geqslant 1$ and RSI(Σ_n) for $n \geqslant 2$ are inconsistent.

The analysis identifies two distinct obstruction mechanisms with structurally different proofs: a diagonal self-reference obstruction for fragments containing Π_1, formally equivalent to Gödel's first incompleteness theorem, and a recursion-theoretic complexity obstruction used here for the higher existential fragments.

Among the standard existential and universal levels, Σ_1 is the unique one at which recursive self-identification is possible.

Keywords

recursive self-identification, epistemic predicates, arithmetical hierarchy, factivity, definability, uniform reducibility, proof-theoretic definability, diagonal lemma, Gödelian incompleteness

1. Introduction

Gödel's incompleteness theorem [3] yields, in modern formulations, that no consistent, recursively axiomatized extension Θ of Robinson arithmetic Q proves all true arithmetical sentences. This result is standardly analyzed proof-theoretically, via provability predicates and reflection principles [2, 1]. The present paper takes a different, purely semantic perspective. Rather than focusing on formal theories, we analyze arbitrary arithmetical predicates intended to represent epistemic knowledge.

A broader philosophical question motivates this shift. Gödel's theorem excludes the possibility that all arithmetical truth be captured by any single consistent, recursively axiomatized theory. But a more nuanced possibility remains open: that every true arithmetical sentence might nevertheless admit, at least in principle, some convincing justification, possibly of a metamathematical kind, that the mathematical community could recognize as sound. The Gödel sentence, asserting its own unprovability in Θ, provides the classical paradigm. Although it is not provable in Θ, its truth may be accepted on the basis of a stronger metamathematical argument.

This suggests a distinction between two questions. One is the broad, non-uniform question whether every arithmetical truth might be knowable by some sufficiently compelling argument. The other is whether such knowledge can be captured in a uniform mechanical way by a single arithmetical predicate. The present paper addresses the second question, in fragment-relative form. It does not attempt to settle whether every arithmetical truth is knowable in the broader non-uniform sense. Rather, it asks under what conditions truth in a fixed fragment can be uniformly recognized by a recursively presented and semantically sound predicate. This distinction is implicit in Gödel's own analysis [4]. The RSI framework provides a precise semantic criterion for the uniform mechanical version of the question.

The guiding question is therefore:

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> Under what structural conditions can a recursively presented, semantically sound predicate K contain all true sentences from a given fragment of the arithmetical hierarchy?

From a computational-logic perspective, the predicate K may be viewed as an abstract recursively presentable knowledge base whose accepted sentences are required to be semantically sound. RSI then asks when such a component can be complete for truth over a prescribed syntactic fragment, thereby isolating a minimal semantic configuration for uniform symbolic truth-recognition.

We isolate three conditions on an arithmetical predicate $K(x)$:

1. *Recursive presentability*: the extension of K is recursively enumerable;
2. *Factivity*: every sentence K accepts is true in the standard model;
3. *Fragment-level truth alignment*: K contains every true sentence in a specified class Γ.

Their conjunction is called *Recursive Self-Identification at level* Γ, abbreviated RSI(Γ). The term *self-identification* refers to the extensional coincidence of K's extension with the truth set at level Γ. More precisely, let

$$\mathrm{ext}(K) = \{\ulcorner\varphi\urcorner : \mathbb{N} \models K(\ulcorner\varphi\urcorner)\}$$

denote the set of Gödel codes accepted by K, and let

$$\mathsf{True}_\Gamma = \{\ulcorner\varphi\urcorner : \varphi \in \Gamma \text{ and } \mathbb{N} \models \varphi\}$$

denote the set of codes of true sentences in Γ. Then RSI(Γ) holds precisely when

$$\mathrm{ext}(K) \cap S_\Gamma = \mathsf{True}_\Gamma,$$

where $S_\Gamma = \{\ulcorner\varphi\urcorner : \varphi \in \Gamma\}$ is the set of codes of all sentences in Γ. Factivity ensures $\mathrm{ext}(K) \cap S_\Gamma \subseteq \mathsf{True}_\Gamma$ (nothing false is accepted), while Γ-truth-alignment ensures $\mathsf{True}_\Gamma \subseteq \mathrm{ext}(K) \cap S_\Gamma$ (nothing true is missed). This should not be confused with syntactic self-reference, which enters only in the diagonal arguments of Sections 4 and 6.

At an intuitive level, the positive result at Σ_1 reflects the fact that true existential arithmetical sentences come with finite certificates: a concrete witness whose correctness is decidable by bounded computation. True Π_1 sentences, by contrast, express universal correctness of a decidable condition, and the diagonal construction shows that no recursively enumerable factive predicate can uniformly capture all such truths.

We call the set True_Γ of true Γ-sentences *the true* Γ*-theory*. The unifying criterion of the paper is a general *definability-theoretic classification*:

> For every syntactically decidable fragment Γ of arithmetical sentences, RSI(Γ) holds if and only if the true Γ-theory is recursively enumerable.

Thus recursive self-identification at level Γ is possible precisely when the truth set True_Γ is recursively enumerable, equivalently Σ_1 as a relation on codes. The general classification theorem in turn provides a uniform account of the fragment-level results established earlier in the paper.

Main Results

The paper establishes the following facts, which are later subsumed under a general classification theorem:

- Σ_1 **Stability.** Since True_{Σ_1} is itself recursively enumerable, RSI(Σ_1) is consistent (i.e., the conditions defining RSI(Σ_1) are jointly realized). Existential arithmetical truth thus admits uniform recursive recognition: true Σ_1 sentences can be enumerated by searching for finite witnesses. The provability predicate of any sound, recursively axiomatized extension of Q (e.g., Peano arithmetic PA) provides an example.

- **Universal Collapse.** For every $n \geqslant 1$, RSI(Π_n) is inconsistent. The obstruction is diagonal: the fixed-point construction applied to $\neg K(x)$ produces a true Π_1 sentence, and Π_1 is the first standard universal fragment at which the classical Gödelian mechanism becomes available. This result is formally equivalent to Gödel's first incompleteness theorem. More generally, every fragment containing Π_1 falls under this sufficient condition for diagonal inconsistency.

- **Higher Existential Collapse.** For every $n \geqslant 2$, RSI(Σ_n) is inconsistent. The proof uses a recursion-theoretic obstruction: True_{Σ_n} is not recursively enumerable, and therefore cannot be captured by any recursively enumerable factive predicate.

- **Uniform Reducibility Criterion.** If a syntactically decidable fragment Γ is computably reducible to Σ_1 via Q-provable equivalences, then True_Γ is recursively enumerable and RSI(Γ) is consistent (Proposition 6.3). This subsumes Σ_1 and Δ_0 and motivates the proof-theoretic notion Δ_1^{Q}.

Taken together, these results yield a sharp conclusion: *among the standard existential and universal levels of the arithmetical hierarchy, Σ_1 is the unique one at which recursive self-identification is consistent.*

Relation to Classical Results

As noted above, the Π_1 collapse is *formally equivalent* to Gödel's first incompleteness theorem [3]. This will be established in Proposition 4.5 via an explicit construction. The Σ_n collapse for $n \geqslant 2$ follows from the arithmetical hierarchy theorem. No strengthening of these classical results is claimed.

The contribution of the present work is therefore structural rather than foundationally revisionary: it recasts these classical limitations as constraints on recursively presentable, semantically sound mechanisms for symbolic truth-recognition. The RSI framework isolates the exact semantic configuration under which fragment-level uniform recognizability is possible or impossible: recursive presentation, factivity, and fragment-level truth alignment. Two obstruction mechanisms with structurally different proofs emerge:

- a diagonal self-reference obstruction, applying to fragments containing Π_1;

- a recursion-theoretic inexhaustibility obstruction, sufficient to establish inconsistency for higher existential fragments.

These mechanisms are distinct as proof methods. For the literal syntactic Σ_2 fragment, the standard fixed point of $\neg K(x)$ is produced in Π_1 form, so the direct diagonal argument does not apply without additional closure assumptions. Nevertheless, True_{Σ_2} is not recursively enumerable.

Unlike reflection principles, which express internal provability conditions [2, 1], the impossibility results established here are entirely semantic, operating in the standard model $\mathbb{N}$ without appeal to any internal reflection schema. Furthermore, the analysis applies to arbitrary recursively enumerable factive predicates, not specifically to provability predicates of formal theories.

Philosophical Significance

The Σ_1/Π_1 boundary acquires a natural epistemic meaning in relation to the broader knowability question motivating the paper. Existential arithmetical truth admits finite witnessing and therefore recursive certification. Once one reaches Π_1, the classical diagonal obstruction becomes available: no recursively enumerable and factive predicate can uniformly recognize all true Π_1 sentences. At higher existential levels, the present proof of failure is non-diagonal: truth is no longer recursively enumerable, so uniform recursive recognition is blocked by a complexity obstruction.

Thue analysis thus identifies a precise boundary, not between truth and untruth, but between fragments whose truth can still be uniformly and mechanically recognized and those whose truth outruns any such recognition. Among the standard existential and universal levels of the arithmetical hierarchy, Σ_1 is the unique one at which arithmetical truth remains compatible with recursive semantic certification.

This bears on discussions of mechanized knowledge [11, 12], arithmetical truth [6, 7], and the limits of formal epistemic operators. The limitation does not arise from recursiveness alone; it arises from the conjunction of recursiveness, semantic soundness, and unrestricted truth alignment over a fragment.

The RSI framework thereby complements the discussion of idealized epistemic theories in [8], by identifying at the fragment level the exact conditions under which recursive presentability, factivity, and truth alignment can or cannot coexist.

The present results should accordingly be read as clarifying one precise aspect of the broader knowability problem: they show exactly when a fragment of arithmetic admits a *uniform mechanical surrogate* for mathematically compelling semantic recognition, and exactly where such a surrogate must fail.

Outline

Section 2 introduces the framework. Sections 3–5 prove the fragment-level stability and collapse results. Section 6 gives the general classification and related criteria. Section 7 draws conclusions and briefly discusses open directions.

2. Formal Framework

2.1. Language and Meta-Theoretic Setting

We work in the language of first-order arithmetic $\mathscr{L}_A = \{0, S, +, \cdot\}$ with a Gödel numbering fixed once and for all [3]. The Gödel number of a formula φ is denoted $\ulcorner\varphi\urcorner$. Throughout, a *sentence* is a *closed formula*, i.e., a formula of $\mathscr{L}_A$ with no free variables; a formula with one or more free variables is called an *open formula*. Unless explicitly stated otherwise, truth and provability assertions in the sentence-based part of the paper concern closed formulas. Robinson arithmetic Q [9] serves as the standard weak base theory for representability and diagonalization.

Meta-theoretic convention. All semantic reasoning takes place in the standard model $\mathbb{N}$. We write $\mathbb{N} \vDash \varphi$ for truth in $\mathbb{N}$ and $\Theta \vdash \varphi$ for provability in a theory Θ. The distinction between semantic truth and syntactic provability is maintained throughout. All impossibility results below are *semantic*: they assert the non-existence of certain arithmetical predicates satisfying given conditions simultaneously in $\mathbb{N}$.

2.2. The Arithmetical Hierarchy

We use the arithmetical hierarchy in its standard recursion-theoretic sense [10]. A *relation* is a subset $R \subseteq \mathbb{N}^k$ for some $k \geqslant 1$. A formula is Δ_0 if each quantifier is bounded, i.e., of the form $\exists x < t$ or $\forall x < t$.

Definition 2.1 (Arithmetical hierarchy). *A relation $R \subseteq \mathbb{N}^k$ is:*

- Σ_1 *if it is recursively enumerable;*
- Π_1 *if its complement $\mathbb{N}^k \setminus R$ is Σ_1;*
- Δ_1 *if it is both Σ_1 and Π_1, equivalently, if it is decidable;*
- Σ_{n+1} *if there exists a Π_n relation $S \subseteq \mathbb{N}^{k+1}$ such that $R(\vec{n}) \iff \exists m\, S(\vec{n}, m)$;*
- Π_{n+1} *if its complement is Σ_{n+1};*
- Δ_{n+1} *if it is both Σ_{n+1} and Π_{n+1}.*

The same classification applies to arithmetical formulas, but one must then distinguish carefully between the syntactic and the definability-theoretic uses of the hierarchy.

Syntactically, a formula is Σ_n (resp. Π_n) if, up to bounded quantifiers, it has a prenex form whose unbounded quantifier prefix begins with an existential block (resp. a universal block), has at most n alternating quantifier blocks, and is followed by a Δ_0 matrix. These syntactic classes are mechanically recognizable.

From the definability-theoretic point of view, a formula is Σ_n (resp. Π_n) if it is equivalent in $\mathbb{N}$ to a syntactic Σ_n-formula (resp. a syntactic Π_n-formula).

The notion of Δ_n requires separate treatment for open formulas and sentences. For *open formulas* (formulas with free variables), a formula is Δ_n if it is equivalent in $\mathbb{N}$ to both a syntactic Σ_n-formula and a syntactic Π_n-formula; this is the standard definability-theoretic notion for relations. For *sentences* (closed formulas), semantic equivalence in $\mathbb{N}$ cannot serve as the criterion: every true sentence is equivalent in $\mathbb{N}$ to $0 = 0$ and every false sentence to $0 = 1$, both of which are Δ_0, collapsing the hierarchy.

Definition 2.2 (Δ_n^{Q})**.** *A sentence φ is Δ_n^{Q} if there exist a syntactic Σ_n sentence σ and a syntactic Π_n sentence π such that $\mathrm{Q} \vdash \varphi \leftrightarrow \sigma$ and $\mathrm{Q} \vdash \varphi \leftrightarrow \pi$.*

Remark 2.3 (The semantic and proof-theoretic notions diverge)**.** *The proof-theoretic condition Δ_n^{Q} is strictly stronger than semantic Δ_n-equivalence for sentences. To see this, consider*

$$\varphi_0 := \forall x(0 + x = x).$$

Since φ_0 is true in $\mathbb{N}$, it is semantically equivalent in $\mathbb{N}$ to $0 = 0$, which is $\Delta_0 \subseteq \Sigma_1 \cap \Pi_1$. So φ_0 is semantically Δ_1 in the trivial sentence-level sense.

We claim, however, that $\varphi_0 \notin \Delta_1^{\mathrm{Q}}$. Since φ_0 is itself a syntactic Π_1 sentence, it suffices to show that it has no Q-provably equivalent syntactic Σ_1 sentence.

Suppose, toward contradiction, that

$$\mathrm{Q} \vdash \varphi_0 \leftrightarrow \sigma$$

for some syntactic Σ_1 sentence σ. By soundness of Q and truth of φ_0 in $\mathbb{N}$, the sentence σ is true in $\mathbb{N}$. Every true Σ_1 sentence is provable in Q: if $\sigma = \exists x\psi(x)$ with $\psi \in \Delta_0$, then some numeral $\overline{m}$ satisfies $\mathbb{N} \models \psi(\overline{m})$, and Q decides true Δ_0 sentences [5, Thm. I.1.22],, whence $\mathrm{Q} \vdash \psi(\overline{m})$ and $\mathrm{Q} \vdash \sigma$. Together with $\mathrm{Q} \vdash \varphi_0 \leftrightarrow \sigma$, this gives $\mathrm{Q} \vdash \varphi_0$.

But $\mathrm{Q} \nvdash \forall x(0 + x = x)$. Indeed, Q defines addition recursively on the second argument: it proves $x + 0 = x$ and $x + S(y) = S(x + y)$, but has no induction principle from which the left-identity law follows. Equivalently, there are nonstandard models of Q in which $\forall x(0 + x = x)$ fails [5, Ch. I].. This contradiction shows that no such σ exists.

Therefore φ_0 is semantically Δ_1 but not Δ_1^{Q}.

The two points of view are connected by the arithmetical hierarchy theorem: a relation is Σ_n in the recursion-theoretic sense if and only if it is definable by a syntactic Σ_n-formula. In particular, True_{Σ_n}, the set of codes of true sentences in the syntactic class Σ_n, is itself a Σ_n relation in the recursion-theoretic sense.

In what follows, Σ_n and Π_n refer to the *syntactic* classes when applied to formulas, and to the *recursion-theoretic* classes (Definition 2.1) when applied to sets of codes, such as True_{Σ_n} and True_{Π_n}. The proof-theoretic classes Δ_n^{Q} (Definition 2.2) are treated separately.

2.3. Epistemic Predicates

Definition 2.4. *An* epistemic predicate *is an arithmetical formula $K(x)$ with one free variable. Intuitively, $K(x)$ represents the set $\{\ulcorner\varphi\urcorner : \mathbb{N} \models K(\ulcorner\varphi\urcorner)\}$ of Gödel codes of sentences accepted as known.*

No deductive closure, modal axioms, or reflection principles are assumed. The framework is purely extensional.

2.4. The Three Conditions

Recursive self-identification rests on three requirements: recursive enumerability, factivity, and truth alignment over a chosen fragment Γ. The first two are stated as assumptions on K; the third is formulated as a fragment-relative alignment condition.

Assumption 2.5 (Recursive Enumerability)**.** *$K(x)$ is a Σ_1 formula. Equivalently, there exists a primitive recursive relation $R(x,s)$ such that $\mathbb{N} \models K(x) \leftrightarrow \exists s\, R(x,s)$.*

Assumption 2.6 (Factivity)**.** *For every arithmetical sentence φ,*

$$\mathbb{N} \models K(\ulcorner \varphi \urcorner) \;\Rightarrow\; \mathbb{N} \models \varphi.$$

Factivity is a purely semantic condition. No internal reflection schema of the form $K(\ulcorner \varphi \urcorner) \to \varphi$ is assumed provable in any theory.

Definition 2.7 (Fragment-level truth alignment)**.** *Let Γ be a class of arithmetical sentences. The predicate K is* Γ-truth-aligned *if for every $\varphi \in \Gamma$,*

$$\mathbb{N} \models \varphi \;\Rightarrow\; \mathbb{N} \models K(\ulcorner \varphi \urcorner).$$

The truth-alignment condition is semantic. Combined with factivity, it implies that the restriction of $\mathrm{ext}(K)$ to Γ-codes coincides exactly with $\mathrm{True}_\Gamma = \{\ulcorner \varphi \urcorner : \varphi \in \Gamma,\ \mathbb{N} \models \varphi\}$.

2.5. Recursive Self-Identification

Definition 2.8 (Recursive Self-Identification)**.** *Let Γ be a fragment of the arithmetical hierarchy. An epistemic predicate K satisfies* Recursive Self-Identification at level Γ, *written* $\mathrm{RSI}(\Gamma)$, *if K satisfies Assumptions 2.5 and 2.6 and is Γ-truth-aligned.*

The guiding question is: for which fragments Γ can there exist an epistemic predicate satisfying $\mathrm{RSI}(\Gamma)$?

2.6. Diagonalization

The collapse arguments use the standard Diagonal Lemma.

Theorem 2.9 (Diagonal Lemma [3])**.** *For every $\mathscr{L}_A$-formula $\theta(x)$ with one free variable, there exists a sentence G such that*

$$\mathsf{Q} \vdash G \leftrightarrow \theta(\ulcorner G \urcorner).$$

To transfer syntactic fixed-point equivalences to $\mathbb{N}$, we assume:

Assumption 2.10 (Soundness of Q)**.** *Q is sound: if $\mathsf{Q} \vdash \varphi$ then $\mathbb{N} \models \varphi$.*

This mild assumption ensures $\mathbb{N} \models G \leftrightarrow \theta(\ulcorner G \urcorner)$ whenever G is obtained via the Diagonal Lemma.

3. Σ_1 Stability

Theorem 3.1 (Σ_1 Stability)**.** *There exists an epistemic predicate satisfying* $\mathrm{RSI}(\Sigma_1)$.

Proof. Let Θ be any sound, recursively axiomatized extension of Q (for example, Peano arithmetic PA). Set $K(x) := \mathrm{Prov}_\Theta(x)$, the standard Σ_1 provability predicate for Θ [3].

(R) Recursive Enumerability. $\mathrm{Prov}_\Theta(x)$ is Σ_1, since proof-checking is primitive recursive.

(F) Factivity. If $\mathbb{N} \models K(\ulcorner \varphi \urcorner)$, then $\Theta \vdash \varphi$. By soundness of Θ, $\mathbb{N} \models \varphi$.

(C_{Σ_1}) Σ_1-Truth-Alignment. Let $\varphi = \exists n\, \psi(n)$ be a true Σ_1 sentence, with ψ a Δ_0 formula. Since φ is true in $\mathbb{N}$, there exists $m \in \mathbb{N}$ with $\mathbb{N} \models \psi(m)$. Since ψ is Δ_0 and $\mathbb{N} \models \psi(\overline{m})$, by Δ_0-completeness of Q [5, Thm. I.1.22] we have $\mathsf{Q} \vdash \psi(\overline{m})$, hence $\Theta \vdash \psi(\overline{m})$ and therefore $\Theta \vdash \varphi$. Thus $\mathbb{N} \models K(\ulcorner \varphi \urcorner)$.

All three conditions of Definition 2.8 hold for $\Gamma = \Sigma_1$. $\square$

Remark 3.2 (Why Σ_1 works)**.** *The argument exploits the* witness property*: a true Σ_1 sentence $\exists n\,\psi(n)$ is verified by exhibiting a concrete numerical witness, that serves as a finite certificate. Witness-checking is Δ_0, hence decidable by Q, and a recursively enumerable predicate can enumerate all such witnesses.*

At the level of complexity, the key observation is that True_{Σ_1} is itself Σ_1: the sentence $\exists n\,\psi(n)$ (with ψ Δ_0) is true in $\mathbb{N}$ iff $\exists n\,\psi(n)$, a Σ_1 condition. Hence an r.e. predicate can capture True_{Σ_1} without exceeding its own complexity. For $n \geqslant 2$, the set True_{Σ_n} is no longer Σ_1, and Section 5 shows that no r.e. factive predicate can then capture it.

Remark 3.3 (RSI at level Δ_0)**.** *The argument of Theorem 3.1 applies a fortiori to Δ_0: since True_{Δ_0} is decidable (every Δ_0 sentence is decided by Q), any factive predicate K with $\mathrm{ext}(K) \cap S_{\Delta_0} = \mathsf{True}_{\Delta_0}$ may itself be taken decidable. Thus $\mathrm{RSI}(\Delta_0)$ is consistent and witnesses a stronger form of recognition: decidability rather than mere recursive enumerability.*

Remark 3.4 (Characterization of $\mathrm{RSI}(\Sigma_1)$ witnesses)**.** *A predicate K satisfies $\mathrm{RSI}(\Sigma_1)$ if and only if K is recursively enumerable and*

$$\mathsf{True}_{\Sigma_1} \subseteq \mathrm{ext}(K) \subseteq \mathsf{True}.$$

The lower bound expresses Σ_1-truth-alignment; the upper bound expresses factivity. The class of $\mathrm{RSI}(\Sigma_1)$ witnesses therefore consists precisely of all r.e. factive predicates whose extension contains True_{Σ_1}. In particular, $\mathrm{RSI}(\Sigma_1)$ is not witnessed only by standard provability predicates: any r.e. factive predicate whose extension contains True_{Σ_1} qualifies.

4. Universal Collapse

We show that RSI is impossible at every universal level Π_n, $n \geqslant 1$.

4.1. The Diagonal Fixed Point

Let K be any Σ_1 epistemic predicate (Assumption 2.5). Then $\neg K(x)$ is Π_1. Applying the diagonal construction (Theorem 2.9) to $\theta(x) := \neg K(x)$ yields a sentence G such that

$$\mathrm{Q} \vdash G \leftrightarrow \neg K(\ulcorner G \urcorner).$$

More explicitly, the standard proof of the Diagonal Lemma can be carried out with preservation of arithmetical complexity: if $\theta(x)$ is a syntactic Π_1 formula, its fixed point may be chosen in syntactic Π_1 form. In the present case $\theta(x) = \neg K(x)$ is Π_1, since $K(x)$ is Σ_1, so G may be chosen to be a syntactic Π_1 sentence.

Remark 4.1 (Syntactic Π_1-complexity of G)**.** *The fixed point used above can be chosen in syntactic Π_1 form. Diagonal substitution is primitive recursive, and composing a Π_1 formula with the corresponding numeral substitution preserves Π_1 complexity. The argument therefore produces a fixed point belonging to the syntactic Π_1 class, not merely a sentence semantically equivalent to a Π_1 sentence.*

By soundness of Q (Assumption 2.10),

$$\mathbb{N} \models G \leftrightarrow \neg K(\ulcorner G \urcorner). \tag{1}$$

4.2. Π_1 Collapse

Theorem 4.2 (Π_1 Collapse)**.** *No epistemic predicate satisfies $\mathrm{RSI}(\Pi_1)$.*

Proof. Assume toward contradiction that K satisfies $\mathrm{RSI}(\Pi_1)$. Thus K is Σ_1, factive, and Π_1-truth-aligned. Let G be the Π_1 fixed point of (1).

Step 1: $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$. Suppose $\mathbb{N} \models K(\ulcorner G \urcorner)$. By factivity, $\mathbb{N} \models G$. By (1), $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$. Contradiction. Hence $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$.

Step 2: $\mathbb{N} \models G$. From Step 1 and (1), $\mathbb{N} \models G$.

Step 3. By Remark 4.1, G is a Π_1 sentence; by Step 2 it is true. By Π_1-truth-alignment, $\mathbb{N} \models K(\ulcorner G \urcorner)$. This contradicts Step 1. Therefore no such K exists. □

4.3. General Π_n Collapse

The general case reduces immediately to Π_1.

Corollary 4.3 (General Π_n Collapse)**.** *For every $n \geqslant 1$, no epistemic predicate satisfies* RSI(Π_n)*.*

Proof. The fixed point G of Section 4.1 is $\Pi_1 \subseteq \Pi_n$ for every $n \geqslant 1$ (Remark 4.1). The argument of Theorem 4.2 applies verbatim: Steps 1 and 2 yield G true and $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$, while Π_n-truth-alignment forces $\mathbb{N} \models K(\ulcorner G \urcorner)$. Contradiction. □

Remark 4.4. *The same fixed point witnesses impossibility at all universal levels. The obstruction at Π_1 is inherited by every Π_n because $\Pi_1 \subseteq \Pi_n$; no genuinely Π_n-specific construction is required.*

Proposition 4.5 (Equivalence with Gödel's First Incompleteness Theorem)**.** *The following statements are equivalent:*

(i) (RSI formulation) *No recursively enumerable factive predicate is Π_1-truth-aligned* (Theorem 4.2).

(ii) (Gödel formulation) *For every recursively enumerable, sound extension Θ of* Q *(and hence consistent), there exists a true Π_1 sentence φ such that $\Theta \nvdash \varphi$.*

Proof. (i)⇒(ii). The provability predicate Prov_Θ of any sound r.e. extension Θ of Q is Σ_1 and factive. By (i), it is not Π_1-truth-aligned, so some true Π_1 sentence is not provable in Θ.

(ii)⇒(i). Let K be r.e. and factive, and define

$$\Theta_K := \mathrm{Q} \cup \{\varphi : \mathbb{N} \models K(\ulcorner \varphi \urcorner)\}.$$

Since K is r.e., the set of additional axioms is r.e., so Θ_K is an r.e. extension of Q. By factivity, every additional axiom is true in $\mathbb{N}$, so Θ_K is sound. By (ii), there is a true Π_1 sentence φ with $\Theta_K \nvdash \varphi$. If $\mathbb{N} \models K(\ulcorner \varphi \urcorner)$, then φ is an axiom of Θ_K, giving $\Theta_K \vdash \varphi$, a contradiction. Therefore $\mathbb{N} \not\models K(\ulcorner \varphi \urcorner)$, so K is not Π_1-truth-aligned. □

Remark 4.6 (On the role of soundness)**.** *Both directions use soundness rather than mere consistency. This is not a weakness but a precise match: factivity of K is exactly the semantic soundness of K as an epistemic predicate, and provability predicates of sound theories are factive. A version of Gödel's theorem based on consistency alone (or ω-consistency for the Π_1 formulation) does not correspond to a weaker form of the RSI framework, since dropping factivity would trivialize the alignment condition. The two formulations are therefore equivalent at the same level of semantic assumption.*

5. Higher Existential Collapse

We now show that recursive self-identification is impossible at every existential level Σ_n for $n \geqslant 2$. Unlike the universal case, the proof does not use diagonalization. It is purely definability-theoretic: the existential truth sets eventually exceed Σ_1-complexity.

5.1. Existential Truth Sets and the Arithmetical Hierarchy

For $n \geqslant 1$, define

$$\mathsf{True}_{\Sigma_n} = \{ \ulcorner \varphi \urcorner : \varphi \text{ is a } \Sigma_n \text{ sentence and } \mathbb{N} \models \varphi \}.$$

Theorem 5.1 (Arithmetical Hierarchy; see [10, Ch. 14] and [5, Ch. I]). *For every $n \geqslant 1$, True_{Σ_n} is Σ_n-complete. In particular:*

(i) True_{Σ_1} *is Σ_1 (and hence recursively enumerable).*

(ii) For $n \geqslant 2$, True_{Σ_n} *is not Σ_1.*

The case $n = 1$ reflects the witness property: truth of a Σ_1 sentence is verified by a concrete numerical instance. For $n \geqslant 2$, evaluatiing Σ_n truth requires alternating quantifier blocks, placing the truth set strictly above Σ_1 in the arithmetical hierarchy.

5.2. Σ_n Collapse for $n \geqslant 2$

Theorem 5.2 (Σ_n Collapse). *For every $n \geqslant 2$, no epistemic predicate satisfies* $\mathrm{RSI}(\Sigma_n)$*.*

Proof. Assume toward contradiction that K satisfies $\mathrm{RSI}(\Sigma_n)$ for some $n \geqslant 2$.

Let $S_n := \{\ulcorner \varphi \urcorner : \varphi \text{ is a } \Sigma_n \text{ sentence}\}$. The set S_n is primitive recursive, since syntactic classification of Σ_n formulas is decidable.

By recursive enumerability of K, its extension $\mathrm{ext}(K)$ is Σ_1. Hence the intersection $K^{\Sigma_n} := \mathrm{ext}(K) \cap S_n$ is Σ_1.

Claim. $K^{\Sigma_n} = \mathsf{True}_{\Sigma_n}$.

($\supseteq$): By Σ_n-truth-alignment, every true Σ_n sentence has its code in $\mathrm{ext}(K)$, so $\mathsf{True}_{\Sigma_n} \subseteq K^{\Sigma_n}$.

($\subseteq$): By factivity, every sentence whose code lies in $\mathrm{ext}(K)$ is true in $\mathbb{N}$. Hence $K^{\Sigma_n} \subseteq \mathsf{True}_{\Sigma_n}$.

Thus True_{Σ_n} is Σ_1. This contradicts Theorem 5.1(ii). □

5.3. Definability-Theoretic Obstruction

The higher existential collapse is therefore a definability mismatch.

Recursive self-identification at level Γ requires that True_Γ coincide with a Σ_1 set (namely the restriction of $\mathrm{ext}(K)$). For $n \geqslant 2$, True_{Σ_n} provably exceeds Σ_1-complexity, so no recursively enumerable factive predicate can capture it.

Remark 5.3 (Contrast with the Universal Case). *The proofs of the universal and the higher existential collapse use structurally different methods.*

- *The* universal collapse *(Section 4) is proved by a diagonal argument: since K is Σ_1, the formula $\neg K(x)$ is Π_1, and a fixed-point construction yields a true Π_1 sentence that K cannot include without contradiction.*

- *The* higher existential collapse *is proved without self-reference, using solely the arithmetical hierarchy:* True_{Σ_n} *is not recursively enumerable for $n \geqslant 2$.*

The RSI framework thus identifies two proof methods establishing inconsistency at different levels of the hierarchy. For the syntactic classes Σ_n with $n \geqslant 2$, the proof given here is the complexity-theoretic one. The standard diagonal argument used for Π_1 does not transfer directly to these literal syntactic classes; one possible remedy would be to add closure under suitable provable equivalences, but such assumptions are not part of the present definition of RSI.

Remark 5.4 (Sharpness at Σ_1). *The contrast with $n = 1$ is exact.* True_{Σ_1} *is itself Σ_1, so recursive self-identification is possible. For every other standard existential or universal fragment, the arguments above show that alignment fails: by diagonalization for universal levels, and by definability complexity for higher existential levels.*

Remark 5.5 (Relation to Tarski's undefinability theorem)**.** *Tarski's undefinability theorem [13] establishes that no arithmetical formula can define, in the standard model, the set of Gödel codes of all true arithmetical sentences. The RSI obstructions arise at a strictly lower level: they concern recursively enumerable predicates, a proper subclass of arithmetically definable ones. In particular, the* Π_1 *collapse (Theorem 4.2) shows that the obstruction already appears within the r.e. level, well below the Tarskian threshold, and identifies* Π_1 *as the minimal fragment at which a diagonal construction is available.*

6. Complete Definability-Theoretic Classification

The fragment-by-fragment analysis of the preceding sections admits a uniform recursion-theoretic formulation. We now state and prove a general classification theorem that subsumes all earlier stability and collapse results.

6.1. Syntactic Fragments and Truth Sets

Definition 6.1. *A class* Γ *of arithmetical sentences is* syntactically decidable *if the set*

$$S_\Gamma = \{ \ulcorner\varphi\urcorner : \varphi \in \Gamma \}$$

of Gödel codes of sentences in Γ *is decidable.*

Any syntactically decidable class of arithmetical sentences will be called an *arithmetical fragment.*

In particular, the standard syntactic classes of sentences literally in syntactic Σ_n-form or syntactic Π_n-form are syntactically decidable. When we refere below to the literal syntactic Σ_n fragment, we mean membership in such a syntactic class, not merely equivalence in $\mathbb{N}$ or provable equivalence over Q to a Σ_n sentence.

Recall that for syntactically decidable Γ, the *truth set* True_Γ is defined as in the introduction:

$$\mathsf{True}_\Gamma = \{ \ulcorner\varphi\urcorner : \varphi \in \Gamma \text{ and } \mathbb{N} \models \varphi \}.$$

Similarly, $S_\Gamma = \{ \ulcorner\varphi\urcorner : \varphi \in \Gamma \}$ denotes the set of codes of all sentences in Γ.

6.2. General Classification Theorem

Theorem 6.2 (General RSI Classification)**.** *Let* Γ *be a syntactically decidable class of arithmetical sentences. Then the following are equivalent:*

(i) There exists an epistemic predicate K *satisfying* RSI(Γ).

(ii) The set True_Γ *is recursively enumerable.*

Proof. (*i*) $\Rightarrow$ (*ii*). Assume K satisfies RSI(Γ). Let $W_K = \{ x : \mathbb{N} \models K(x) \}$. Since K is recursively enumerable, W_K is recursively enumerable.

We show $\mathsf{True}_\Gamma = W_K \cap S_\Gamma$.

If $\ulcorner\varphi\urcorner \in \mathsf{True}_\Gamma$, then $\varphi \in \Gamma$ and $\mathbb{N} \models \varphi$. By Γ-truth-alignment, $\mathbb{N} \models K(\ulcorner\varphi\urcorner)$, so $\ulcorner\varphi\urcorner \in W_K$ and hence $\ulcorner\varphi\urcorner \in W_K \cap S_\Gamma$.

Conversely, if $x \in W_K \cap S_\Gamma$, then $x = \ulcorner\varphi\urcorner$ for some $\varphi \in \Gamma$ and $\mathbb{N} \models K(\ulcorner\varphi\urcorner)$. By factivity, $\mathbb{N} \models \varphi$, so $x \in \mathsf{True}_\Gamma$.

Thus $\mathsf{True}_\Gamma = W_K \cap S_\Gamma$. Since W_K is recursively enumerable and S_Γ is decidable, True_Γ is recursively enumerable.

(*ii*) $\Rightarrow$ (*i*). Assume True_Γ is recursively enumerable. Since True_Γ is recursively enumerable, there is a Σ_1 formula $K_\Gamma(x)$ whose extension is exactly True_Γ. Set $K(x) := K_\Gamma(x)$. Then K is recursively enumerable, factive by definition, and Γ-truth-aligned. Thus K satisfies RSI(Γ). $\square$

The proof is intentionally direct: the simplicity of the argument reflects the fact that RSI(Γ) is, by design, a semantic reformulation of the recursion-theoretic condition on True_Γ. The mathematical content lies in determining, for each specific fragment Γ, whether True_Γ is recursively enumerable — a question answered by the arithmetical hierarchy theorem and the diagonal arguments of the preceding sections.

The classification theorem is not intended as a technically deep equivalence. Its role is organizational: it isolates the exact semantic content of RSI and separates it from the independent recursion-theoretic analysis of the truth sets of particular fragments.

Proposition 6.3 (Uniform Q-reduction to Σ_1)**.** *Let Γ be a syntactically decidable fragment. Suppose there exists a computable function f such that, for every $\varphi \in \Gamma$, $f(\ulcorner\varphi\urcorner)$ is the Gödel code of a syntactic Σ_1 sentence σ_φ, and*

$$\mathrm{Q} \vdash \varphi \leftrightarrow \sigma_\varphi.$$

Then True_Γ *is recursively enumerable, and hence* RSI(Γ) *is consistent.*

Proof. By soundness of Q, $\mathrm{Q} \vdash \varphi \leftrightarrow \sigma_\varphi$ implies $\mathbb{N} \vDash \varphi \leftrightarrow \sigma_\varphi$ for every $\varphi \in \Gamma$. Hence

$$\mathsf{True}_\Gamma = \left\{ \ulcorner\varphi\urcorner \in S_\Gamma \;:\; \mathbb{N} \vDash \sigma_\varphi \right\} = \left\{ \ulcorner\varphi\urcorner \in S_\Gamma \;:\; f(\ulcorner\varphi\urcorner) \in \mathsf{True}_{\Sigma_1} \right\}.$$

The map $\ulcorner\varphi\urcorner \mapsto f(\ulcorner\varphi\urcorner)$ is computable by hypothesis. Since True_{Σ_1} is recursively enumerable (Theorem 5.1(i)), True_Γ is recursively enumerable as the preimage of True_{Σ_1} under a computable map. By Theorem 6.2, RSI(Γ) is consistent. □

Remark 6.4. *Proposition 6.3 gives a sufficient condition, but neither the uniformity of f nor the existence of Q-provable Σ_1-equivalences is necessary for the consistency of* RSI(Γ)*. A fragment Γ may have recursively enumerable truth set for entirely different reasons. If Γ is finite, for instance, then* True_Γ *is finite and hence recursively enumerable regardless of the logical complexity of its members. The fragment $\Gamma = \{\mathrm{Con(PA)}\}$ illustrates this:* True_Γ *is finite, yet* Con(PA) *has no* Q*-provably equivalent Σ_1 sentence, by Gödel's second incompleteness theorem. Indeed, if* $\mathrm{Q} \vdash \mathrm{Con(PA)} \leftrightarrow \sigma$ *for some syntactic Σ_1 sentence σ, then, assuming* PA *is consistent (so that* Con(PA) *is true in $\mathbb{N}$), σ is true by soundness of* Q*, hence* $\mathrm{Q} \vdash \sigma$ *by Σ_1-completeness, hence* $\mathrm{Q} \vdash \mathrm{Con(PA)}$*; since* PA *extends* Q*, this gives* $\mathrm{PA} \vdash \mathrm{Con(PA)}$*, contradicting Gödel's second incompleteness theorem.*

Within the scope of Proposition 6.3, the uniformity of f can itself be weakened: it suffices that, for each $\varphi \in \Gamma$, there merely exist *a syntactic Σ_1 sentence σ and a* Q*-proof of $\varphi \leftrightarrow \sigma$. Enumerating all such proofs together with witnesses to the truth of σ yields a recursive enumeration of* True_Γ *without any uniform map. Corollary 6.5(i) is already an instance of this non-uniform argument: for each true Δ_1^{Q} sentence φ, the required Σ_1 sentence σ exists by definition, and its truth is witnessed by a numeral.*

The uniform version stated in Proposition 6.3 is retained because it admits a more compact formulation and because for $\Gamma = \Sigma_1$ and $\Gamma = \Delta_0$, the function f is simply the identity and no search for equivalences is needed. It is therefore the natural criterion when a direct syntactic presentation of the fragment is available.

6.3. Recovery of the Fragment Results

The earlier fragment-specific theorems now follow immediately.

- True_{Σ_1} is recursively enumerable, hence RSI(Σ_1) is consistent.
- RSI(Π_1) is inconsistent by Theorem 4.2. Equivalently, by Theorem 6.2, True_{Π_1} is not recursively enumerable.
- For $n \geqslant 2$, True_{Σ_n} is not recursively enumerable, hence RSI(Σ_n) is inconsistent.

Thus, among the standard existential and universal hierarchy levels, Σ_1 is the unique one for which the set of true sentences at that level is recursively enumerable, and therefore the unique one at which recursive self-identification is possible.

The class Δ_n^{Q} need not itself be syntactically decidable, so the following corollary does not follow directly from Theorem 6.2. Instead, the two cases are established by direct arguments using the soundness of Q and the diagonal mechanism.

Corollary 6.5 (RSI at the Δ_n^{Q} levels, proof-theoretic over Q)**.** *Let Δ_n^{Q} be as in Definition 2.2. Then:*

(i) $\mathrm{RSI}(\Delta_1^{\mathrm{Q}})$ *is consistent.*

(ii) For every $n \geqslant 2$, $\mathrm{RSI}(\Delta_n^{\mathrm{Q}})$ *is inconsistent.*

Proof. (i). Let $\varphi \in \Delta_1^{\mathrm{Q}}$. Then there are a syntactic Σ_1 sentence σ and a syntactic Π_1 sentence π such that

$$\mathrm{Q} \vdash \varphi \leftrightarrow \sigma \qquad \text{and} \qquad \mathrm{Q} \vdash \varphi \leftrightarrow \pi.$$

By soundness of Q, $\mathbb{N} \models \varphi$ iff $\mathbb{N} \models \sigma$. Enumerate all tuples consisting of a sentence φ, sentences $\sigma \in \Sigma_1$ and $\pi \in \Pi_1$, Q-proofs of the two equivalences above, and a witness to the truth of σ. This gives a recursive enumeration of the true Δ_1^{Q} sentences. The corresponding predicate is r.e., factive by soundness of Q, and Δ_1^{Q}-truth-aligned.

(ii). Let $n \geqslant 2$. Every syntactic Π_1 sentence φ belongs to Δ_n^{Q}. Indeed, since z does not occur in φ,

$$\mathrm{Q} \vdash \varphi \leftrightarrow \exists z\, \varphi \qquad \text{and} \qquad \mathrm{Q} \vdash \varphi \leftrightarrow \forall z\, \varphi.$$

If φ is syntactically Π_1, then $\exists z\, \varphi$ is syntactically Σ_2 and hence belongs to Σ_n, while $\forall z\, \varphi$ is syntactically Π_n. Thus $\Pi_1 \subseteq \Delta_n^{\mathrm{Q}}$. Any predicate satisfying $\mathrm{RSI}(\Delta_n^{\mathrm{Q}})$ would therefore be Π_1-truth-aligned, contradicting Theorem 4.2. ☐

Remark 6.6. *By soundness of* Q *(Assumption 2.10), every Δ_1^{Q} sentence is semantically Δ_1: if $\mathrm{Q} \vdash \varphi \leftrightarrow \sigma$ with $\sigma \in \Sigma_1$, then $\mathbb{N} \models \varphi \leftrightarrow \sigma$. The converse fails: Remark 2.3 exhibits a sentence that is semantically Δ_1 but not Δ_1^{Q}. Thus $\Delta_1^{\mathrm{Q}} \subsetneq \{\varphi : \varphi \text{ is semantically } \Delta_1\}$.*

6.4. A Sufficient Fragment Condition for Diagonal Collapse

The recursion-theoretic classification identifies two proof methods with structurally different arguments. We now isolate a sufficient fragment condition under which the diagonal mechanism operates.

Theorem 6.7 (Diagonal Collapse Above Π_1)**.** *Let Γ be a syntactically decidable class of arithmetical sentences. If every syntactic Π_1 sentence belongs to Γ, then no recursively enumerable factive predicate can be Γ-truth-aligned.*

Proof. Let K be any recursively enumerable factive predicate, and suppose K is Γ-truth-aligned.

Apply the diagonal construction to $\neg K(x)$. Since $K(x)$ is Σ_1, the formula $\neg K(x)$ is Π_1, and by Remark 4.1 the resulting fixed point G may be chosen to be a syntactic Π_1 sentence. By hypothesis, $G \in \Gamma$.

If $\mathbb{N} \models K(\ulcorner G \urcorner)$, then by factivity $\mathbb{N} \models G$, and hence by the fixed-point equivalence $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$, a contradiction. So $\mathbb{N} \models \neg K(\ulcorner G \urcorner)$, whence $\mathbb{N} \models G$. Since $G \in \Gamma$ and K is Γ-truth-aligned, we obtain $\mathbb{N} \models K(\ulcorner G \urcorner)$, a contradiction. ☐

Remark 6.8. *The preceding theorem isolates the fragment range in which the classical diagonal argument applies. It is not the general criterion for the consistency or inconsistency of* $\mathrm{RSI}(\Gamma)$*; that is given by Theorem 6.2, which shows that* $\mathrm{RSI}(\Gamma)$ *holds if and only if* True_Γ *is recursively enumerable. The paper thus identifies two proof methods for inconsistency: a diagonal self-reference argument for fragments containing Π_1, and a recursion-theoretic inexhaustibility argument when* True_Γ *is not recursively enumerable.*

Corollary 6.9 (Scope of the standard diagonal mechanism)**.** *Let K be any recursively enumerable predicate. The fixed point produced by applying Theorem 2.9 to $\neg K(x)$ is syntactically Π_1 (Remark 4.1).*

Therefore Theorem 6.7 applies immediately to fragments containing all syntactic Π_1 sentences. If a fragment Γ does not contain all Π_1 sentences, the theorem gives no automatic diagonal obstruction: one would have to verify separately that the particular Π_1 fixed point constructed from $\neg K(x)$ belongs to Γ.

For the literal syntactic classes Σ_n with $n \geqslant 2$, the fixed point produced by the standard construction is in Π_1 form rather than in Σ_n form. The direct diagonal proof used for universal fragments therefore does not apply to these literal syntactic classes, and the inconsistency result is obtained by the complexity argument of Theorem 5.2.

Fragment Γ	**RSI consistent?**	**Reason / Obstruction**	**Proof used here**
Σ_1	Yes	True_{Σ_1} r.e.	—
Δ_1^Q	Yes	$\mathsf{True}_{\Delta_1^Q}$ r.e.	—
Π_n ($n \geqslant 1$)	No	True_{Π_n} not r.e.	Diagonal
Σ_n ($n \geqslant 2$)	No	True_{Σ_n} not r.e.	Complexity
Δ_n^Q ($n \geqslant 2$)	No	$\Pi_1 \subseteq \Delta_n^Q$	Diagonal

7. Conclusion

This paper introduced the notion of *Recursive Self-Identification*: the conjunction of recursive presentability, factivity, and truth alignment over a fragment Γ of the arithmetical hierarchy.

The organizing principle is a complete definability-theoretic classification: for every syntactically decidable fragment Γ, RSI(Γ) holds if and only if the true Γ-theory is recursively enumerable. This yields a uniform account of when a recursively enumerable factive predicate can coincide with truth over a given fragment. The simplicity of this criterion is deliberate: its value lies not in the depth of its proof but in its role as a unifying framework that subsumes all fragment-level results and connects them to the recursion-theoretic properties of truth sets.

The classification has several immediate consequences. It yields the consistency of RSI(Σ_1), where truth is recursively witnessable; the inconsistency of RSI(Π_n) for every $n \geqslant 1$, by the diagonal Π_1 obstruction; and the inconsistency of RSI(Σ_n) for every $n \geqslant 2$, by the non-recursive-enumerability of the corresponding truth sets. It also shows that the Π_1 obstruction is formally equivalent to Gödel's first incompleteness theorem and identifies uniform Q-reducibility to Σ_1 as a sufficient condition for the consistency of RSI(Γ) beyond the standard hierarchy levels.

Taken together, these results show that, among the standard existential and universal levels of the arithmetical hierarchy, Σ_1 is the unique one at which recursive self-identification is possible.

Conceptually, the analysis employs two structurally different proof methods. For fragments containing Π_1, inconsistency is established through a classical diagonal argument. For higher existential fragments, the proof follows a different route: the truth set itself already exceeds recursive enumerability.

These two methods are structurally distinct, as illustrated by the literal syntactic class Σ_2: inconsistency there follows from the complexity-theoretic argument, since the standard diagonal construction does not produce a fixed point in literal Σ_2 form (Corollary 6.9). The RSI framework unifies both methods within a single semantic scheme.

These results should also be read as clarifying one precise aspect of the broader problem of arithmetical knowability. They do not show that some true arithmetical sentences can never be justified by any mathematically compelling argument, possibly of a metamathematical kind. Rather, they show exactly where such recognition cannot be uniformly captured by a single recursively enumerable factive predicate. The analysis therefore identifies the boundary not of knowability as such, but of uniform mechanical semantic recognition. This distinction parallels the one Gödel himself draws in [4], where he separates the broad question of whether every arithmetical truth might admit some convincing metamathematical justification from the stronger requirement of uniform mechanical recognition. The RSI framework addresses only the second question, and its negative results concern that question alone.

Open directions. Several directions remain open. One may weaken recursive enumerability, factivity, or truth alignment, and ask whether analogous classifications survive under these relaxations. Another direction is to add epistemic closure principles or reasoning-theoretic constraints, bringing the framework closer to formal knowledge representation and automated reasoning. Finally, one may study relativized and higher-order versions of RSI, or characterize more precisely which fragments admit a genuinely diagonal proof of inconsistency rather than only a complexity-theoretic one.

Viewed in computational-logic terms, the classification gives a limit theorem for recursively presentable knowledge bases or reasoning mechanisms required to be semantically sound: their possible fragment-level completeness is bounded by recursive enumerability of the corresponding truth set.

Beyond that point, either diagonal self-reference or recursion-theoretic complexity forces failure.

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Declaration on Generative AI

During the preparation of this work, the authors used ChatGPT, powered by GPT-5.5, to paraphrase and reword text, improve writing style, and check grammar and spelling. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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