Lattice Spaces and Formal Concept Analysis

Uta Priss

Fakultät Informatik, Ostfalia University, Wolfenbüttel, Germany

Abstract

This paper introduces lattice spaces with logical operations in an analogy to vector spaces as the main structure related to Boolean matrices instead of matrices over fields. Different types of Boolean matrices, their operations, algebraic and lattice theoretical properties are discussed. The focus of this paper is more on connecting Boolean matrix theory with Formal Concept Analysis with respect to representations and potential applications than on developing new mathematical theory but some novel mathematical results are also presented.

Keywords

Boolean matrix, Formal Concept Analysis, Lattice Space

1. Introduction

In linear algebra, geometric properties are translated into properties of vector spaces as represented by matrices over a field of scalars. Boolean matrices with 0s, 1s do not form vector spaces if logical AND and OR are used as operations because the field conditions are not fulfilled. Nevertheless a fair amount of the usual vector space terminology still has meaning for Boolean matrices. The resulting structures, however, are lattices instead of vector spaces and are called 'lattice spaces' in this paper in order to emphasise the analogy. Lattice spaces use 'C-matrices' instead of unit vectors and have reduced matrices as bases. Their main operation is a Galois connection, but union, intersection and matrix multiplication are also relevant. Permutations as the only kind of invertible Boolean matrices replace linear transformations. While vector spaces have dual spaces, there are two types of dualities for Boolean matrices: matrix transposition and complementation, which changes union into intersection and posets (i.e. partially ordered sets) into distributive lattices.

Knowledge of Formal Concept Analysis (FCA) is not strictly required for this paper but the discussion is aimed at FCA notions and applications. The definitions of FCA can be found in the textbook by Ganter & Wille (1999) and are not included in this paper. One important difference between lattice spaces and FCA is that for matrices the formal objects and attributes are just positions. Each row of a matrix corresponds to an object and each column to an attribute, but otherwise objects and attributes are not mentioned. While FCA discusses Boolean matrices for context constructions, some translations and proofs, matrices are not usually considered in FCA as elements of an algebraic theory themselves. It is quite possible that some of the many algebraic properties of Boolean matrices discussed by Kim (1982) and others (Maddux 1996) might have yet undiscovered relevance for FCA. It is therefore of interest to combine matrix theory and FCA. Kim (1982) presents a coherent theory of Boolean matrices but his terminology is somewhat different from FCA and lacks some of FCA's insights, such as the Basic Theorem (Ganter & Wille 1999), which implies a more succinct description of the relationship between matrices and lattices.

Both Kim (1982) and Knowledge Space Theory (Doignon & Falmagne 1999) employ unions as primary operation instead of intersections in FCA. A standard view of the difference between intersection and union is to switch to the complement of a matrix. While complements are interesting for mathematical modelling, for an application they involve a discussion of negated objects or negated attributes, which may not be useful. Instead, combining intersections as actually shared attributes and unions as potential combinations of attributes might be relevant for conceptual exploration and is presented as a more

3rd International Joint Conference on Conceptual Knowledge Structures
https://www.upriss.org.uk (U. Priss)
0000-0003-0375-429X (U. Priss)

useful example in this paper. Other Boolean matrix properties, such as relation algebra, which can be the foundation of any form of relational FCA (Priss, 2009) are left for another paper.

The goal of this paper is to provide a unified terminology for combining these different approaches to Boolean matrices and lattices. Most of the mathematical results in this paper are already contained somewhere in the literature but not necessarily well connected. Some results about a different type of dimension are new in this paper. The focus of this paper, however, is on establishing connections and developing suitable representations for applications. Boolean matrices appear to present a universal language for many topics of discrete mathematics with lattice spaces at the core. Furthermore Boolean matrices are straightforward to implement and process in a computer with immediate algorithms that are sufficient at least for smallish matrices.

The next section introduces Boolean matrices and lattice spaces. Section 3 discusses how to generate matrices from C-matrices and different kinds of completions. Section 4 presents details of Boolean matrix multiplication, decomposition and ranks. Section 5 considers permutation matrices as transformations amongst bases. In Section 6 triangular matrices are introduced as full representations of posets. Last but not least, Section 7 discusses the connection between the algebraic properties of idempotence, transitivity and distributivity in the context of Boolean matrices and lattice spaces.

2. Boolean Matrices and Lattice Spaces

The structure $(\{0,1\}, \vee, \wedge)$ with the usual logical operations ($0 \vee 0 = 0, 0 \vee 1 = 1 \vee 1 = 1, 0 \wedge 0 = 0 \wedge 1 = 0, 1 \wedge 1 = 1$) forms a commutative semiring because it is closed, associative and commutative under both operations and is a distributive bounded lattice. Together with the complement operation $\neg 1 = 0$ and $\neg 0 = 1$ the structure $(\{0,1\}, \vee, \wedge, \neg)$ constitutes the 2-element Boolean algebra. The definitions in this paper are mostly adapted from Kim (1982) but translated into a language and notation that is better suited for FCA.

Definition 1. *1) A* Boolean m, n-matrix *is a matrix with m rows and n columns that contains only 0s and 1s.*
2) The set of all Boolean m, n-matrices is denoted by $\mathbf{M}_{mn}$ and by $\mathbf{M}_n$ if $m = n$.
3) A matrix $I \in \mathbf{M}_{1n}$ is a row matrix *and $I \in \mathbf{M}_{m1}$ is a* column matrix.
4) The set of matrices of graphs is denoted by $\mathbf{G}_n$, i.e. $\mathbf{G}_n = \mathbf{M}_n$ but with an interpretation of row i and column i both belonging to vertex i.

In the remainder of this paper, Boolean matrices are just called ‘matrices’ and the adjective ‘Boolean’ is usually omitted. All sets and matrices in this paper are finite. Boolean matrix operations are the usual matrix operations but with logical $\vee$ for addition and $\wedge$ for multiplication. For FCA, matrices in $\mathbf{M}_{mn}$ relate to formal contexts whereas Hasse diagrams[1] can be represented with matrices in $\mathbf{G}_n$. The difference between $\mathbf{M}_n$ and $\mathbf{G}_n$ is of a semantic nature and determines which matrix operations are meaningful in each case. In the remainder of this paper, *permuting* means for $I \in \mathbf{G}_n$ that rows and columns are permuted simultaneously in the same manner whereas for $\mathbf{M}_{mn}$ and $\mathbf{M}_n$ rows and columns can be permuted independently.

Definition 2. *For $I, J \in \mathbf{M}_{mn}$ and $R \in \mathbf{M}_m$ the following matrices are defined:*
$\overline{I}$: negated or complemented matrix that changes all 0s to 1s and vice versa
I^d : transposed (or dual) matrix (mirroring I along the top left/bottom right-diagonal)
$I \cup J$: matrix with 1s where at least one of I or J has a 1
$I \cap J$: matrix with 1s where both I and J have a 1
$R \circ J$: matrix multiplication of R and J with $\vee$ as addition and $\wedge$ as multiplication
$\mathbb{1}_n \in \mathbf{M}_n$ matrix with 1s on the top left/bottom right-diagonal and 0s otherwise
$\mathbb{0}_{mn} \in \mathbf{M}_{mn}$ and $\mathbb{0}_n \in \mathbf{M}_n$ matrices with only 0s

[1]The diagrams should not be named after Hasse because he did not invent them but the name is established in the literature.

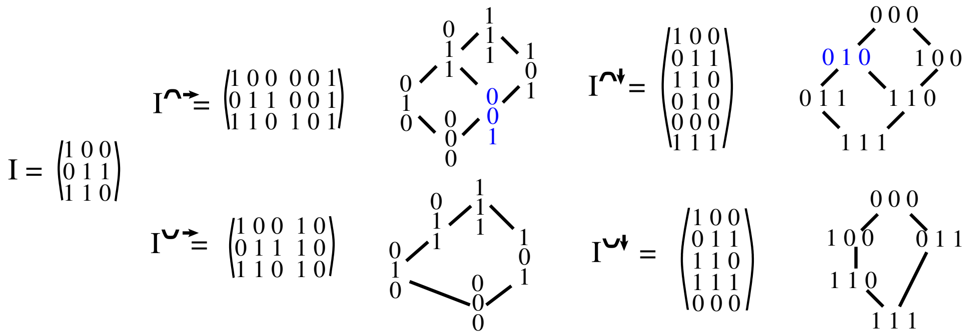

Figure 1: Matrices and lattices of row and column sets for the running example I

$I \subseteq J \iff J$ *has a* 1 *wherever* I *has a* 1
I *is* invertible *if there exists a* I^{-1} *with* $I \circ I^{-1} = I^{-1} \circ I = \mathbb{1}$.
I *is* idempotent $\iff I \circ I = I$.
I *is* sparse *if less than half of all entries are 1s. Otherwise* I *is* dense.

Because $(\{0,1\}, \vee)$ and $(\{0,1\}, \wedge)$ are idempotent commutative monoids with identity elements 0 and 1, $(\mathbf{M}_{mn}, \cup)$ and $(\mathbf{M}_{mn}, \cap)$ are also commutative monoids with identity elements $\mathbb{0}_{mn}$ and $\overline{\mathbb{0}_{mn}}$, respectively. Only for square matrices, $(\mathbf{M}_n, \circ)$ is a monoid with identity element $\mathbb{1}_n$ because $\circ$ is not defined for all pairs of matrices in $\mathbf{M}_{mn}$. The structure $(\mathbf{M}_n, \cup, \circ, \mathbb{0}_n, \mathbb{1}_n)$ forms a semiring, but $\circ$ is not commutative. The structure $(\mathbf{M}_{mn}, \cup, \cap, ^-)$ is itself a Boolean algebra and $(\mathbf{M}_n, \cup, \circ, ^-, ^d, \mathbb{0}_n, \mathbb{1}_n)$ is a relation algebra. Kim (1982) calls row and column matrices 'vectors' and uses the notion 'vector space' with $\cup$ as addition even though $(\{0,1\}, \vee, \wedge)$ is not a field as there are no additive inverses. If exclusive OR ($\oplus$) is used instead of $\vee$ then $(\{0,1\}, \oplus, \wedge)$ is a Boolean ring where each element is an additive inverse of itself ($0 \oplus 0 = 0$ and $1 \oplus 1 = 0$) and a multiplicative idempotent ($0 \wedge 0 = 0$ and $1 \wedge 1 = 1$), thus allowing to define a finite vector space. While such vector spaces have applications in computer science they might have less relevance for FCA and are not discussed in this paper.

Definition 3. *1) The* $\cap$*-row set* $\{I^{\cap\downarrow}\}$ *of a matrix* I *is a set of all rows of* I *and all their intersections including the intersection over the empty set, which yields a row of 1s.* $\{I^{\cap\rightarrow}\}$, $\{I^{\cup\downarrow}\}$ *and* $\{I^{\cup\rightarrow}\}$ *are defined analogously.*
2) Matrices I *and* J *are* $\cap$*-row-equivalent if* $\{I^{\cap\downarrow}\} = \{J^{\cap\downarrow}\}$*. Similar definitions hold for* $\{I^{\cap\rightarrow}\}$, $\{I^{\cup\downarrow}\}$ *and* $\{I^{\cup\rightarrow}\}$.
3) $I^{\cap\downarrow}$ *denotes a matrix whose sets of rows are complete with respect to intersections.* $I^{\cap\rightarrow}$, $I^{\cup\downarrow}$ *and* $I^{\cup\rightarrow}$ *are defined analogously.*

Figure 1 shows row and column sets of a matrix I which serves as a running example in this paper. As stated in the following lemma, it is well known from FCA and Kim (1982) that the row and column sets form lattices which are isomorphic independently of whether rows or columns are used[2]. Employing unions instead of intersections corresponds to using intersections for the complement of the original matrix.

Lemma 1. *1) The equivalence relations of Def.3.2 are indeed equivalence relations.*
2) $\{I^{\cap\downarrow}\}$ *with* $\cap$ *as join,* $\{I^{\cap\rightarrow}\}$ *with* $\cap$ *as meet,* $\{I^{\cup\downarrow}\}$ *with* $\cup$ *as meet and* $\{I^{\cup\rightarrow}\}$ *with* $\cup$ *as join and the other operations accordingly are lattices.*
3) $\{I^{\cap\downarrow}\}$ *with* $\cap$ *as join and* $\{I^{\cap\rightarrow}\}$ *with* $\cap$ *as meet are isomorphic lattices.*
4) $\{I^{\cup\downarrow}\}$ *with* $\cup$ *as meet and* $\{I^{\cup\rightarrow}\}$ *with* $\cup$ *as join are isomorphic lattices.*

[2]More precisely the lattices are anti-isomorphic, but the convention in FCA is to read rows/objects and columns/attributes in opposite directions resulting in isomorphic lattices.

Proof: 1) Kim (1982) Lemma 1.3.1
2) Kim (1982, p. 12)
3),4) Kim (1982) Corollary 1.2.4, Ganter & Wille (1999) Basic Theorem

Because of the isomorphisms the arrows can be dropped from the notation. Isomorphism between lattice spaces is denoted by $\cong$ in this paper.

Definition 4. *1) A $\cap$-lattice space $\langle I^{\cap} \rangle$ of a Boolean matrix I is a lattice $(\{I^{\cap}\}, \leq)$ as in Lemma 1.3 and a $\cup$-lattice space $\langle I^{\cup} \rangle$ is a lattice $(\{I^{\cup}\}, \leq)$ as in Lemma 1.4 where the $\leq$-relation is a row or column matrix $\subseteq$ or $\supseteq$-relation, respectively.*
2) The $\cap$-lattice space of a matrix or the smallest $\cap$-lattice space containing a set of row or column matrices is called the $\cap$-span *of the matrix or set. (Analogously for $\cup$.)*

In other words, $\langle I^{\cap} \rangle$ is the $\cap$-span of I. As shown in the next lemma, a switch from $\cap$ to $\cup$ is equivalent to turning all 0s into 1s and vice versa.

Lemma 2. *1) $\langle \overline{I}^{\cap} \rangle = \langle \overline{I^{\cup}} \rangle$ and $\langle I^{\cup} \rangle = \langle \overline{\overline{I}^{\cap}} \rangle$ and $\langle \overline{I}^{\cup} \rangle = \langle \overline{I^{\cap}} \rangle$ and $\langle I^{\cap} \rangle = \langle \overline{\overline{I}^{\cup}} \rangle$.*
2) For a matrix I there are 2 different lattice spaces $\langle I^{\cap} \rangle \cong \langle \overline{I}^{\cup} \rangle$ and $\langle I^{\cup} \rangle \cong \langle \overline{I}^{\cap} \rangle$.
3) It is possible to have $\langle I^{\cap} \rangle \cong \langle J^{\cap} \rangle$ and $\langle I^{\cup} \rangle \ncong \langle J^{\cup} \rangle$.

Proof: 1) and 2) hold because of de Morgan's law for Boolean logic ($\overline{a} \cup \overline{b} = \overline{a \cap b}$).
3) For example, $I = \begin{pmatrix} 0111 \\ 1010 \\ 1100 \end{pmatrix}$ and $J = \begin{pmatrix} 011 \\ 101 \\ 110 \end{pmatrix}$.

The lemma states that isomorphic lattice spaces need not have isomorphic lattice spaces for their complements. When complements are involved it is therefore usually helpful to specify whether the matrices are fully represented or reduced as defined next.

Definition 5. *1) A row matrix I is* $\cap$-row-independent *of a matrix J if $I \notin \{J^{\cap\downarrow}\}$. (Analogously: $\cup$-row-independent, $\cap$- or $\cup$-column-independent.)*
2) A matrix I is $\cap$-reduced *if each row is $\cap$-row-independent of any matrix of a subset of the remaining rows and each column is $\cap$-column-independent of any matrix of a subset of the remaining columns. (Analogously:* $\cup$-reduced*)*
3) A $\cap$-reduced matrix is considered a basis *for its $\cap$-lattice space. (Analogously for $\cup$)*
4) A matrix $I^{\cap}$ is fully represented *if $I^{\cap} = I^{\cap\downarrow} = I^{\cap\rightarrow}$ (after permuting rows and columns accordingly) and it has no duplicate rows or columns. (Analogously for $I^{\cup}$)*

If it is clear whether $\cap$ or $\cup$ is the operation in question a matrix can also be referred to as *reduced*. The top and bottom elements of a lattice are reducible in most applications. Therefore rows and columns with only 0s or only 1s can often be omitted.

Lemma 3. *Reduced matrices are unique apart from row and column permutations.*

Proof: According to Kim (1982, p 16) the basis consists exactly of the join/meet irreducible lattice elements and is thus uniquely determined as a set. This also follows from the Basic Theorem of FCA (Ganter & Wille 1999).

3. Base, Ferrers and L-matrices

The lattice operations in the previous section were based on either row or column matrices. An idea in this section is to represent lattice spaces with C-matrices (or concept matrices) that simultaneously contain row and column information. They correspond to concepts in the sense of FCA because they contain a maximum rectangle of rows (objects) and columns (attributes). While C-matrices have some advantages they also have the disadvantage that the lattice operations become more difficult to read compared to row- or column matrices. But at least for reduced matrices it is possible to read the operations. Two further types of matrices are also introduced and shown to be related to a different notion of dimension in the next section.

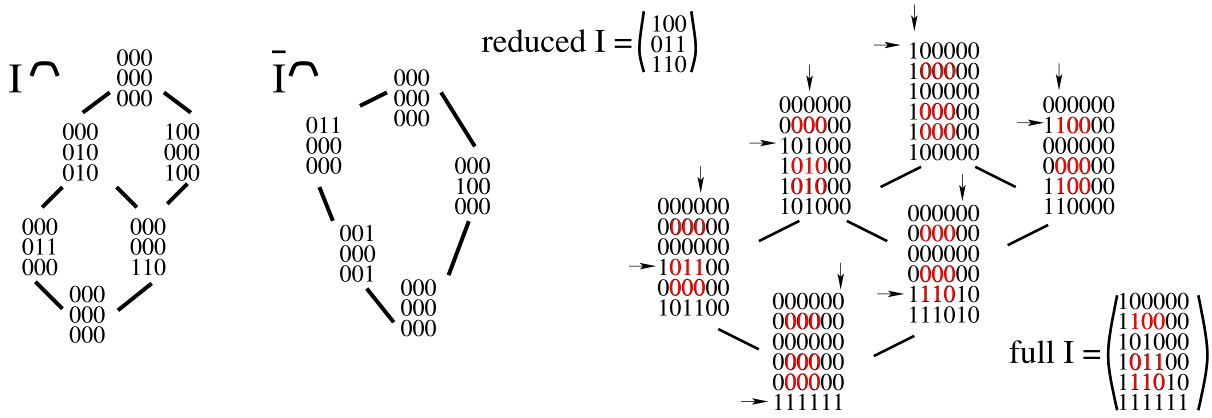

Figure 2: C-matrices for a reduced I from Fig.1 (left) and for a full representation (right)

Definition 6. *1) A matrix which can be permuted into a 'rectangle of 1s' is called a* C-matrix*, i.e. all its non-zero rows are identical (and called* defining row*) and all its non-zero columns are identical (and called* defining column*).*
2) A matrix which can be permuted into a 'triangle of 1s' is called a Ferrers matrix*, i.e. a matrix where any submatrix of 2 rows and 2 columns never contains a pattern* $\begin{pmatrix} 10 \\ 01 \end{pmatrix}$.
3) A matrix which can be permuted so that no column contains a 0 between two 1s is called an L-matrix.

Lemma 4. *1)* I *is a C-matrix* $\Longrightarrow$ I *is a Ferrers matrix* $\Longrightarrow$ I *is an L-matrix.*
2) I *is a Ferrers matrix* $\Longleftrightarrow$ $\overline{I}$ *is a Ferrers matrix.*
3) I *has at most 2 columns* $\Longrightarrow$ I *is an L-matrix.*

Proof: 1) Holds because non-zero rows are identical thus I cannot contain $\begin{pmatrix} 10 \\ 01 \end{pmatrix}$. A Ferrers matrix can always be permuted to contain a triangle of 1s without 0s in between.
2) If I does not contain $\begin{pmatrix} 10 \\ 01 \end{pmatrix}$ (up to permutation) then neither does $\overline{I}$.
3) The rows of a 2-column matrix can be permuted to $(\ 00\ ) \ldots (\ 10\ ) \ldots (\ 11\ ) \ldots (\ 01\ )$.

Lattice spaces of Ferrers matrices are chains (Ganter & Glodeanu 2012) and lattice spaces of L-matrices correspond to linear Euler diagrams (Priss & Dürrschnabel 2024). The left hand side of Fig.2 show the lattice spaces $\langle I^{\cap}\rangle$ and $\langle \overline{I}^{\cap}\rangle$ for a reduced I from Fig.1 using C-matrices. The right hand side shows $\langle I^{\cap}\rangle$ for a fully represented I where each lattice element has a unique defining pair of row and column as indicated by the arrows (which correspond in FCA to the object and attribute that would be attached to that concept in the Hasse diagram). The red 0s and 1s are elements of the reduced matrices. For all C-matrices apart from the bottom element of a lattice space the defining row and column are the same as in Fig.1. Moving up in a lattice space, the C-matrices have fewer non-0-columns, moving down they have fewer non-0-rows. Therefore both the top and bottom elements are $\mathbb{O}$-matrices in the reduced case.

Lemma 5. *For a representation of a lattice with C-matrices as in Fig.2:*
1) C-matrix $C_i \leq C_j$ *if* $r_j \subseteq r_i$ *for defining rows and* $c_i \subseteq c_j$ *for defining columns.*
2) If $C_1 < C_2 < \ldots < C_s$ *then* $C_1 \cup C_2 \cup \ldots \cup C_s$ *is a Ferrers matrix.*
3) The union of all C-matrices of $\langle I^{\cap}\rangle$ *equals* I*.*
4) The complement of the union of all C-matrices of $\langle \overline{I}^{\cap}\rangle$ *equals* I*.*
5) The disjoint union of two L-matrices is an L-matrix where disjoint union means that no row/column that has a 1 in one matrix also has a 1 in the other matrix.
6) If the $\cap$*-lattice space of a union of 2 Ferrers matrices only contains elements which are elements of either of the individual lattice spaces or, if there is no row of just 1s, neighbours of the bottom element of the union, the union is an L-matrix.*

$$\begin{pmatrix}100\\011\\110\end{pmatrix}=\begin{pmatrix}000\\010\\010\end{pmatrix}\cup\begin{pmatrix}100\\000\\100\end{pmatrix}\cup\begin{pmatrix}000\\011\\000\end{pmatrix}\cup\begin{pmatrix}000\\000\\110\end{pmatrix}=\overline{\begin{pmatrix}011\\000\\000\end{pmatrix}\cup\begin{pmatrix}001\\000\\001\end{pmatrix}\cup\begin{pmatrix}000\\100\\000\end{pmatrix}}$$

$$\begin{pmatrix}000\\010\\010\end{pmatrix}=\begin{pmatrix}000\\011\\000\end{pmatrix}\vee\begin{pmatrix}000\\000\\110\end{pmatrix}=(\begin{pmatrix}000\\011\\000\end{pmatrix}\cup\ldots\cup\begin{pmatrix}000\\000\\110\end{pmatrix})\cap(\begin{pmatrix}011\\011\\011\end{pmatrix}\cap\begin{pmatrix}110\\110\\110\end{pmatrix})$$

$$\begin{pmatrix}000\\000\\000\end{pmatrix}=\begin{pmatrix}000\\011\\000\end{pmatrix}\wedge\begin{pmatrix}000\\000\\110\end{pmatrix}=(\begin{pmatrix}000\\011\\000\end{pmatrix}\cup\ldots\cup\begin{pmatrix}000\\000\\110\end{pmatrix})\cap(\begin{pmatrix}000\\111\\000\end{pmatrix}\cap\begin{pmatrix}000\\000\\111\end{pmatrix})$$

$$\begin{pmatrix}100\\011\\110\end{pmatrix}=\begin{pmatrix}000\\011\\010\end{pmatrix}\cup\begin{pmatrix}100\\000\\100\end{pmatrix}=\begin{pmatrix}100\\111\\110\end{pmatrix}\cap\begin{pmatrix}111\\011\\111\end{pmatrix}$$

Figure 3: The matrix I from Fig.1 as a union of C-matrices, the lattice $\wedge$- and $\vee$-operations and I as a union and intersection of Ferrers matrices

Proof: 1) Holds because $\leq$ represents the order of the closed sets of a Galois connection.
2) Holds because of 1).
3) Holds because from an FCA viewpoint each C-matrix contains 1s for the incidence relation of a concept and every 1 of I must belong to at least one concept.
4) Dual to 3).
5) Holds because then the condition for the columns does not change.
6) Extending Petersen's (2010) conditions for linearising a lattice, the lattice space of an L-matrix is planar with a path from top to bottom and back (possibly visiting neighbours of the bottom node) that visits all nodes exactly once. The bottom node can be visited several times as long as there is no row of just 1s in the matrix.

Thus, unions of a containment chain of C-matrices yield Ferrers matrices and unions of L- or Ferrers matrices yield L-matrices under certain conditions, which are unfortunately somewhat complex. Dürrschnabel & Priss (2024) provide a more precise mathematical condition and an algorithm for checking the conditions of L-matrices. The top equation of Fig.3 contains an example for point 3 and 4 of the lemma. The next equation shows that for the lattice space of a reduced matrix the join of C-matrices equals a union of all lower neighbours of the join intersected with matrices that contain copies of the defining rows of the original matrices. The last equation of Fig.3 displays I as a union and intersection of Ferrers matrices.

An obvious question arising from Fig.3 is the minimum number of C-matrices required to create I as a union, which Kim (1982) calls the Schein rank of a matrix. According to Kim the row/column rank of a matrix is the number of rows/columns left after $\cup$-reducing the matrix. The column rank does not need be equal to the row rank and the Schein rank is always smaller or equal to the minimum of the row and column rank. The next section utilises matrix multiplication in order to illustrate the Schein rank. Belohlavek & Vychodil (2010) state that computing the Schein rank is NP-hard and prove that the factors for the Schein rank must always be formal concepts in the sense of FCA, which means in the terminology of this paper that they correspond to C-matrices.

The rest of this section shows how union and intersection might contribute to investigating data in an application. Computing intersections in a lattice space produces information about what is shared amongst objects or attributes in FCA terminology. Computing unions ensures that all possible combinations are represented. The following lemma explains what happens if one first computes intersections and then unions. Birkhoff completions are distributive lattices.

Lemma 6. *1) $\langle I^{\cap}\rangle$ is isomorphic to the concept lattice of I.*
2) If I is $\cup$-reduced then $\langle (I^{\cup\rightarrow})^{\cap}\rangle$ and $\langle (I^{\cup\downarrow})^{\cap}\rangle$ are Birkhoff completions of $\langle I^{\cap}\rangle$.
3) $\langle ((I^{\cup\rightarrow})^{\cup\downarrow})^{\cap}\rangle$ and $\langle ((I^{\cup\downarrow})^{\cup\rightarrow})^{\cap}\rangle$ are potentially completions as free completely distributive lattices.

Proof: 1) Ganter & Wille (1999).

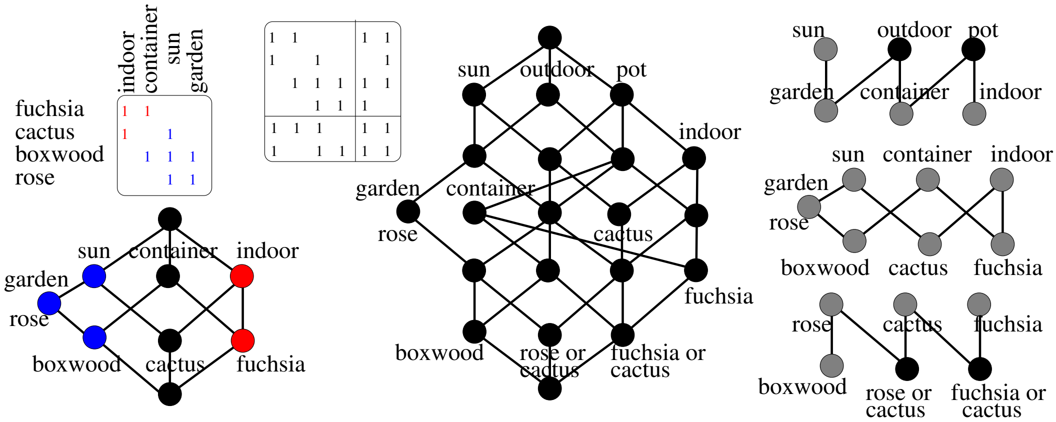

Figure 4: Combining union and intersection

$$\begin{pmatrix} 1010 \\ 1100 \\ 0010 \end{pmatrix} \circ \begin{pmatrix} a_1b_1c_1 \\ a_2b_2c_2 \\ a_3b_3c_3 \\ a_4b_4c_4 \end{pmatrix} = \begin{pmatrix} a_1 \vee a_3 & b_1 \vee b_3 & c_1 \vee c_3 \\ a_1 \vee a_2 & b_1 \vee b_2 & c_1 \vee c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \qquad \begin{pmatrix} 0100 \\ 0000 \\ 0100 \end{pmatrix} \circ \begin{pmatrix} 0000 \\ 1110 \\ 0000 \\ 0000 \end{pmatrix} = \begin{pmatrix} 1110 \\ 0000 \\ 1110 \end{pmatrix}$$

Figure 5: Matrix multiplication: row-column principle (left) and C-matrix principle (right)

2) cf. Abdulla et al. (2024)
3) as shown in Fig.4., also cf. Ganter & Wille (1999, Chapter 1.4). But it depends on the sparseness of the matrices. For dense matrices adding unions might not change anything.

The lemma is illustrated in Fig.4, which contains a lattice of an example of some plants and their locations in a garden, indoors, in sun or a container. The coloured nodes are explained in the next section. The lattice on the left of Fig.4 is not distributive. If one forms $I^{\cup}$ and then builds the $\cap$-lattice space one obtains a distributive lattice (Abdulla et al. 2024). According to Lemma 2.3 above even if $\langle I^{\cup\rightarrow}\rangle \cong \langle I^{\cup\downarrow}\rangle$, $\langle (I^{\cup\rightarrow})^{\cap}\rangle$ and $\langle (I^{\cup\downarrow})^{\cap}\rangle$ need not be isomorphic. Transposing $I^{\cup\rightarrow}$ yields the lattices shown on the right hand side of Fig.4 without their top and bottom elements. The elements that are drawn as black nodes are not in the original lattice and correspond to OR-combinations, such as 'outdoor' = 'garden OR container' and 'pot' = 'container OR indoor'. The OR-combinations for the objects are not useful in this case. In general, it might be most useful to consider $\cap$-combinations as the space below the attributes and $\cup$-combinations as the space above, which might need conceptual exploration. Adding both attribute- and object-OR combinations does not seem a good idea because the size of the lattice can increase rapidly as demonstrated in the middle of Fig.4 where $\langle ((I^{\cup\rightarrow})^{\cup\downarrow})^{\cap}\rangle$ was formed for the gardening example, which includes a free distributive lattice of 3 generators. The relationship between the right and the middle lattices is further discussed in the next section with respect to matrix decomposition.

4. Matrix Multiplication, Decomposition and Rank

Two principles which are essential for an understanding of Boolean matrix multiplication are shown in Fig.5. The row-column principle states that each row of a product matrix contains Boolean ORs of the rows of the right matrix according to the positions of 1s in the rows of the left matrix. Dually, a product matrix contains Boolean ORs of columns of the left matrix according to the 1s in the right matrix. The C-matrix principle states that the k-th column of the left matrix together with the k-th row of the right matrix generate a C-matrix as a product, which is the only contribution of the k-th row and column to a product.

A long list of properties of Boolean matrix multiplication is known from the literature (e.g. Maddux (1996)). A matrix $I \in \mathbf{G}_n$ is reflexive if $\mathbb{1}_n \subseteq I$, symmetric if $I = I^d$, antisymmetric if $I \cap I^d \subseteq \mathbb{1}_n$ and

Table 1
Rank/dimension of matrices in relation to unions, decompositions and embeddings. According to Ganter & Wille (1999), Kim (1982) and Ganter & Glodeanu (2012). BA_n: Boolean algebra with n atoms, C_n: product of n chains, I_f: fully represented version of I.

	min nr of	min nr of	embedding
Schein rank of I, 2-dim of $\overline{I}$	I as $\cup$ of C-matrices	Boolean factors of I	$\langle \overline{I}^{\cap} \rangle$ in BA_n
Schein rank of $\overline{I}$, 2-dim of I	$\overline{I}$ as $\cup$ of C-matrices	Boolean factors of $\overline{I}$	$\langle I^{\cap} \rangle$ in BA_n
order dimension of $\langle I^{\cap} \rangle$	I as $\cap$ of Ferrers matrices	ordinal factors of $\overline{I}$	$\langle I^{\cap} \rangle$ in C_n
order dimension of $\langle \overline{I}^{\cap} \rangle$	I as $\cup$ of Ferrers matrices	ordinal factors of I	$\langle \overline{I}^{\cap} \rangle$ in C_n
E-dim of $\langle I^{\cap} \rangle$	I_f as $\cup$ of L-matrices	L-factors of I	Euler diagram

transitive if $I \circ I \subseteq I$. Idempotent matrices are further discussed in Section 7 and need not be reflexive because they might contain rows and columns of just 0s. The following properties are immediately apparent from the two principles. If Lemma 7 is applied to matrices in $\mathbf{M}_n$ instead of $\mathbf{G}_n$ then notions such as 'transitive' are purely structural properties without an immediate conceptual interpretation. Matrix multiplication can indicate a form of similarity. A result of $\mathbb{O}$ means that the rows/columns have nothing in common.

Lemma 7. *1)* $I \circ J = \mathbb{O} \iff$ *there does not exist a* k *so that both the* k*-th column in* I *and the* k*-th row in* J *contain a* 1*.*
2) $I \circ J = \mathbb{1}_n \iff I$ *and* J *must be mutually inverse* permutation matrices $\in \mathbf{M}_n$, *(where permutation matrices contain exactly one* 1 *in each row and column).*
3) I *is reflexive and transitive* $\Longrightarrow I \circ I = I \Longrightarrow I$ *is transitive.*
4) I *is a poset matrix (reflexive, antisymmetric, transitive)* $\iff I \cap I^d = \mathbb{1}$ *and* $I \circ I = I$*.*
5) If $I = J \circ K$ *then* $\{I\} \subseteq \{K^{\cup\downarrow}\}$ *and* $\{I\} \subseteq \{J^{\cup\rightarrow}\}$*.*

Proof: 1) and 2) Evident from the row-column principle and the C-matrix principle.
3) reflexive $I \Longrightarrow I \circ J \supseteq J$ because of the row-column principle. I transitive: $I \circ I \subseteq I$.
4) Follows from 3) and the definitions.
5) Follows from the row-column principle.

From the C-matrix principle it follows that a matrix product equals the union of n C-matrices for two matrices J and K with $J \in \mathbf{M}_{xn}$, $K \in \mathbf{M}_{ny}$, such as

$$J_1 \circ J_2 = \begin{pmatrix} 0100 \\ 1010 \\ 1101 \end{pmatrix} \circ \begin{pmatrix} 010 \\ 100 \\ 011 \\ 110 \end{pmatrix} = \begin{pmatrix} 000 \\ 010 \\ 010 \end{pmatrix} \cup \begin{pmatrix} 100 \\ 000 \\ 100 \end{pmatrix} \cup \begin{pmatrix} 000 \\ 011 \\ 000 \end{pmatrix} \cup \begin{pmatrix} 000 \\ 000 \\ 110 \end{pmatrix} = \begin{pmatrix} 100 \\ 011 \\ 110 \end{pmatrix} = I.$$

Therefore in analogy to natural numbers where a product might equal a sum (such as 2 + 3 + 4 = 3 * 3), matrices can also be computed either as a product or a sum. The Schein rank of a matrix thus also equals the minimal number of columns (called factors) of the left matrix in a *matrix decomposition*, which is also called a *set representation* of a matrix in FCA. Another equivalent definition of the Schein rank is the minimal number of vectors that $\cup$-span the row or column set of a matrix (Kim 1982), which is called 2-dimension of $\overline{I}$ by Ganter & Glodeanu (2012) as it is equivalent to the number of atoms of the smallest Boolean algebra that admits an order embedding of $\langle \overline{I}^{\cup} \rangle$. Table 1 provides an overview of the different ranks and dimensions, not all the details of which are fully discussed in this paper. Figure 6 contains an example of embeddings and natural-number-valued matrices, which are also not further discussed in this paper. The use of the complement $\overline{I}$ means that $\cup$ is considered instead of $\cap$. For FCA applications, an embedding via $\cap$ potentially replaces all attributes with new ones whereas an embedding of a fully represented matrix with $\cup$ maintains the original attributes and is thus more suitable for many applications. The E-dim refers to the dimension of an Euler diagram that can represent a lattice space (Priss & Dürrschnabel 2024). Simple Euler diagrams usually imply an E-dim ≤ 2.

If factors are not minimally chosen because the defining columns of the C-matrices are not $\cap$-independent, a decomposition does not really provide any useful information. In the example of

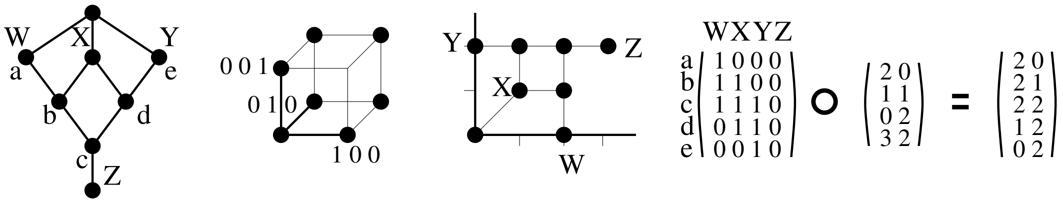

Figure 6: A lattice with embeddings into BA_3 (without bottom node) and C_2 with matrices with integer multiplication and maximum as addition

$J_1 \circ J_2 = I$ above, either the first or the last C-matrix of the union is not needed and the Schein rank of I is 3. Therefore the row set of I is spanned by 3 row matrices (100), (011) and (110), which are the defining rows of 3 C-matrices and can be order embedded into a lattice space $\cup$-spanned by (100), (010) and (001). In this case the Schein rank equals the row and column rank because $\langle I^{\cup} \rangle$ has 3 join-irreducible elements and 3 meet-irreducible elements. Because of the last row in Fig.3, it can be concluded that the order dimension of $\langle I^{\cap} \rangle$ and $\langle \overline{I}^{\cap} \rangle$ is 2.

Lemma 8. *1) For a lattice space: order dimension of* $\langle \overline{I}^{\cap} \rangle \leq$ *Schein rank* $\leq$ *minimum of* $\cup$*-row or column rank.*
2) The Schein ranks of I *and* $\overline{I}$ *can be different.*
3) The E-dim for a fully represented I_f *is the smallest number of L-matrices whose union equals* I_f.

Proof: 1) Holds because a C-matrix is also a Ferrers matrix and a matrix I can always be factorised as $I \circ \mathbb{1}$. Kim (1982) proves that the Schein rank is smaller or equal to the $\cup$-row- and column ranks. For $\cap$ the matrix I equals the union of the C-matrices of the irreducible elements.
2) B_6 has a Schein rank of 4. Its complement has a Schein rank of 6.
3) It is to be shown that $\langle I_f^{\cap} \rangle$ can be drawn as an E-dim Euler diagram. Each row of I_f is a union of rows of L-matrices thus belongs to an exact zone for each L-matrix. Because I_f is fully represented it is not possible to have rows in $I^{\cap\downarrow}$ which are not unions of rows of L-matrices.

If the union of two Ferrers matrices was always an L-matrix, then the E-dim would be half the order dimension of $\langle \overline{I} \rangle$. If I is a disjoint union of two Ferrers matrices then $\langle I \rangle$ and $\langle \overline{I} \rangle$ both have an order dimension of 2. But if I is not fully represented then the $\cap$-lattice space of a union can contain more elements than a union of the elements of the individual lattice spaces. Therefore, while a single L-matrix has E-dim = 1, the union of two L-matrices can have E-dim > 2. Although a fully represented I_f may be impractical, it is also possible to just add all rows to I_f which pertain to 'interesting combinations' (i.e. what Priss & Dürrschnabel call non-supplemental concepts) in order to be able to represent more lattices with Euler diagrams. If I_f is a union of two L-matrices whose columns are either all 0s or equal a column of I_f then a representation as a tabular Euler diagram is possible. A more detailed discussion of this topic will be left for a different paper.

Figure 7 shows a decomposition of a Boolean lattice with 4 atoms into two lattices with factor concepts W, X, Y and Z. Because the union of the C-matrices of the factors results in the original matrix, the complete information is preserved in the factors but potentially more compacted than in a Hasse diagram of a lattice. Although in the example in Fig.7 the row, column and Schein rank are all equal to 4, splitting the lattice on the left into the 2 lattices on the right reduces the complexity somewhat because some concepts are omitted. The reduction of complexity is more significant for larger Boolean lattices because the Schein rank for these increases slowly[3]. If ordinal factors are used instead of Boolean factors, the information is further compacted because fewer factors are required. The coloured nodes in the left Hasse diagram in Fig.4 show the 2 ordinal factors that factorise that lattice. We would argue, though, that while Boolean factors support a horizontal split of a lattice, ordinal factors are more suited to a vertical split. In this case an Euler diagram could be constructed where each colour represents an axis. The 3 Hasse diagrams on the right of Fig.4 represent a decomposition of the form

[3]https://oeis.org/A305233.

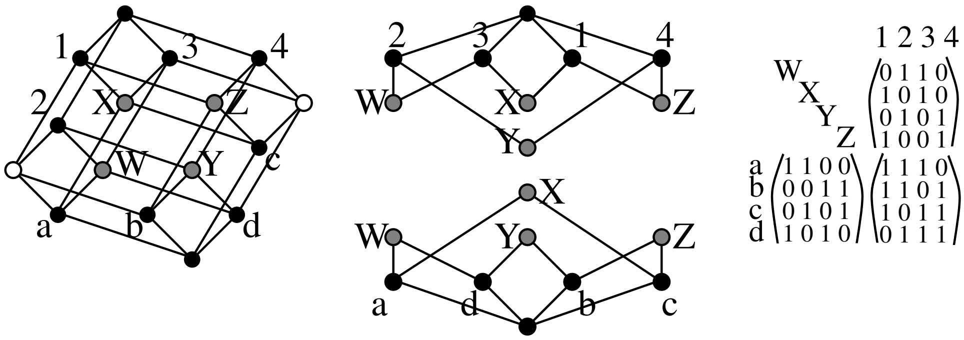

Figure 7: Decomposition of a Boolean lattice with 4 elements

$I_1 \circ I_2 \circ I_3$ of the middle lattice where the grey coloured nodes are shared factors between the lattices. Again the factorised information is more compacted. In particular the 'unlabelled' nodes of the middle lattice are omitted.

A further question is whether relevant information can be maintained by multiplication. In linear algebra, a randomly created square real-valued matrix is most likely invertible and thus will maintain ranks. But for example for I from Fig.1, with $I \circ I = \begin{pmatrix} 100 \\ 000 \\ 100 \end{pmatrix} \cup \begin{pmatrix} 000 \\ 011 \\ 011 \end{pmatrix} \cup \begin{pmatrix} 000 \\ 110 \\ 000 \end{pmatrix} = \begin{pmatrix} 100 \\ 111 \\ 111 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 11 \end{pmatrix} \circ \begin{pmatrix} 100 \\ 011 \end{pmatrix}$ the rank of the product is smaller than the rank of the original matrices. One reason for the loss of information is that the C-matrices are not maximal rectangles of the product matrix and, in FCA terminology, just preconcepts instead of concepts.

Within a group every element can be expressed as a product of every other element. For semigroups that is not the case. The consequences for Boolean matrix multiplication are discussed at length by Kim (1982) using Green's relations but are not further examined in this paper. The following lemma follows from Lemma 6.

Lemma 9. *$\langle (I \circ J)^{\cap} \rangle$ is contained in the smallest distributive lattice that contains $\langle I^{\cap} \rangle$.*

In analogy to $I^{\cap}$ and $I^{\cup}$, which are completions with respect to $\cap$ and $\cup$, it is of interest to discuss what happens to a repeated product of a matrix, in particular if the rows and columns refer to the same set such as the vertices of a graph.

Definition 7. *For $I \in \mathbf{G}_n$:*
1) $I^{\circ} = I \circ I \circ I \circ ...$
2) $I^{\star} = I \cup (I \circ I) \cup (I \circ I \circ I) \cup ...$ (transitive closure of I)
3) $I^{\bullet} = (I \cup \mathbb{1}_n) \circ (I \cup \mathbb{1}_n) \circ ...$ (reachability matrix)

Kim (1982) discusses I°, $I^{\star}$ and $I^{\bullet}$ in detail, which are closely related as shown in the next lemma. For a graph, I has 1s for neighbouring vertices, $I \circ I$ has 1s for vertices that are connected by a 2-step path and so on. The reachability matrix assumes that every vertex also has a 0-step path to itself. Kim states that if a graph is strongly connected, I° has a period or converges to $\mathbb{1}_n$ depending on cycles in the graph. If a graph does not have cycles, I° converges to $\mathbb{0}_n$.

Lemma 10. *$I^{\bullet} = \mathbb{1}_n \cup I^{\star}$*

Proof: It is known from relation algebra that the distributive law holds for $\cup$ and $\circ$. $(I \cup \mathbb{1}_n) \circ (I \cup \mathbb{1}_n) \circ (I \cup \mathbb{1}_n) = (I \circ I \circ I) \cup (I \circ \mathbb{1}_n \circ I) \cup (\mathbb{1}_n \circ I \circ I) \cup (\mathbb{1}_n \circ \mathbb{1}_n \circ I) \cup (I \circ I \circ \mathbb{1}_n) \cup (I \circ \mathbb{1}_n \circ \mathbb{1}_n) \cup (\mathbb{1}_n \circ I \circ \mathbb{1}_n) \cup (\mathbb{1}_n \circ \mathbb{1}_n \circ \mathbb{1}_n)$. Because $I \circ \mathbb{1}_n = I$ even for an infinite sequence only $\mathbb{1}_n \cup I \cup (I \circ I) \cup (I \circ I \circ I) \cup ...$ remains.

$$\begin{array}{c|ccc} & i & a & b \\ \hline i & i & a & b \\ a & a & b & i \\ b & b & i & a \end{array} \quad \begin{array}{c} i \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} a \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c} b \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{array}{c|c} a \circ a & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{d} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Figure 8: Group operations as permutation matrices

5. Matrices of Permutations

Functions and morphisms are probably amongst the most important mathematical structures. While matrices in linear algebra only represent special types of functions (linear transformations), any finite binary relation and thus function can be represented as a Boolean matrix. The analogy to linear transformations for vector spaces are permutation matrices (i.e. matrices of bijective functions) for lattice spaces, which serve to determine whether one matrix can be permuted into another one and thus has the same lattice space. Since bases are representable as reduced matrices, permutations can transform one basis representation into another one. Permutation matrices are considered to be in $\mathbf{G}_n$.

Lemma 11. *1) Matrices with exactly one 1 per row correspond to functions.*
2) Matrices in $\mathbf{G}_n$ with exactly one 1 per row and column are permutations.
3) For a permutation matrix I: $I \circ I^d = I^d \circ I = \mathbb{1}$.
4) Over a Boolean algebra only permutation matrices are invertible.
5) The permutation matrices in $\mathbf{G}_n$ form a group.
6) Any set of permutation matrices that is closed for multiplication ($\circ$) forms a group.

Proof: 1) The 1 in each row associates the row with exactly one column.
2) I.e. it is a function in both directions.
3) If I permutes then I^d permutes in reverse and undoes what I did.
4) cf. Kim (1982) p. 23/24.
5) Holds because the axioms are fulfilled: multiplying permutation matrices yields permutation matrices. $(\mathbf{G}_n, \circ)$ is a monoid. Inverses exist according to 4).
6) Holds because the axioms are fulfilled.

Multiplying a matrix (either from the left or the right) with a permutation matrix does not alter the lattice space of a matrix because a space does not depend on the ordering of rows and columns (according to Lemma 3). If a permutation matrix P is applied to a matrix $I \in \mathbf{G}_n$, it should be applied as $P \circ I \circ P^d$ so that the columns and rows are permuted simultaneously and the i-th row still corresponds to the i-th column.

Using matrices for permutations demonstrates a close connection between groups and permutation matrices. A permutation group is a subgroup of the symmetric group. According to Cayley's theorem every finite group is isomorphic to a permutation group because each group element can be thought of as a permutation. Figure 8 shows an example where each element of a cyclic group with 3 elements is represented as a permutation matrix so that matrix multiplication corresponds to group operation. Each column of a group table is seen as a permutation caused by that element. Contrary to the common representation of groups using vector spaces over complex numbers, the rank of Boolean matrices is higher and equals the number of elements in the group. The matrices in Fig.8 show group operations but not group automorphisms, which would need to map the group identity i onto itself. Kim (1982) provides many further details about group and semigroup properties of Boolean matrices.

6. Triangular Matrices

Definition 8. *A* triangular *matrix contains 1s either only on the diagonal and below (lower triangular) or only on the diagonal and above (upper triangular).*

$$\begin{pmatrix} 10000 \\ 11000 \\ 11100 \\ 10110 \\ 11111 \end{pmatrix} \quad \begin{pmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \end{pmatrix} \quad \begin{pmatrix} 100 \\ 110 \\ 111 \end{pmatrix} \begin{pmatrix} 100 \\ 110 \\ 110 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \\ 111 \end{pmatrix} \begin{pmatrix} 000 \\ 110 \\ 111 \end{pmatrix} \begin{pmatrix} 100 \\ 100 \\ 110 \end{pmatrix} \begin{pmatrix} 000 \\ 110 \\ 110 \end{pmatrix} \begin{pmatrix} 000 \\ 100 \\ 111 \end{pmatrix} \cdots \begin{pmatrix} 000 \\ 000 \\ 000 \end{pmatrix}$$

Figure 9: $\overline{I}^{\cap}$ from Fig.2, antichain and Ferrers matrices (with Catalan number possibilities)

In this paper all triangular matrices are lower triangular matrices and the attribute 'lower' is omitted. Obviously upper and lower triangular matrices are dual to each other. Triangular matrices are usually sparse. While Ferrers matrices were introduced as containing a 'triangle of 1s', their triangle needs to be completely filled with 1s whereas the triangle of a triangular matrix might also contain 0s. Furthermore, Ferrers matrices might contain duplicate rows and columns. Therefore only reduced Ferrers matrices are certain to be representable as triangular matrices. Part 4 of the following lemma is one example of a combinatorial result about matrices and is included in this paper in order to highlight how Boolean matrices relate to various topics of discrete mathematics. Kim (1982) discusses many more combinatorial results. Poset matrices are considered to be in $\mathbf{G}_n$, which means that row i and column i belong to the same element.

Lemma 12. *1) The set of all (lower) triangular matrices in $\mathbf{G}_n$ is closed under union and multiplication and thus forms a subsemiring.*
2) If I is triangular then $I^\bullet$ is a poset matrix.
3) The fully represented matrix of every poset can be arranged into a triangular matrix.
4) The number of possible (lower) triangular matrices in $\mathbf{G}_n$ that do not have 0s to the left side of a 1 is the Catalan number C_n.

Proof: 1) Holds for multiplication because of the C-matrix principle.
2) $I^\bullet$ is reflexive because it contains $\mathbb{1}_n$, antisymmetric because it is triangular and transitive because it contains the transitive closure $I^\star$.
3) Holds because a poset does not contain cycles (Kim (1982) Theorem 5.4.25).
4) cf. Volkov (2025).

Because of Lemma 12.2 and 12.3, triangular matrices present an alternative representation for lattice diagrams other than Hasse diagrams providing a linearisation of the lattice. For example, Fig.9 displays a triangular matrix for $\overline{I}^{\cap}$ from Fig.2. A fully represented triangular $I^\bullet$ with minimal number of 1s has 1s exactly and only on the diagonal and corresponds to the poset of an antichain (2nd from the left in Fig.9). A triangular matrix with maximal number of 1s has 1s on the diagonal and everywhere below and corresponds to a Ferrers matrix (3rd from the left in Fig.9). The matrices described in Lemma 12.4, some of which are shown in Fig.9, are all fully represented, not necessarily reduced Ferrers matrices of row rank ≤ 3. They form a lattice based on inclusion ($\subseteq$), which is not the same construction as in Def.4. If one mirrors and copies the triangle into the other empty triangle, one obtains matrices in $\mathbf{G}_n$ of symmetric relations or undirected graphs (potentially with loops). Ganter & Wille (1999, Chapter 1.4) explain that the lattices of such matrices correspond to complete polarity lattices, that is lattices with an order-reversing bijection that is inverse to itself. The lattice in the middle of Fig.4 is almost of that kind. Unless the data also displays a symmetry, splitting a lattice into 2 symmetric halves may not be relevant for applications.

7. Regular Matrices

Lemma 7 implies that a reflexive fully represented triangular matrix is idempotent exactly if it is transitive and therefore a poset matrix. A non-reflexive idempotent might contain rows and columns of just 0s (Kim 1982, Theorem 2.1.13). But often such rows and columns are removed when a matrix is reduced. Therefore reduced idempotent matrices can be reflexive. But an idempotent matrix can be permuted to lose all its properties and not be reflexive, antisymmetric, transitive or idempotent (for example permuting $\left(\begin{smallmatrix}10\\01\end{smallmatrix}\right)$ into $\left(\begin{smallmatrix}01\\10\end{smallmatrix}\right)$). In that case the matrix is only regular instead of idempotent as defined below. The $\cup$-lattice space of a regular matrix is a distributive lattice and every (finite) distributive lattice is a $\cup$-lattice space of a regular matrix. Thus in some sense, the complements of reflexive, transitive matrices are correlated with distributive lattice spaces. Idempotent matrices tend to be sparse and their complements dense. The details can be found in Kim (1982) but because he favours union over intersection, his definitions and theorems usually apply to complements.

Definition 9. I *is* regular $\iff \exists K : I \circ K \circ I = I$.

Lemma 13. *For matrices $I, J \in \mathbf{M}_n$ and K as in Def. 9:*
1) I is idempotent $\Longrightarrow$ I is regular.
2) J is regular $\Longrightarrow$ a product of J of the same rank exists which is idempotent.
3) J is a permutation of an idempotent $I \Longrightarrow J$ is regular with K as a permutation.
4) J is regular $\iff$ there exists an idempotent, triangular I that is reflexive (except for rows/columns with just 0s) with $\langle \overline{J}^{\cap} \rangle \cong \langle \overline{I}^{\cap} \rangle$.
5) 2 matrices with isomorphic lattice spaces are either both regular or both not regular.
6) J is regular $\iff$ the lattice space $\langle \overline{J}^{\cap} \rangle$ is distributive.
7) For regular matrices the Schein rank equals the row and column rank of $I^{\cup}$.
8) Matrix J is regular $\iff J \subseteq J \circ \overline{J^d \circ \overline{J} \circ J^d} \circ J$

Proof: 1) choose $K = \mathbb{1}$
2) $J \circ K \circ J = J \Longrightarrow \exists x, y : J \circ x \circ y \circ J = J \Longrightarrow y \circ J \circ x \circ y \circ J \circ x = y \circ J \circ x$.
Multiplying with x or y cannot reduce the rank because otherwise $J \circ x \circ y \circ J \neq J$.
3) $J = P_1 \circ I \circ P_2$ with permutations P_1, P_2 and choose $K = P_2^{-1} \circ P_1^{-1}$,
$J \circ K \circ J = P_1 \circ I \circ P_2 \circ P_2^{-1} \circ P_1^{-1} \circ P_1 \circ I \circ P_2 = P_1 \circ I \circ I \circ P_2 = J$
4) According to Kim (1982) Theorem 2.1.13.
5) Follows from 2)
6-8) Kim (1982) Theorem 2.1.29, Corollary 2.1.11., Theorem 2.1.26

The matrix I in Lemma 13.4 is a matrix of a poset. Therefore $\langle \overline{I}^{\cap} \rangle$ is the lattice space of a contraordinal scale according to Ganter & Wille (1999), who also state that 'every finite distributive lattice is isomorphic to the concept lattice of a contraordinal scale'. The main structure of an idempotent matrix is already contained in its corresponding poset. Thus decomposition may not reduce complexity much (because of $I \circ I = I$). On the contrary, the lattice space of the complement of an idempotent matrix is distributive and thus contains potentially many more elements. Its decomposition might reduce complexity as shown in the examples in Fig.4 and Fig.7.

In conclusion, this paper presents the main concepts of a combination of Boolean matrix theory and FCA. Potential applications are different forms of representations (as Hasse diagrams or matrices), combinations of $\cap$- and $\cup$-constructions for the space above and below the attributes and different forms of reductions using decomposition or union/intersection of C-or Ferrers matrices. Because Boolean matrices are related to many topics of discrete mathematics there are numerous algebraic properties that are not included in this paper. Various further extensions are possible: 3 dimensional matrices for triadic FCA, matrices with integers representing counts such as Repeated Attribute Lattices (Priss 2026), relation algebra, and so on.

Declaration on Generative AI

The author did not employ any Generative AI tools for writing this paper.

References

[1] Abdulla, M., Hirth, J., Stumme, G. (2024). The Birkhoff completion of finite lattices. International Joint Conference, CONCEPTS, Springer, LNAI 14914, p. 20-35.

[2] Belohlavek, R.; Vychodil, V. (2010). Discovery of optimal factors in binary data via a novel method of matrix decomposition. Journal of Computer and System Sciences 76, 1, p. 3-20.
[3] Doignon, J. P.; Falmagne, J. C. (1999). Knowledge Spaces, Springer.
[4] Dürrschnabel, D.; Priss, U. (2024). Realizability of Rectangular Euler Diagrams. International Joint Conference, CONCEPTS, Springer, LNAI 14914, p. 149-165.
[5] Ganter, B.; Wille, R. (1999). Formal Concept Analysis - Mathematical Foundations. Springer.
[6] Ganter, B.; Glodeanu, C. (2012). Ordinal factor analysis. International Conference on Formal Concept Analysis, ICFCA, Springer, LNCS 7278, p. 128–139.
[7] Kim, K. H. (1982). Boolean Matrix Theory and Applications. Marcel Dekker, New York.
[8] Maddux R. (1996). Relation-algebraic semantics. Theoretical Computer Science, 160, p. 1-85.
[9] Petersen, W. (2010). Linear Coding of Non-Linear Hierarchies: Revitalization of an Ancient Classification Method. In A. Fink et al., (eds.), Advances in Data Analysis, Data Handling and Business Intelligence, Springer, p. 307-316.
[10] Priss, U. (2009). Relation Algebra Operations on Formal Contexts. In: Rudolph; Dau; Kuznetsov (eds.), Proceedings of ICCS'09, Springer, LNCS 5662, p. 257-269.
[11] Priss, U.; Dürrschnabel, D. (2024). Rectangular Euler Diagrams and Order Theory. In: Lemanski et al. (eds.), Proceedings of Diagrams 2024, Springer, LNAI 14981, p. 165-181.
[12] Priss, U. (2026). Alternative Diagrams for Concept Lattices. International Joint Conference, CONCEPTS, Springer, LNAI, to appear.
[13] Volkov, M. (2025). A new Boolean matrix representation for Catalan semirings. arXiv preprint arXiv:2510.05045.