A proof-search procedure for Kuznetsov-Muravitski Logic

Mauro Ferrari[1,*], Camillo Fiorentini[2] and Paolo Giardini[1]

[1]Dep. of Theoretical and Applied Sciences, Università degli Studi dell'Insubria, Italy
[2]Dep. of Computer Science, Università degli Studi di Milano, Italy

Abstract
We present a sequent calculus and a proof-search procedure for the *Kuznetsov–Muravitsky Logic* KM, an intuitionistic modal logic extending the Intuitionistic Strong Löb Logic with the Cantor–Bendixson axiom. The proof-search procedure is based on a sequent calculus that ensures strong termination and supports countermodel extraction. To support practical experimentation, we provide an implementation of both the proof-search and countermodel-extraction procedures.

1. Introduction

Within the framework of intuitionistic modal logics, systems that consider only the $\Box$ modality and satisfy the *normality principle* (k) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and the *coreflection principle* (c) $\varphi \to \Box\varphi$ have attracted considerable attention due to their connections with provability interpretations [1, 2, 3, 4, 5] and epistemic interpretations [6, 7, 8], as well as their applications in formal verification [9, 10, 11]. Fig. 1 provides a schematic overview of these systems, including their axiomatizations, mutual relationships, and the main references in the literature; see [12] for a systematic review. Adopting the normality and coreflection principles has important consequences for the usual Kripke birelational semantics of intuitionistic modal logics. In the resulting models, called *strong*, the modal accessibility relation R is constrained by the intuitionistic one ($R \subseteq \leq$); as a consequence, the persistence of forcing, which characterizes the $\leq$ relation in intuitionistic Kripke models, also holds for R.

Within this landscape, we aim at developing terminating sequent calculi that support constructive reasoning for all the logics in Fig. 1, within the framework of labelled calculi for intuitionistic logic developed in [13, 14]. In this direction, we have already treated the *minimal coreflection logic* iCK4 [15] (also known as IEL^- in the epistemic setting) and the Intuitionistic Strong Löb Logic iSL [16]. The latter extends iCK4 with the Gödel-Löb axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ and is a natural candidate for the Σ_1-provability logic of Heyting Arithmetic [5, 17].

In this paper, we study the *Kuznetsov–Muravitsky Logic* KM, obtained by extending iSL with the Cantor–Bendixson axiom $\Box\varphi \to ((\psi \to \varphi) \lor \psi)$, and semantically characterized by Kripke models in which the modal accessibility relation R coincides with $<$ (the strict order included in the intuitionistic relation $\leq$). This logic has been extensively investigated in the literature for its connections with the provability logics [18, 19]; for a comprehensive survey on KM, see [20].

We present a sequent calculus $\mathcal{E}$-KM that is strongly terminating, meaning that backward proof search terminates, regardless of the strategy employed. The calculus also supports countermodel extraction: when a formula α is not provable, a model witnessing that α does not belong to KM can be extracted from a failed proof search. Following [13, 14, 16], our approach relies on decorating sequents with suitable labels to control termination and on equipping the calculus with an evaluation relation guiding rule application. Completeness of $\mathcal{E}$-KM is established via a dual refutation calculus, $\mathcal{R}$-KM, which enables countermodel extraction whenever proof search in $\mathcal{E}$-KM fails.

In the literature, some calculi for KM have been proposed. The sequent calculus introduced in [25] is not terminating and is therefore unsuitable for proof search. The calculus presented in [26] is multi-

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*Corresponding author.
mauro.ferrari@uninsubria.it (M. Ferrari); fiorentini@di.unimi.it (C. Fiorentini); pcgiardini@uninsubria.it (P. Giardini)
0000-0002-7904-1125 (M. Ferrari); 0000-0003-2152-7488 (C. Fiorentini); 0009-0004-6857-9095 (P. Giardini)

$(k)\quad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

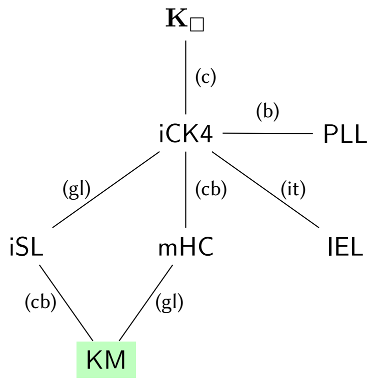
$(c)\quad \varphi \to \Box\varphi$

$(cb)\quad \Box\varphi \to ((\psi \to \varphi) \lor \psi)$

$(it)\quad \Box\varphi \to \neg\neg\varphi$

$(gl)\quad \Box(\Box\varphi \to \varphi) \to \Box\varphi$

$(b)\quad \Box\Box\varphi \to \Box\varphi$



IPL Intuitionistic Propositional Logic

$\mathbf{K}_\Box$ = IPL + (k) Smallest IML ($\Box$-fragment)

iCK4 = IEL$^-$ = $\mathbf{K}_\Box$ + (c) Minimal Coreflection Logic [6, 8, 21, 22]

mHC = iCK4 + (cb) Modalized-Heyting Calculus [11, 23]

IEL = iCK4 + (it) Logic of Intuitionistic Knowledge [6, 21, 22]

iSL = iCK4 + (gl) Intuitionisitic Strong-Löb Logic [1, 2, 4, 5, 11, 24]

KM = iSL + (cb) = mHC + (gl) Kuznetsov-Muravitski Logic [7, 11, 20]

PLL = iCK4 + (b) Lax Logic [9, 10, 11]

Figure 1: Coreflection logics diagram

succedent and introduces two types of implication. Note that the presence of multiple formulas on the right-hand side increases the nondeterminism, and hence the complexity, of proof search. The system proposed in [27] is based on the terminating multi-succedent sequent calculus G4i for intuitionistic propositional logic [28] and lacks the subformula property.

To complement our theoretical results with an applied contribution, we have implemented both the proof search procedure and the countermodel extraction in the Java framework JTabWB [29]; the implementation is available online at [30].

2. The Logic KM

Formulas, denoted by lowercase Greek letters, are built from an enumerable set of propositional variables $\mathcal{V}$, the constant $\bot$ and the connectives $\land$, $\lor$, $\to$ and $\Box$; $\neg\alpha$ is an abbreviation for $\alpha \to \bot$. Let α be a formula and Γ a multiset of formulas. By $\Box\Gamma$ we denote the multiset $\{\Box\alpha \mid \alpha \in \Gamma\}$. By $\mathrm{Sf}(\alpha)$ we denote the set of the subformulas of α, including α itself; $\mathrm{Sf}(\Gamma)$ is the union of the sets $\mathrm{Sf}(\alpha)$, for every α in Γ. The size of α, denoted by $|\alpha|$, is the number of symbols in α; the size of Γ, denoted by $|\Gamma|$, is the sum of the sizes of formulas α in Γ, taking into account their multiplicity.

A KM-*(Kripke) model* $\mathcal{K}$ is a tuple $\langle W, R, r, V\rangle$ where W is a finite non-empty set (worlds), R (the modal accessibility relation) is an irreflexive and transitive relation over W, r (the root) is the minimum element of W w.r.t. R, V (the valuation function) is a map from W to $2^{\mathcal{V}}$ such that V is persistent, namely: $w_0 R w_1$ implies $V(w_0) \subseteq V(w_1)$. Let $\leq$ be the reflexive closure of R; one can easily check that $\leq$ and R satisfy the following properties:

(M1) if $w_0 \leq w_1$ and $w_1 R w_2$, then $w_0 R w_2$ (normality);

(M2) $R \subset \leq$ (strongness);

(M3) $R = <$, namely: wRw' iff $w \leq w'$ and $w \neq w'$.

Conversely, if $\leq$ and R satisfy properties (M1)-(M3), then R is irreflexive and transitive and $\leq$ is the reflexive closure of R. Accordingly, KM-models characterize the logic KM [11, 7, 20, 4]. Given a KM-model $\mathcal{K}$, the forcing relation $\Vdash$ between worlds of $\mathcal{K}$ and formulas is defined as follows:

$w \Vdash p$ iff $p \in V(w), \forall p \in \mathcal{V}$;

$w \nVdash \bot$;

$w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $\mathcal{K}, w \Vdash \beta$;

$w \Vdash \alpha \vee \beta$ iff $w \Vdash \alpha$ or $w \Vdash \beta$;

$w \Vdash \alpha \to \beta$ iff $\forall w' \geq w$, if $w' \Vdash \alpha$ then $w' \Vdash \beta$;

$w \Vdash \Box\alpha$ iff $\forall w' \in W$, if wRw' then $w' \Vdash \alpha$.

One can easily prove that forcing is persistent, i.e.: if $w \Vdash \varphi$ and wRw', then $w' \Vdash \varphi$. Let Γ be a (multi)set of formulas, by $w \Vdash \Gamma$ we mean that $w \Vdash \varphi$, for every φ in Γ.

Since KM-models are finite, the following property holds:

Lemma 1 *Let $\mathcal{K} = \langle W, R, r, V\rangle$ be a* KM*-model and $w \in W$. If $w \nVdash \Box\alpha$, then there exists w' such that wRw' and $w' \nVdash \alpha$ and $w' \Vdash \Box\alpha$.*

The KM-consequence relation $\models_{\mathsf{KM}}$ is defined as follows:

$$\Gamma \models_{\mathsf{KM}} \varphi \quad \text{iff} \quad \forall \mathcal{K}\, \forall w \in \mathcal{K}\ (\, w \Vdash \Gamma \implies w \Vdash \varphi\,)\ .$$

The logic KM is the set of formulas φ such that $\emptyset \models_{\mathsf{KM}} \varphi$. Accordingly, if $\varphi \notin$ KM, there exists a KM-model $\mathcal{K}$ such that $r \nVdash \varphi$, with r the root of $\mathcal{K}$; we call $\mathcal{K}$ a *countermodel* for φ.

As discussed in [20] and originally proved in [7], the logic KM is axiomatized as described in Fig. 1. Note that [7, 20] refer to a different formulation of Kripke semantics, which is nevertheless equivalent to our definition of KM-models.

3. The Sequent Calculus $\mathcal{E}$-KM

Following the approach introduced in [13, 14, 16], our calculus for KM makes use of an evaluation relation to control termination. The evaluation relation is a purely syntactic relation that allows us to make explicit some information that is implicitly contained in a set of formulas.

Definition 2 (Evaluation) *Let Γ be a multiset of formulas and φ a formula. We say that Γ* evaluates *φ, written $\Gamma \triangleright \varphi$, iff φ matches the following BNF:*

$$\varphi \coloneqq \gamma \mid \varphi \wedge \varphi \mid \varphi \vee \alpha \mid \alpha \vee \varphi \mid \alpha \to \varphi \mid \Box\varphi \qquad \textit{with } \gamma \in \Gamma \textit{ and } \alpha \textit{ any formula.}$$

By $\Gamma \triangleright \Delta$ we mean that $\Gamma \triangleright \delta$, for every $\delta \in \Delta$. We state some properties of evaluation which can be easily proved by induction on the structure of φ (see [16]).

Lemma 3

(i) If $\Gamma \triangleright \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \triangleright \varphi$.

(ii) If $\Gamma \cup \Delta \triangleright \varphi$ and $\Gamma' \triangleright \Delta$, then $\Gamma \cup \Gamma' \triangleright \varphi$.

(iii) If $\Gamma \triangleright \varphi$, then $\Gamma \cap \mathrm{Sf}(\varphi) \triangleright \varphi$.

(iv) If $\Gamma \triangleright \varphi$ and $w \Vdash \Gamma$, then $w \Vdash \varphi$.

The sequent calculus $\mathcal{E}$-KM acts on labelled sequents having either the form $\Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ or $\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \delta\,;\,\psi$, where Γ is a multiset of formulas and δ, ψ are formulas. The set Γ is called the *left-hand side* of σ and is also denoted by $\mathrm{lhs}(\sigma)$; the formula δ is called the *right-hand side* of σ; the formula ψ of b-sequents is called *context formula.*

The rules of calculus $\mathcal{E}$-KM are displayed in Fig. 2. The calculus consists of the axiom rules $\mathrm{Ax}^{\triangleright}$ and $L\bot$, together with left/right rules for each propositional connective and right rules for $\Box$; left occurrences of $\Box$ are implicitly treated by the right rules for $\Box$ and by the rules $R\stackrel{\mathrm{u}}{\to}$ and $R\stackrel{\mathrm{b}}{\to}$. With $\Gamma \stackrel{l}{\Rightarrow} \delta\,;\,\psi$ we denote a generic l-sequent, with $l \in \{\mathrm{b}, \mathrm{u}\}$, that can be concretized as an u-sequent or a b-sequent. In particular, if $l = \mathrm{u}$ then φ is missing. As an example, the possible applications of rule $R\wedge$ are:

$$\frac{\Gamma \stackrel{\mathrm{u}}{\Rightarrow} \alpha \qquad \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \beta}{\Gamma \stackrel{\mathrm{u}}{\Rightarrow} \alpha \wedge \beta}\ R\wedge \qquad\qquad \frac{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha\,;\,\psi \qquad \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \beta\,;\,\psi}{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha \wedge \beta\,;\,\psi}\ R\wedge$$

$$\frac{}{\Gamma \stackrel{l}{\Rightarrow} \alpha \,;\, \psi}\ \mathrm{Ax}^{\triangleright} \text{ if } \Gamma \triangleright \alpha \qquad \frac{}{\bot, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\ L\bot$$

$$\frac{\alpha, \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\ L\wedge \qquad \frac{\Gamma \stackrel{l}{\Rightarrow} \alpha \,;\, \psi \qquad \Gamma \stackrel{l}{\Rightarrow} \beta \,;\, \psi}{\Gamma \stackrel{l}{\Rightarrow} \alpha \wedge \beta \,;\, \psi}\ R\wedge$$

$$\frac{\alpha, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta \qquad \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \vee \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\ L\vee \qquad \frac{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha_k \,;\, \alpha_0 \vee \alpha_1}{\Gamma \stackrel{\mathrm{u}}{\Rightarrow} \alpha_0 \vee \alpha_1}\ R^{\mathrm{u}}_{\vee_k} \qquad \frac{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha_k \,;\, \psi}{\Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha_0 \vee \alpha_1 \,;\, \psi}\ R^{\mathrm{b}}_{\vee_k}$$

$$\frac{\alpha \to \beta, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha \,;\, \delta \qquad \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\alpha \to \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\ L\to \qquad \frac{\Gamma \stackrel{l}{\Rightarrow} \beta \,;\, \psi}{\Gamma \stackrel{l}{\Rightarrow} \alpha \to \beta \,;\, \psi}\ R_{\stackrel{\triangleright}{\to}} \text{ if } \Gamma \triangleright \alpha$$

$$\frac{\alpha, \Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \beta \quad \alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \beta}{\Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha \to \beta}\ R_{\stackrel{\mathrm{u}}{\to}} \text{ if } \Gamma \cup \Box\Theta \not\triangleright \alpha \qquad \frac{\alpha, \Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \psi \quad \alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \beta}{\Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \alpha \to \beta \,;\, \psi}\ R_{\stackrel{\mathrm{b}}{\to}} \text{ if } \Gamma \cup \Box\Theta \not\triangleright \alpha$$

$$\frac{\Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \Box\alpha}\ R^{\Box}_{\mathrm{u}} \qquad \frac{\Box\alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha \,;\, \psi}\ R^{\Box}_{\mathrm{b}} \text{ if } \Gamma \cup \Box\Theta \not\triangleright \Box\alpha$$

Figure 2: The calculus $\mathcal{E}$-KM ($l \in \{\mathrm{b}, \mathrm{u}\}$, $k \in \{0, 1\}$).

For calculi and derivations we use the definitions and notations of [31]. Applications of rules are depicted as trees, called *$\mathcal{E}$-trees*, whose nodes are sequents. Moreover, given an $\mathcal{E}$-tree $\mathcal{D}$, we call the *root sequent of $\mathcal{D}$* the sequent σ at the root of $\mathcal{D}$, and the *root rule of $\mathcal{D}$* the rule having σ as its conclusion. An *$\mathcal{E}$-derivation* is an $\mathcal{E}$-tree in which every leaf is an *axiom sequent*, that is, a sequent obtained by the application of a zero-premise rule. A sequent σ is provable in $\mathcal{E}$-KM, denoted $\vdash_{\mathcal{E}\text{-KM}} \sigma$, if there exists an $\mathcal{E}$-derivation whose root sequent is σ.

The calculus $\mathcal{E}$-KM is oriented to backward proof search, where rules are applied bottom-up. If the conclusion of a rule has label b (where b stands for *blocked* and u for unblocked), the (bottom-up) application of left rules is blocked. There are three rules for right implication, namely $R_{\stackrel{\triangleright}{\to}}$, $R_{\stackrel{\mathrm{u}}{\to}}$ and $R_{\stackrel{\mathrm{b}}{\to}}$; the choice between them is settled by the evaluation relation $\triangleright$ and the label of the conclusion. Right $\Box$-formulas are handled by rules $R^{\Box}_{\mathrm{u}}$ and $R^{\Box}_{\mathrm{b}}$; here the choice is determined by the label of the conclusion. We remark that if $\sigma = \Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha \,;\, \psi$ and $\Gamma \cup \Box\Theta \triangleright \Box\alpha$, then σ is an axiom sequent (see rule $\mathrm{Ax}^{\triangleright}$) and an application of rule $R^{\Box}_{\mathrm{b}}$ to σ is prevented by the side condition of $R^{\Box}_{\mathrm{b}}$. Rule $R^{\Box}_{\mathrm{b}}$ introduces in the left-hand side of the premise a copy of the main formula $\Box\alpha$ (also called *diagonal formula*); in rule $R^{\Box}_{\mathrm{u}}$ such a duplication is not required. In backward proof search, a b-sequent starts the construction of a branch only containing b-sequents, where only right rules are applied. This phase ends either when an axiom sequent is obtained or when no rule can be applied or when one of the rules turning a label b into u is applied (namely, rules $R_{\stackrel{\mathrm{b}}{\to}}$ and $R^{\Box}_{\mathrm{b}}$).

Example 4 Let $\alpha = \Box p \to ((q \to p) \vee q)$ be an instance of Cantor–Bendixson axiom and $\beta = (\Box p \to p) \to p$ an instance of the *strong Gödel Löb axiom* [4]; in Fig. 3 we show an $\mathcal{E}$-derivation of the u-sequent $\sigma_0 = \stackrel{\mathrm{u}}{\Rightarrow} \alpha \wedge \beta$. Sequents in the derivation are marked with an index (n) so that they can be referred to as σ_n. In the applications of rule $R_{\stackrel{\mathrm{u}}{\to}}$ having conclusions σ_1 and σ_6 respectively, the left and right premises coincide, hence only one of them is displayed. This example illustrates some of the key features of $\mathcal{E}$-KM. In particular, it shows how the presence of the context formula in the premise σ_3 of rule $R^{\mathrm{u}}_{\vee_0}$, together with the treatment of context formulas in rule $R_{\stackrel{\mathrm{b}}{\to}}$, is crucial for deriving the Cantor–Bendixson axiom. Both disjuncts of $(p \to q) \vee q$ are essential. σ_5 is an axiom due to the first disjunct which is stored in the left hand side of σ_3, whereas σ_4 is an axiom due to the second disjunct, which is stored in the context of σ_3 and moved to the right-hand side by the application of $R_{\stackrel{\mathrm{b}}{\to}}$. An analogous result can be obtained using a standard multi-succedent calculus, as shown in [27]. However, the presence of multiple formulas on the right-hand side increases the nondeterminism, and hence the complexity, of proof search. As for the role of b labels, note that in backward proof search σ_8 is

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathrm{Ax}^{\triangleright}$}
\UnaryInfC{$q, \Box p \stackrel{\mathrm{u}}{\Rightarrow} (q \to p) \vee q_{(4)}$}
\AxiomC{}
\RightLabel{$\mathrm{Ax}^{\triangleright}$}
\UnaryInfC{$q, p \stackrel{\mathrm{u}}{\Rightarrow} p_{(5)}$}
\RightLabel{$R\!\xrightarrow{\mathrm{b}}$}
\BinaryInfC{$\Box p \stackrel{\mathrm{b}}{\Rightarrow} q \to p \,;\, (q \to p) \vee q_{(3)}$}
\RightLabel{$R^{\mathrm{u}}_{\vee_0}$}
\UnaryInfC{$\Box p \stackrel{\mathrm{u}}{\Rightarrow} (q \to p) \vee q_{(2)}$}
\RightLabel{$R\!\xrightarrow{\mathrm{u}}$}
\UnaryInfC{$\stackrel{\mathrm{u}}{\Rightarrow} \Box p \to ((q \to p) \vee q)_{(1)}$}
\AxiomC{}
\RightLabel{$\mathrm{Ax}^{\triangleright}$}
\UnaryInfC{$\Box p, \Box p \to p \stackrel{\mathrm{b}}{\Rightarrow} \Box p \,;\, p_{(10)}$}
\AxiomC{}
\RightLabel{$\mathrm{Ax}^{\triangleright}$}
\UnaryInfC{$\Box p, p \stackrel{\mathrm{u}}{\Rightarrow} p_{(11)}$}
\RightLabel{$L\!\to$}
\BinaryInfC{$\Box p, \Box p \to p \stackrel{\mathrm{u}}{\Rightarrow} p_{(9)}$}
\RightLabel{$R^{\Box}_{\mathrm{b}}$}
\UnaryInfC{$\Box p \to p \stackrel{\mathrm{b}}{\Rightarrow} \Box p \,;\, p_{(8)}$}
\AxiomC{}
\RightLabel{$\mathrm{Ax}^{\triangleright}$}
\UnaryInfC{$p \stackrel{\mathrm{u}}{\Rightarrow} p_{(12)}$}
\RightLabel{$L\!\to$}
\BinaryInfC{$\Box p \to p \stackrel{\mathrm{u}}{\Rightarrow} p_{(7)}$}
\RightLabel{$R\!\xrightarrow{\mathrm{u}}$}
\UnaryInfC{$\stackrel{\mathrm{u}}{\Rightarrow} (\Box p \to p) \to p_{(6)}$}
\RightLabel{$R\wedge$}
\BinaryInfC{$\stackrel{\mathrm{u}}{\Rightarrow} (\Box p \to ((q \to p) \vee q)) \wedge ((\Box p \to p) \to p)_{(0)}$}
\end{prooftree}
$$

Figure 3: The $\mathcal{E}$-derivation $\mathcal{D}$ of $\sigma_0 = \stackrel{\mathrm{u}}{\Rightarrow} (\Box p \to ((q \to p) \vee q)) \wedge ((\Box p \to p) \to p)$ (see Ex. 4).

obtained by an application of rule $L\!\to$ to σ_7. The label b in σ_8 is crucial to block further applications of rule $L\!\to$, which would otherwise generate an infinite branch. The sequent σ_9 is obtained by applying rule $R^{\Box}_{\mathrm{b}}$ to σ_8. In this case, the key feature is the presence of the diagonal formula $\Box p$; without it, σ_9 would be $\Box p \to p \stackrel{\mathrm{u}}{\Rightarrow} p$ and, after applying $L\!\to$ (the only applicable rule), the left premise would be $\sigma'_{10} = \Box p \to p \stackrel{\mathrm{b}}{\Rightarrow} \Box p \,;\, p$, yielding a loop (i.e., $\sigma'_{10} = \sigma_8$). ◊

The calculus $\mathcal{E}$-KM is *terminating*, that is: there exists a well-founded relation $\prec$ such that every rule of $\mathcal{E}$-KM is decreasing with respect to $\prec$. More precisely, for every application ρ of a rule of $\mathcal{E}$-KM, if σ is the conclusion of ρ and σ' is any premise of ρ, then $\sigma' \prec \sigma$. In the following we associate to the right part of a sequent $\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi$ the formula $\delta \vee \psi$; if $l = \mathrm{u}$ we stipulate that $\delta \vee \psi$ coincides with δ (the formula ψ is missing).

We state the main properties of $\mathcal{E}$-KM.

Theorem 5

(i) $\mathcal{E}$-KM *has the subformula property.*

(ii) $\mathcal{E}$-KM *is terminating.*

(iii) $\vdash_{\mathcal{E}\text{-KM}} \Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi$ *implies* $\Gamma \models_{\mathsf{KM}} \delta \vee \psi$ *(Soundness).*

(iv) $\Gamma \models_{\mathsf{KM}} \delta$ *implies* $\vdash_{\mathcal{E}\text{-KM}} \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ *(Completeness).*

We remark that in soundness l is any label; instead, in completeness the label is set to u. For instance, since $p \vee q \models_{\mathsf{KM}} q \vee p$, completeness guarantees that the u-sequent $\sigma^{\mathrm{u}} = p \vee q \stackrel{\mathrm{u}}{\Rightarrow} q \vee p$ is provable in $\mathcal{E}$-KM. A $\mathcal{E}$-derivation of σ^{u} is obtained by first (upwards) applying rule $L\vee$ to σ^{u} and then one of the rules $R^{\mathrm{u}}_{\vee_0}$ or $R^{\mathrm{u}}_{\vee_1}$; if we first apply a right rule, we are stuck (e.g., if we apply $R^{\mathrm{u}}_{\vee_0}$ to σ^{u}, we get the unprovable sequent $p \vee q \stackrel{\mathrm{b}}{\Rightarrow} q \,;\, p \vee q$). On the contrary, the b-sequent $p \vee q \stackrel{\mathrm{b}}{\Rightarrow} q \vee p \,;\, \psi$ is not provable in $\mathcal{E}$-KM, since the label b inhibits the application of rule $L\vee$ and forces the application of a right rule.

The subformula property of $\mathcal{E}$-KM can be easily checked by inspecting the rules; termination is discussed below and completeness in the next section. Soundness can be proved relying on the fact that rules preserve the consequence relation $\models_{\mathsf{KM}}$:

Lemma 6 *Let ρ be an instance of a rule of $\mathcal{E}$-KM and let $\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi$ be the conclusion of ρ. If, for every premise $\Gamma' \stackrel{l'}{\Rightarrow} \delta' \,;\, \psi'$ of ρ, it holds that $\Gamma' \models_{\mathsf{KM}} \delta' \vee \psi'$, then $\Gamma \models_{\mathsf{KM}} \delta \vee \psi$.*

Proof. The proof goes by a case analysis on ρ. We only consider the most relevant cases.

The case of the axiom rules $\mathrm{Ax}^{\triangleright}$ immediately follows from Lemma 3(iv), while the case of $L\bot$ trivially holds since $\bot$ occurs in the left-hand side.

Let ρ be an instance of $R\!\xrightarrow{\mathrm{b}}$, i.e.:

$$
\frac{\alpha, \Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \psi \qquad \alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \beta}{\Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \alpha \to \beta \,;\, \psi}\; R\!\xrightarrow{\mathrm{b}} \qquad \Gamma \cup \Box\Theta \not\triangleright \alpha
$$

We assume $\Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} (\alpha \to \beta) \vee \psi$ and we show that either $\{\alpha\} \cup \Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} \psi$ or $\{\alpha\} \cup \Gamma \cup \Theta \not\models_{\mathsf{KM}} \beta$. Since $\Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} (\alpha \to \beta) \vee \psi$, there exists a KM-model $\mathcal{K} = \langle W, R, r, V \rangle$ and a world $w \in W$ such that $w \Vdash \Gamma \cup \Box\Theta$, $w \nVdash \psi$, and $w \nVdash \alpha \to \beta$. The latter implies that there exists a world w' such that $w \leq w'$, $w' \Vdash \alpha$, and $w' \nVdash \beta$. If $w = w'$, we immediately get $\{\alpha\} \cup \Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} \psi$. Otherwise, it holds that wRw'; since $w \Vdash \Box\Theta$, we get $w' \Vdash \Theta$. It follows that $w' \Vdash \{\alpha\} \cup \Gamma \cup \Theta$ and $w' \nVdash \beta$, hence $\{\alpha\} \cup \Gamma \cup \Theta \not\models_{\mathsf{KM}} \beta$.

Let ρ be in instance of $R^{\Box}_{\mathrm{b}}$, i.e.:

$$\frac{\Box\alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \Box\alpha \,;\, \psi}\ R^{\Box}_{\mathrm{b}} \qquad \Gamma \cup \Box\Theta \not\rhd \Box\alpha$$

We assume $\Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} \Box\alpha \vee \psi$ and we show $\{\Box\alpha\} \cup \Gamma \cup \Theta \not\models_{\mathsf{KM}} \alpha$. Since $\Gamma \cup \Box\Theta \not\models_{\mathsf{KM}} \Box\alpha \vee \psi$, there exists a KM-model $\mathcal{K} = \langle W, R, r, V \rangle$ and $w \in W$ such that $w \Vdash \Gamma \cup \Box\Theta$ and $w \nVdash \Box\alpha \vee \psi$. Since $w \nVdash \Box\alpha$, by Lemma 1, there exists w' such that wRw' and $w' \Vdash \Box\alpha$ and $w' \nVdash \alpha$. Since $w' \Vdash \Gamma \cup \Theta$, we conclude $\{\Box\alpha\} \cup \Gamma \cup \Theta \not\models_{\mathsf{KM}} \alpha$. $\Box$

To prove termination of $\mathcal{E}$-KM, we introduce a suitable well-founded relation $\prec_{\mathrm{bu}}$ on labelled sequents. The main difficulty arises from rule $L\!\to$, which admits applications where the sequent $\alpha \to \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \alpha$ is the conclusion and $\alpha \to \beta, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha \,;\, \alpha$ is the left premise. The labeling mechanism is designed precisely to prevent such configurations from causing loops in backward proof search (see, e.g., [16] for details). So, let σ and σ' be the conclusion and the left premise of an application of rule $L\!\to$, we stipulate that $\sigma' \prec_{\mathrm{bu}} \sigma$, since σ' has label b while σ has label u; thus, b is assigned a lower weight than u. Now, we need a way out to accommodate rules $R\xrightarrow{\mathrm{b}}$ and $R^{\Box}_{\mathrm{b}}$ that, read bottom-up, switch b with u. In both cases, we observe that the left-hand side of the premises evaluate a new formula; e.g., in the application of rule $R\xrightarrow{\mathrm{b}}$ having premises $\alpha, \Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \psi$ and $\alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \beta$ and conclusion $\Gamma, \Box\Theta \stackrel{\mathrm{b}}{\Rightarrow} \alpha \to \beta \,;\, \psi$, it holds that $\Gamma \cup \Box\Theta \not\rhd \alpha$ (side condition) and $\Gamma \cup \Box\Theta \cup \{\alpha\} \rhd \alpha$ and $\Gamma \cup \Theta \cup \{\alpha\} \rhd \alpha$ (definition of $\rhd$); this suggests that we can exploit the evaluation relation to define $\prec_{\mathrm{bu}}$ in these cases. Let Ev be defined as follows:

$$\mathrm{Ev}(\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi) \;=\; \{\, \varphi \mid \varphi \in \mathrm{Sf}(\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi) \text{ and } \Gamma \rhd \varphi \,\}$$

where $\mathrm{Sf}(\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi) = \mathrm{Sf}(\Gamma \cup \{\delta, \psi\})$. Note that $\mathrm{Ev}(\sigma) \subseteq \mathrm{Sf}(\sigma)$. Finally, to treat the other rules we have to take into account the size of a sequents, where $|\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi| = |\Gamma| + |\delta| + |\psi|$. This leads to the definition of $\prec_{\mathrm{bu}}$:

Definition 7 **($\prec_{\mathrm{bu}}$)** $\sigma' \prec_{\mathrm{bu}} \sigma$ *iff one of the following conditions holds:*

(a) $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$*;*

(b) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$*;*

(c) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ *and* $\mathrm{label}(\sigma') = \mathrm{b}$ *and* $\mathrm{label}(\sigma) = \mathrm{u}$*;*

(d) $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ *and* $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$ *and* $\mathrm{label}(\sigma') = \mathrm{label}(\sigma)$ *and* $|\sigma'| < |\sigma|$.

Proposition 8 *The relation $\prec_{\mathrm{bu}}$ is well-founded.*

Proof. Assume, by contradiction, that there is an infinite descending chain of the kind $\ldots \prec_{\mathrm{bu}} \sigma_1 \prec_{\mathrm{bu}} \sigma_0$. Since $\mathrm{Sf}(\sigma_0) \supseteq \mathrm{Sf}(\sigma_1) \supseteq \ldots$ and $\mathrm{Sf}(\sigma_0)$ is finite, the sets $\mathrm{Sf}(\sigma_j)$ eventually stabilize, namely: there is $k \geq 0$ such that $\mathrm{Sf}(\sigma_j) = \mathrm{Sf}(\sigma_k)$ for every $j \geq k$. Since $\mathrm{Ev}(\sigma_j) \subseteq \mathrm{Sf}(\sigma_j)$, we get $\mathrm{Ev}(\sigma_k) \subseteq \mathrm{Ev}(\sigma_{k+1}) \subseteq \ \ldots\ \subseteq \mathrm{Sf}(\sigma_k)$. Since $\mathrm{Sf}(\sigma_k)$ is finite, there is $m \geq k$ such that $\mathrm{Ev}(\sigma_j) = \mathrm{Ev}(\sigma_m)$ for every $j \geq m$. This implies that there exists $n \geq m$ such that all the sequents $\sigma_n, \sigma_{n+1}, \ldots$ have the same label; accordingly $|\sigma_n| > |\sigma_{n+1}| > |\sigma_{n+2}| > \ldots \geq 0$, a contradiction. We conclude that $\prec_{\mathrm{bu}}$ is well-founded. $\Box$

The next lemma shows that, given a rule $\mathcal{R}$ of $\mathcal{E}$-KM, the set of formulas evaluated by the left-hand side of the conclusion is a subset of the set of formulas evaluated by the left-hand side of any of the premises.

Lemma 9 *Let ρ be an instance of a rule of $\mathcal{E}$-KM, let σ be the conclusion of ρ and σ' any of its premises. For every formula φ, if $\mathrm{lhs}(\sigma) \rhd \varphi$ then $\mathrm{lhs}(\sigma') \rhd \varphi$.*

Proof. The assertion can be proved by applying Lemma 3. For instance, let us consider an instance of the rule $R_{\mathrm{u}}^{\Box}$:

$$\frac{\Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\Gamma, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \Box\alpha} \; R_{\mathrm{u}}^{\Box}$$

and let us assume that $\Gamma \cup \Box\Theta \rhd \varphi$. Since $\Theta \rhd \Box\Theta$, by Lemma 3(ii) we get $\Gamma \cup \Theta \rhd \varphi$. $\Box$

Proposition 10 *Every rule of the calculus $\mathcal{E}$-KM is decreasing w.r.t. $\prec_{\mathrm{bu}}$.*

Proof. Let ρ be an instance of any of the rules of $\mathcal{E}$-KM and let σ and σ' be the conclusion and one of the premises of ρ. Note that, by the subformula property, $\mathrm{Sf}(\sigma') \subseteq \mathrm{Sf}(\sigma)$; moreover, if $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$, by Lemma 9 we get $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. We can prove $\sigma' \prec_{\mathrm{bu}} \sigma$ by a case analysis on ρ; we only detail two significant cases.

Let ρ be an instance of rule $L\!\to$:

$$\frac{\sigma' = \alpha \to \beta, \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha \,;\, \delta \qquad \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}{\sigma = \alpha \to \beta, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta} \; L\!\to$$

If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (a) of Def. 7. Otherwise, it holds that $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supseteq \mathrm{Ev}(\sigma)$. If $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (b) of Def. 7; otherwise, it holds that $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') = \mathrm{Ev}(\sigma)$, hence $\sigma' \prec_{\mathrm{bu}} \sigma$ follows by point (c) of Def. 7.

Let ρ be an instance of rule $R_{\mathrm{b}}^{\Box}$:

$$\frac{\sigma' \;=\; \Box\alpha, \Gamma, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \alpha}{\sigma \;=\; \Gamma, \Box\Theta \stackrel{l}{\Rightarrow} \Box\alpha \,;\, \psi} \; R_{\mathrm{b}}^{\Box} \qquad \Gamma \cup \Box\Theta \not\rhd \Box\alpha$$

If $\mathrm{Sf}(\sigma') \subset \mathrm{Sf}(\sigma)$, then $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (a) of Def. 7. Otherwise, $\mathrm{Sf}(\sigma') = \mathrm{Sf}(\sigma)$ and $\mathrm{Ev}(\sigma') \supset \mathrm{Ev}(\sigma)$ since $\Box\alpha \in \mathrm{Ev}(\sigma')$ and, by the side condition, $\Box\alpha \notin \mathrm{Ev}(\sigma)$. Hence $\sigma' \prec_{\mathrm{bu}} \sigma$ by point (b) of Def. 7. $\Box$

By Prop. 8 and 10, we conclude that the calculus $\mathcal{E}$-KM is terminating (point (iv) of Th. 5).

4. The Refutation Calculus $\mathcal{R}$-KM

A common technique to prove the completeness of a sequent calculus consists in showing that, whenever a sequent σ is not provable in it, then a countermodel for σ can be built; we prove the completeness of $\mathcal{E}$-KM according with this plan. Following the ideas in [32, 13, 14, 33], we formalize the notion of "non-provability in $\mathcal{E}$-KM" by introducing the refutation calculus $\mathcal{R}$-KM, a dual calculus to $\mathcal{E}$-KM.

Sequents of $\mathcal{R}$-KM, called *antisequents*, have the same form of the sequents of $\mathcal{E}$-KM where the sequent symbol $\Rightarrow$ is replaced by $\not\Rightarrow$. The notations and terminology we introduced for sequents is extended to antisequents. Given a sequent σ (an antisequent, respectively) we denote with $\overline{\sigma}$ the corresponding antisequent (sequent, respectively). Intuitively, a derivation in $\mathcal{R}$-KM of $\Gamma \stackrel{l}{\not\Rightarrow} \delta \,;\, \psi$ witnesses that the sequent $\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi$ is refutable, that is, not provable, in $\mathcal{E}$-KM.

Henceforth, Γ^{at} denotes a finite multiset of propositional variables, $\Gamma^{\to}$ denotes a finite multiset of $\to$-formulas (i.e., formulas of the kind $\alpha \to \beta$). The axioms of $\mathcal{R}$-KM are the *irreducible antisequents*, namely the antisequents $\Gamma \stackrel{l}{\not\Rightarrow} \delta \,;\, \psi$ such that the corresponding dual sequents $\Gamma \stackrel{l}{\Rightarrow} \delta \,;\, \psi$ are not the conclusion of any of the rules of $\mathcal{E}$-KM; formally:

Definition 11 *An antisequent σ is* irreducible *iff $\sigma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \stackrel{l}{\not\Rightarrow} \delta \,;\, \psi$ and: (i) $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$; (ii) $l = \mathrm{b}$ or $\Gamma^{\to} = \emptyset$.*

Γ^{at} is a multiset of propositional variables, $\Gamma^{\rightarrow}$ is a multiset of $\rightarrow$-formulas

$$\frac{}{\sigma}\,\mathrm{Irr}\quad \text{if } \sigma \text{ is irreducible} \qquad \frac{\alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,\mathsf{W} \qquad \frac{\alpha, \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha \wedge \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L\wedge \qquad \frac{\Gamma \overset{l}{\not\Rightarrow} \alpha_k \,;\, \psi}{\Gamma \overset{l}{\not\Rightarrow} \alpha_0 \wedge \alpha_1 \,;\, \psi}\,R\wedge_k$$

$$\frac{\alpha_k, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha_0 \vee \alpha_1, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L\vee_k \qquad \frac{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \psi \qquad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \beta \,;\, \psi}{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \vee \beta \,;\, \psi}\,R\vee \qquad \frac{\beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\alpha \rightarrow \beta, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,L\rightarrow$$

$$\frac{\Gamma \overset{l}{\not\Rightarrow} \beta \,;\, \psi}{\Gamma \overset{l}{\not\Rightarrow} \alpha \rightarrow \beta \,;\, \psi}\,R\overset{\triangleright}{\rightarrow}\quad \Gamma \triangleright \alpha \qquad \frac{\alpha, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \alpha \rightarrow \beta}\,R\overset{\mathrm{u}}{\rightarrow} \qquad \frac{\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \alpha \rightarrow \beta \,;\, \psi}\,R\overset{\mathrm{b}}{\rightarrow}$$

$$\frac{\Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha \,;\, \psi}\,R^{\Box}_{\mathrm{b}} \qquad \frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \delta \,\}_{\alpha\rightarrow\beta \in \Gamma^{\rightarrow}}}{\underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \delta}\,\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}\ \begin{array}{l}\Gamma^{\rightarrow} \neq \emptyset \\ \delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}\end{array}$$

$$\frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \delta_0 \vee \delta_1 \,\}_{\alpha\rightarrow\beta \in \Gamma^{\rightarrow}} \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_0 \,;\, \delta_0 \vee \delta_1 \quad \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \delta_1 \,;\, \delta_0 \vee \delta_1}{\underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \delta_0 \vee \delta_1}\,\mathrm{S}^{\vee}_{\mathrm{u}}$$

$$\frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \gamma \rightarrow \delta \,\}_{\alpha\rightarrow\beta \in \Gamma^{\rightarrow}} \quad \gamma, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \gamma \rightarrow \delta}\,\mathrm{S}^{\rightarrow}_{\mathrm{u}} \qquad \frac{\{\, \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \Box\delta \,\}_{\alpha\rightarrow\beta \in \Gamma^{\rightarrow}} \quad \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta}{\underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta}_{\Gamma} \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta}\,\mathrm{S}^{\Box}_{\mathrm{u}}$$

Figure 4: The refutation calculus $\mathcal{R}$-KM ($l \in \{\mathrm{b}, \mathrm{u}\}$, $k \in \{0, 1\}$).

The rules of $\mathcal{R}$-KM are displayed in Fig. 4. The definitions of $\mathcal{R}$-trees and $\mathcal{R}$-derivations for the calculus $\mathcal{R}$-KM are similar to the corresponding notions of the calculus $\mathcal{E}$-KM. An antisequent σ is provable in $\mathcal{R}$-KM, and we write $\vdash_{\mathcal{R}\text{-KM}} \sigma$, if there exists an $\mathcal{R}$-derivation having the antisequent σ as root. Rules $\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$, $\mathrm{S}^{\vee}_{\mathrm{u}}$, $\mathrm{S}^{\rightarrow}_{\mathrm{u}}$ and $\mathrm{S}^{\Box}_{\mathrm{u}}$ are called *Succ rules* since in countermodel construction generate new worlds. In Succ rules, the notation $\{\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \psi\}_{\alpha\rightarrow\beta\in\Gamma^{\rightarrow}}$ means that, for every $\alpha \rightarrow \beta \in \Gamma^{\rightarrow}$, the b-antisequent $\Gamma \overset{\mathrm{b}}{\not\Rightarrow} \alpha \,;\, \psi$ is a premise of the rule. Note that all of the Succ rules have at least one premise (in rule $\mathrm{S}^{\mathrm{At}}_{\mathrm{u}}$ this is imposed by the condition $\Gamma^{\rightarrow} \neq \emptyset$). The next theorem, proved below, states the soundness of $\mathcal{R}$-KM:

Theorem 12 (Soundness of $\mathcal{R}$-KM) *If* $\vdash_{\mathcal{R}\text{-KM}} \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$*, then* $\Gamma \not\models_{\mathsf{KM}} \delta$*.*

Example 13 Let $\varphi = \Box(\Box p \vee \Box q) \rightarrow ((p \rightarrow \Box q) \vee (q \rightarrow \Box p))$. Fig. 5 displays the $\mathcal{R}$-derivation $\mathcal{D}$ of the u-antisequent $\sigma_0 = \overset{\mathrm{u}}{\not\Rightarrow} \varphi$. The (backward) application of rule $\mathrm{S}^{\vee}_{\mathrm{u}}$ to σ_1 has only two premises since the left-hand side of σ_1 does not contain implicative formulas. The same holds for the applications of $\mathrm{S}^{\Box}_{\mathrm{u}}$ acting on σ_4 and σ_8. By Th. 12, we get $\not\models_{\mathsf{KM}} \varphi$, namely $\varphi \notin \mathsf{KM}$. $\Diamond$

Countermodel extraction. A KM-model $\mathcal{K}$ with root r is a *countermodel for* $\sigma = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ iff $r \Vdash \Gamma$ and $r \nVdash \delta$; thus $\mathcal{K}$ certifies that $\Gamma \not\models_{\mathsf{KM}} \delta$. Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of an u-antisequent σ^{u}_0; we show that from $\mathcal{D}$ we can extract a countermodel $\mathrm{Mod}(\mathcal{D})$ for σ^{u}_0.

An u-antisequent σ of $\mathcal{D}$ is *prime* iff σ is the conclusion of rule Irr or of a Succ rule. We introduce the relations $\preceq$ and $\prec$ between antisequents occurring in $\mathcal{D}$:

- $\sigma_1 \prec \sigma_2$ iff σ_1 and σ_2 belong to the same branch of $\mathcal{D}$ and σ_1 is below σ_2;
- $\sigma_1 \preceq \sigma_2$ iff either $\sigma_1 = \sigma_2$ or $\sigma_1 \prec \sigma_2$.

We define $\mathrm{Mod}(\mathcal{D})$ as the structure $\langle W, R, \sigma^{\mathrm{u}}_r, V\rangle$ where:

- W is the set of the prime antisequents of $\mathcal{D}$;

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p, p \overset{\mathrm{u}}{\not\Rightarrow} q_{(5)}\ \bullet}\ \mathrm{Irr}}{p, \Box p \overset{\mathrm{u}}{\not\Rightarrow} \Box q_{(4)}\ \bullet}\ \mathrm{S}^{\Box}_{\mathrm{u}}}{p, \Box p \lor \Box q \overset{\mathrm{u}}{\not\Rightarrow} \Box q_{(3)}}\ L{\lor_0}}{\Box(\Box p \lor \Box q) \overset{\mathrm{b}}{\not\Rightarrow} p \to \Box q\, ;\, \psi_{(2)}}\ R\overset{\mathrm{b}}{\to} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{q, q \overset{\mathrm{u}}{\not\Rightarrow} p_{(9)}\ \bullet}\ \mathrm{Irr}}{q, \Box q \overset{\mathrm{u}}{\not\Rightarrow} \Box p_{(8)}\ \bullet}\ \mathrm{S}^{\Box}_{\mathrm{u}}}{q, \Box p \lor \Box q \overset{\mathrm{u}}{\not\Rightarrow} \Box p_{(7)}}\ L{\lor_1}}{\Box(\Box p \lor \Box q) \overset{\mathrm{b}}{\not\Rightarrow} q \to \Box p\, ;\, \psi_{(6)}}\ R\overset{\mathrm{b}}{\to}}{\Box(\Box p \lor \Box q) \overset{\mathrm{u}}{\not\Rightarrow} \psi_{(1)}\ \bullet}\ \mathrm{S}^{\lor}_{\mathrm{u}}}{\overset{\mathrm{u}}{\not\Rightarrow} \Box(\Box p \lor \Box q) \to \underbrace{((p \to \Box q) \lor (q \to \Box p))}_{\psi}{}_{(0)}}\ R\overset{\mathrm{u}}{\to}
$$

Figure 5: The $\mathcal{R}$-derivation $\mathcal{D}$ of $\sigma_0 = \overset{\mathrm{u}}{\not\Rightarrow} \Box(\Box p \lor \Box q) \to ((p \to \Box q) \lor (q \to \Box p))$ (see Ex. 13).

- R is the restriction of $\prec$ to W;
- σ_r^{u} is the $\prec$-minimum prime antisequent of $\mathcal{D}$;
- $V(\,\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta\,) \;=\; \Gamma \cap \mathcal{V}$.

It is easy to check that $\mathrm{Mod}(\mathcal{D})$ is a KM-model. In particular, the root σ_r^{u} exists since the antisequent at the root of $\mathcal{D}$ has label u. Persistence of V holds since the propositional variables in the left-hand side of a sequent are preserved by backward application of rules of $\mathcal{R}$-KM.

We introduce a *canonical map* Ψ between the u-antisequents of $\mathcal{D}$ and the worlds of $\mathrm{Mod}(\mathcal{D})$:

- $\Psi(\sigma^{\mathrm{u}}) = \sigma_p^{\mathrm{u}}$ iff σ_p^{u} is the $\preceq$-minimum prime antisequent σ such that $\sigma^{\mathrm{u}} \preceq \sigma$.

One can easily check that Ψ is well-defined and $\Psi(\sigma_p) = \sigma_p$, for every prime σ_p. We state the main properties of $\mathrm{Mod}(\mathcal{D})$.

Theorem 14 *Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of an* u*-antisequent σ_0^{u}.*

(i) For every u*-antisequent $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$, $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.*

(ii) $\mathrm{Mod}(\mathcal{D})$ is a countermodel for σ_0^{u}.

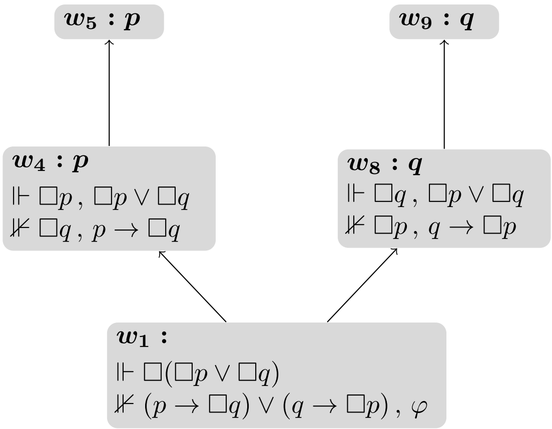
Point (ii) follows from (i) and the fact that $\Psi(\sigma_0^{\mathrm{u}})$ is the root of $\mathrm{Mod}(\mathcal{D})$. The proof of (i) is deferred below. We remark that point (ii) of Th. 14 immediately implies the soundness of $\mathcal{R}$-KM (Th. 12).

Example 15 In Fig. 6 we display the countermodel $\mathrm{Mod}(\mathcal{D})$ for the formula φ of Ex. 13 extracted from the $\mathcal{R}$-derivation $\mathcal{D}$ of Fig. 5, where prime sequents are labelled by the $\bullet$ symbol; the figure also reports the canonical map Ψ. The worlds of $\mathcal{K}$ are w_1 (the root), w_4, w_5, w_8, w_9. The arrows represent the relation R. In each world w_k, the first line displays the value of $V(w_k)$, the remaining lines report (separated by commas) some of the formulas forced and not forced in w_k. Since $w_1 \nVdash \varphi$, $\mathcal{K}$ is a countermodel for φ. $\Diamond$

Proof search. We investigate more deeply the duality between $\mathcal{E}$-KM and $\mathcal{R}$-KM. Given an l-sequent $\sigma = \Gamma \overset{l}{\Rightarrow} \delta\, ;\, \psi$ we denote with $\overline{\sigma}$ the l-antisequent $\Gamma \overset{l}{\not\Rightarrow} \delta\, ;\, \psi$. Let σ be an u-sequent; in the next proposition we show that either σ is provable in $\mathcal{E}$-KM or $\overline{\sigma}$ is provable in $\mathcal{R}$-KM. The proof conveys a proof search strategy to build the proper derivation, based on backward application of the rules of $\mathcal{E}$-KM. We give priority to the *invertible rules* of $\mathcal{E}$-KM, namely: $L\land$, $R\land$, $L\lor$, $R\overset{\triangleright}{\to}$; as discussed in the proof of Prop. 16, the application of such rules does not require backtracking. If the search for an $\mathcal{E}$-derivation of σ fails, we get an $\mathcal{R}$-derivation of $\overline{\sigma}$.

Proposition 16

(i) Let σ be a u*-sequent. Then $\vdash_{\mathcal{E}\text{-KM}} \sigma$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma}$.*



$$\begin{aligned}
&\Psi(\sigma_0) = \Psi(\sigma_1) = w_1\\
&\Psi(\sigma_2) = \Psi(\sigma_3) = \Psi(\sigma_4) = w_4\\
&\Psi(\sigma_6) = \Psi(\sigma_7) = \Psi(\sigma_8) = w_8\\
&\Psi(\sigma_5) = w_5\\
&\Psi(\sigma_9) = w_9
\end{aligned}$$

Figure 6: The countermodel $\mathrm{Mod}(\mathcal{D})$ for φ (see Ex. 15).

(ii) Let $\sigma = \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \delta\,;\, \psi$, where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta$. Then $\vdash_{\mathcal{E}\text{-KM}} \sigma$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma}$ or $\vdash_{\mathcal{R}\text{-KM}} \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \psi$.

Proof. Since $\prec_{\mathrm{bu}}$ is well-founded (Prop. 8), we can inductively assume that the assertion holds for every sequent σ' such that $\sigma' \prec_{\mathrm{bu}} \sigma$ (IH).

Proof of point (*i*). If σ or $\overline{\sigma}$ is an axiom (in the respective calculus), the assertion immediately follows.

If an invertible rule ρ of $\mathcal{E}$-KM is (backward) applicable to σ, we can build the proper derivation by applying ρ or its dual image in $\mathcal{R}$-KM. For instance, let us assume that rule $L\vee$ of $\mathcal{E}$-KM is applicable with conclusion $\sigma = \alpha_0 \vee \alpha_1, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ and premises $\sigma_0 = \alpha_0, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$ and $\sigma_1 = \alpha_1, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta$. Since, by Prop. 10, $\sigma_k \prec_{\mathrm{bu}} \sigma$ for $k \in \{0,1\}$, by (IH) $\vdash_{\mathcal{E}\text{-KM}} \sigma_k$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_k}$. According to the case, we can build one of the following derivations:

$$\dfrac{\begin{matrix}\vdots\\ \sigma_0 = \alpha_0, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta\end{matrix} \qquad \begin{matrix}\vdots\\ \sigma_1 = \alpha_1, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta\end{matrix}}{\sigma = \alpha_0 \vee \alpha_1, \Gamma \stackrel{\mathrm{u}}{\Rightarrow} \delta}\ L\vee \qquad \dfrac{\begin{matrix}\vdots\\ \overline{\sigma_0} = \alpha_0, \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta\end{matrix}}{\overline{\sigma} = \alpha_0 \vee \alpha_1, \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta}\ L\vee_0 \qquad \dfrac{\begin{matrix}\vdots\\ \overline{\sigma_1} = \alpha_1, \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta\end{matrix}}{\overline{\sigma} = \alpha_0 \vee \alpha_1, \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta}\ L\vee_1$$

Let us assume that no invertible rule can be applied to σ; then σ has one of the following forms:

(S1) $\sigma = \underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta}_{\Gamma} \stackrel{\mathrm{u}}{\Rightarrow} \delta$ with $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ or $\delta = \delta_0 \vee \delta_1$ or ($\delta = \delta_0 \to \delta_1$ and $\Gamma \not\triangleright \delta_0$) or $\delta = \Box\delta_0$.

Let us consider the case $\delta = \delta_0 \to \delta_1$, where $\Gamma \not\triangleright \delta_0$. Let $\sigma_0 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \delta_1$ and $\sigma_1 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \delta_1$ be the left and right premises of an application of rule $R\stackrel{\mathrm{u}}{\to}$ of $\mathcal{E}$-KM to σ; for every $\alpha \to \beta \in \Gamma^{\to}$, let $\sigma_\alpha = \Gamma \stackrel{\mathrm{b}}{\Rightarrow} \alpha\,;\,\delta$ and $\sigma_\beta = \Gamma \setminus \{\alpha \to \beta\}, \beta \stackrel{\mathrm{u}}{\Rightarrow} \delta$ be the left and right premises of an application of rule $L\to$ of $\mathcal{E}$-KM to σ with main formula $\alpha \to \beta$. By the (IH):

- $\vdash_{\mathcal{E}\text{-KM}} \sigma_k$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_k}$ where $k \in \{0,1\}$;
- for every $\alpha \to \beta \in \Gamma^{\to}$, $\vdash_{\mathcal{E}\text{-KM}} \sigma_\beta$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_\beta}$;
- for every $\alpha \to \beta \in \Gamma^{\to}$, $\vdash_{\mathcal{E}\text{-KM}} \sigma_\alpha$ or $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_\alpha}$ or $\vdash_{\mathcal{R}\text{-KM}} \Gamma \stackrel{\mathrm{u}}{\not\Rightarrow} \delta$.

One of the following cases holds.

(A) $\vdash_{\mathcal{E}\text{-KM}} \sigma_0$ and $\vdash_{\mathcal{E}\text{-KM}} \sigma_1$.

We can build the following $\mathcal{E}$-derivation:

$$\dfrac{\begin{matrix}\vdots\\ \sigma_0 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \stackrel{\mathrm{u}}{\Rightarrow} \delta_1\end{matrix} \qquad \begin{matrix}\vdots\\ \sigma_1 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Theta \stackrel{\mathrm{u}}{\Rightarrow} \delta_1\end{matrix}}{\sigma = \underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta}_{\Gamma} \stackrel{\mathrm{u}}{\Rightarrow} \delta_0 \to \delta_1}\ R\stackrel{\mathrm{u}}{\to}$$

Note that the application of rule $R\overset{\mathrm{u}}{\to}$ is sound since we are assuming $\Gamma \not\rhd \delta_0$.

(B) There is $\alpha \to \beta \in \Gamma^{\to}$ such that $\vdash_{\mathcal{E}\text{-}\mathsf{KM}} \sigma_\alpha$ and $\vdash_{\mathcal{E}\text{-}\mathsf{KM}} \sigma_\beta$.

We can build the following $\mathcal{E}$-derivation:

$$\dfrac{\begin{matrix}\vdots \\ \sigma_\alpha = \Gamma \overset{\mathrm{b}}{\Rightarrow} \alpha\,;\,\delta\end{matrix} \qquad \begin{matrix}\vdots \\ \sigma_\beta = \Gamma \setminus \{\alpha \to \beta\}, \beta \overset{\mathrm{u}}{\Rightarrow} \delta\end{matrix}}{\sigma = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta}\ L{\to}$$

(C) There is $\alpha \to \beta \in \Gamma^{\to}$ such that $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma_\beta}$.

We can build the following $\mathcal{R}$-derivation:

$$\dfrac{\begin{matrix}\vdots \\ \overline{\sigma_\beta} = \Gamma \setminus \{\alpha \to \beta\}, \beta \overset{\mathrm{u}}{\not\Rightarrow} \delta\end{matrix}}{\overline{\sigma} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta}\ L{\to}$$

(D) $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma_0}$.

We can build the following $\mathcal{R}$-derivation:

$$\dfrac{\begin{matrix}\vdots \\ \overline{\sigma_0} = \delta_0, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta_1\end{matrix}}{\overline{\sigma} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta_0 \to \delta_1}\ R\overset{\mathrm{u}}{\to}$$

(E) $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma_1}$ and, for every $\alpha \to \beta \in \Gamma^{\to}$, $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma_\alpha}$.

We can build the following $\mathcal{R}$-derivation:

$$\dfrac{\begin{matrix}\vdots \\ \ldots\ \overline{\sigma_\alpha} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \alpha\,;\,\delta_0 \to \delta_1\ \ldots\end{matrix} \qquad \begin{matrix}\vdots \\ \overline{\sigma_1} = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta_1\end{matrix}}{\overline{\sigma} = \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta_0 \to \delta_1}\ \mathrm{S}^{\to}_{\mathrm{u}}$$

(F) $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$.

In this case we immediately get $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma}$.

The other subcases of (S1) are similar.

Proof of point (*ii*). If σ or $\overline{\sigma}$ is an axiom (in the respective calculus), the assertion immediately follows.

If an invertible rule ρ of $\mathcal{R}$-KM is (backward) applicable to σ, we can build the proper derivation in the same way as in the case (*i*).

Let us assume that no invertible rule can be applied to σ; then:

(T1) $\sigma = \underbrace{\Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta}_{\Gamma} \overset{\mathrm{b}}{\Rightarrow} \delta\,;\,\psi$ with $\delta = \delta_0 \vee \delta_1$ or ($\delta = \delta_0 \to \delta_1$ and $\Gamma \not\rhd \delta_0$) or $\delta = \Box\delta_0$.

Let $\delta = \delta_0 \to \delta_1$, where $\Gamma \not\rhd \delta_0$. Let $\sigma_0 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Box\Theta \overset{\mathrm{u}}{\Rightarrow} \psi$ and $\sigma_1 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\to}, \Theta \overset{\mathrm{u}}{\Rightarrow} \delta_1$ be the left and the right premises of the application of rule $R\overset{\mathrm{b}}{\to}$ of $\mathcal{E}$-KM to σ. By the (IH):

- $\vdash_{\mathcal{E}\text{-}\mathsf{KM}} \sigma_k$ or $\vdash_{\mathcal{R}\text{-}\mathsf{KM}} \overline{\sigma_k}$ where $k \in \{0, 1\}$.

One of the following cases holds:

(A) $\vdash_{\mathcal{E}\text{-KM}} \sigma_0$ and $\vdash_{\mathcal{E}\text{-KM}} \sigma_1$.

We can build the following $\mathcal{E}$-derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \sigma_0 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{u}}{\Rightarrow} \psi\end{array} \qquad \begin{array}{c}\vdots\\ \sigma_1 = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\Rightarrow} \delta_1\end{array}}{\sigma = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\Rightarrow} \delta_0 \rightarrow \delta_1 \,;\, \psi}\; R\overset{\mathrm{b}}{\rightarrow}$$

(B) $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_0}$.

We can build the following $\mathcal{R}$-derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \overline{\sigma_0} = \delta_0, \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \psi\end{array}}{\Gamma \overset{\mathrm{u}}{\not\Rightarrow} \psi}\; \mathrm{W}$$

(C) $\vdash_{\mathcal{R}\text{-KM}} \overline{\sigma_1}$.

We can build the following $\mathcal{R}$-derivation:

$$\dfrac{\begin{array}{c}\vdots\\ \overline{\sigma_1} = \delta_0, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta_1\end{array}}{\overline{\sigma} = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \delta_0 \rightarrow \delta_1 \,;\, \psi}\; R\overset{\mathrm{b}}{\rightarrow}$$

The other subcases of (T1) are similar. $\Box$

Let us assume $\Gamma \models_{\mathsf{KM}} \delta$ and let $\sigma = \Gamma \overset{\mathrm{u}}{\Rightarrow} \delta$. By Soundness of $\mathcal{R}$-KM (Th. 12) $\overline{\sigma}$ is not provable in $\mathcal{R}$-KM, hence, by Prop. 16, σ is provable in $\mathcal{E}$-KM; this proves the completeness of $\mathcal{E}$-KM stated in point (iv) of Th. 5.

Soundness of $\mathcal{R}$-KM. To prove the soundness of $\mathcal{R}$-KM, it remains to prove point (i) of Th. 14. Let $\mathcal{D}$ be an $\mathcal{R}$-derivation having a Succ rule at the root. To display $\mathcal{D}$, we introduce the derivation schema (DS1) below; at the same time, we define the relation $\ll$ between u-antisequents in $\mathcal{D}$.

$$\mathcal{D} = \quad \dfrac{\begin{array}{c}\mathcal{D}_\chi\\ \cdots\ \sigma^{\mathrm{b}}_\chi \ =\ \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \chi \,;\, \delta\ \cdots\end{array} \qquad \begin{array}{c}\vdots\\ \sigma^{\mathrm{u}}_0\end{array}}{\sigma^{\mathrm{u}} \ =\ \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{u}}{\not\Rightarrow} \delta}\; \mathrm{Succ} \tag{DS1}$$

- σ^{b}_χ is any of the premises of Succ having label b.
- σ^{u}_0 is defined only in the following cases:
  - Succ is $\mathrm{S}^{\Box}_{\mathrm{u}}$ and $\sigma^{\mathrm{u}}_0 = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ with $\delta = \Box\alpha$;
  - Succ is $\mathrm{S}^{\rightarrow}_{\mathrm{u}}$ and $\sigma^{\mathrm{u}}_0 = \alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \beta$ with $\delta = \alpha \rightarrow \beta$.

We set $\sigma^{\mathrm{u}} \ll \sigma^{\mathrm{u}}_0$. Let us consider the $\mathcal{R}$-derivation $\mathcal{D}_\chi$ of σ^{b}_χ. Looking at the rules working on b-antisequents, it is easy to see that $\mathcal{D}_\chi$ has the following form.

$$\begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \sigma^{\mathrm{u}}_1\end{array}}{\sigma^{\mathrm{b}}_1}\,\rho_1 \quad \cdots \quad \dfrac{\begin{array}{c}\vdots\\ \sigma^{\mathrm{u}}_m\end{array}}{\sigma^{\mathrm{b}}_m}\,\rho_m \quad \dfrac{}{\tau^{\mathrm{b}}_1}\,\mathrm{Irr} \quad \cdots \quad \dfrac{}{\tau^{\mathrm{b}}_n}\,\mathrm{Irr} \\ \mathcal{T}^{\mathrm{b}}_\chi \\ \sigma^{\mathrm{b}}_\chi \ =\ \Gamma \overset{\mathrm{b}}{\not\Rightarrow} \chi \,;\, \delta \end{array} \qquad \begin{array}{l} m+n \geq 0 \\ \mathcal{T}^{\mathrm{b}}_\chi \text{ only contains} \\ \text{b-antisequents} \\ \Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \end{array} \tag{DS2}$$

where the $\mathcal{R}$-tree $\mathcal{T}^{\mathrm{b}}_\chi$ has root σ^{b}_χ and leaves $\sigma^{\mathrm{b}}_1, \dots, \sigma^{\mathrm{b}}_m, \tau^{\mathrm{b}}_1, \dots, \tau^{\mathrm{b}}_n$ and for every $i \in \{1, \dots, m\}$, either (A) $\rho_i = R\overset{\not\vdash}{\rightarrow}_{\mathrm{b}}$ or (B) $\rho_i = R^{\Box}_{\mathrm{b}}$, namely:

(A) $\dfrac{\sigma_i^{\mathrm{u}} \;=\; \alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \beta}{\sigma_i^{\mathrm{b}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \alpha \rightarrow \beta \,;\, \delta} \; R_{\mathrm{b}}^{\not\triangleright\rightarrow}$ or (B) $\dfrac{\sigma_i^{\mathrm{u}} \;=\; \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \alpha}{\sigma_i^{\mathrm{b}} \;=\; \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta \overset{\mathrm{b}}{\not\Rightarrow} \Box\alpha \,;\, \delta} \; R_{\mathrm{b}}^{\Box}$

In all these cases we set $\sigma^{\mathrm{u}} \ll \sigma_i^{\mathrm{u}}$.

The next technical lemma establishes the crucial property of $\mathcal{R}$-derivations having the form (DS1); it can be proved as the Lemma 4 of [16] (the proof is in App. **??**).

Lemma 17 *Let $\mathcal{D}$ be an $\mathcal{R}$-derivation of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ having form (DS1), where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta$; let $\mathcal{K} = \langle W, R, r, V \rangle$ and $w \in W$ such that:*

(J1) For every $w' \in W$ such that wRw', it holds that $w' \Vdash \Gamma^{\rightarrow} \cup \Theta$.

(J2) For every $\sigma' = \alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \beta$ such that $\sigma^{\mathrm{u}} \ll \sigma'$, there exists $w' \in W$ such that $w \leq w'$ and $w' \Vdash \alpha$ and $w' \nVdash \beta$.

(J3) For every $\sigma' = \Box\alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \alpha$ such that $\sigma^{\mathrm{u}} \ll \sigma'$, there exists $w' \in W$ such that wRw' and $w' \nVdash \alpha$.

(J4) $V(w) = \Gamma^{\mathrm{at}}$.

Then, $w \Vdash \Gamma$ and $w \nVdash \delta$.

We are now in the position to complete the proof of point (i) of Th. 14: given an $\mathcal{R}$-derivation $\mathcal{D}$ of an u-antisequent σ_0^{u}, for every u-antisequent $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$, $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.

Proof. Let ρ be the rule of $\mathcal{R}$-KM having conclusion σ^{u}. We proceed by induction on the depth of $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \delta$ in $\mathcal{D}$. We proceed by a case analysis, only detailing some significant cases.

If $\rho = \mathrm{Irr}$, then $\Gamma = \Gamma^{\mathrm{at}}, \Box\Theta$ and $\delta \in (\mathcal{V} \cup \{\bot\}) \setminus \Gamma^{\mathrm{at}}$ and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Since $V(\sigma^{\mathrm{u}}) = \Gamma^{\mathrm{at}}$ and σ^{u} has no R-successors, it follows that $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \delta$.

Let us assume that $\rho = R^{\triangleright}_{\rightarrow}$. Then, $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \alpha \rightarrow \beta$, where $\Gamma \triangleright \alpha$, and the premise of ρ is $\sigma_1^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \beta$. By the induction hypothesis, $\Psi(\sigma_1^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma_1^{\mathrm{u}}) \nVdash \beta$. Since $\Gamma \triangleright \alpha$, by Lemma 3(iv) we get $\Psi(\sigma_1^{\mathrm{u}}) \Vdash \alpha$, which implies $\Psi(\sigma_1^{\mathrm{u}}) \nVdash \alpha \rightarrow \beta$. Since $\Psi(\sigma^{\mathrm{u}}) = \Psi(\sigma_1^{\mathrm{u}})$, we conclude $\Psi(\sigma^{\mathrm{u}}) \Vdash \Gamma$ and $\Psi(\sigma^{\mathrm{u}}) \nVdash \alpha \rightarrow \beta$.

Let us assume $\rho = \mathrm{S}_{\mathrm{u}}^{\Box}$. We have $\sigma^{\mathrm{u}} = \Gamma \overset{\mathrm{u}}{\not\Rightarrow} \Box\delta$, where $\Gamma = \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Box\Theta$, and $\Psi(\sigma^{\mathrm{u}}) = \sigma^{\mathrm{u}}$. Let $\mathcal{D}^{\mathrm{u}}$ be the subderivation of $\mathcal{D}$ having root sequent σ^{u}; we apply Lemma 17 by setting $\mathcal{D} = \mathcal{D}^{\mathrm{u}}$, $\mathcal{K} = \mathrm{Mod}(\mathcal{D})$ and $w = \sigma^{\mathrm{u}}$. We check that hypotheses *(J1)*–*(J4)* hold.

Hyp. (J1) Let w' be a world of $\mathrm{Mod}(\mathcal{D})$ such that $\sigma^{\mathrm{u}} R w'$. There exists an u-sequent σ' such that $\sigma^{\mathrm{u}} \ll \sigma' \preceq w'$. Let $\sigma' = \Gamma' \overset{\mathrm{u}}{\Rightarrow} \delta'$. Since $\mathrm{depth}(\sigma') < \mathrm{depth}(\sigma^{\mathrm{u}})$, by the induction hypothesis we get $\Psi(\sigma') \Vdash \Gamma'$; by the fact that $\Gamma^{\rightarrow} \cup \Theta \subseteq \Gamma'$, we get $\Psi(\sigma') \Vdash \Gamma^{\rightarrow} \cup \Theta$. Since $\Psi(\sigma') \leq w'$, we conclude $w' \Vdash \Gamma^{\rightarrow} \cup \Theta$, and this proves hypothesis *(J1)*.

Hyp. (J2) Let $\sigma' = \alpha, \Gamma^{\mathrm{at}}, \Gamma^{\rightarrow}, \Theta \overset{\mathrm{u}}{\not\Rightarrow} \beta$ such that $\sigma^{\mathrm{u}} \ll \sigma'$. By the induction hypothesis, $\Psi(\sigma') \Vdash \alpha$ and $\Psi(\sigma') \nVdash \beta$. Since $\sigma^{\mathrm{u}} = \Psi(\sigma^{\mathrm{u}})$ and $\Psi(\sigma^{\mathrm{u}}) \leq \Psi(\sigma')$, hypothesis *(J2)* holds. The proof of hypothesis *(J3)* is similar. Hypothesis *(J4)* holds by the definition of V. By Lemma 17, we get $\sigma^{\mathrm{u}} \Vdash \Gamma$ and $\sigma^{\mathrm{u}} \nVdash \delta$. $\Box$

Conclusions. We have presented a terminating sequent calculus $\mathcal{E}$-KM for KM enjoying the subformula property. If a sequent σ is not derivable in $\mathcal{E}$-KM, then σ is derivable in the dual calculus $\mathcal{R}$-KM, and from the $\mathcal{R}$-KM-derivation we can extract a countermodel for σ. We leave as future work the investigation of cut-admissibility for $\mathcal{E}$-KM; this is a rather tricky task since labels impose strict constraints on the shape of derivations. Moreover, continuing the line of work initiated in [16, 15] and pursued in this paper, we aim at extending our approach to the remaining coreflection logics in Fig. 1, namely IEL, mHC, and PLL.

Declaration on Generative AI

No generative AI tools were used in the preparation of this article.

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