A Kripke Semantics and Tableau for Monadic Wajsberg Logic

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Abstract

We develop a Kripke-style relational semantics and labelled tableau calculus for monadic Wajsberg logic, the implication–negation presentation of the monadic fragment of infinite-valued Łukasiewicz logic. The semantics isolates the order, involution, partial operations for Wajsberg implication, and equivalence relation arising in the dual representations of Martínez and Figallo-Orellano. The corresponding relational conditions become structural tableau rules. We prove soundness and completeness of the tableau with respect to our semantics, and obtain the finite relational countermodel property: every non-valid formula fails in a finite monadic Wajsberg model.

Keywords

Łukasiewicz logic, monadic operators, Kripke semantics, Wajsberg logic, Labelled tableau

1. Introduction

Recent work in fuzzy logics, of which Łukasiewicz logic is a paradigmatic example, has sought to extend Kripke-style relational semantics from intuitionistic and modal logic to constructive fuzzy [1] and substructural logics [2, 3], particularly those surrounding Hájek's Basic Logic (BL)[4, 5, 6] and Esteva and Godo's Monoidal T-Norm Logic (MTL) [7, 8]. The motivation for developing such semantics is to unify disparate fuzzy logics relationally, in a manner analogous to Kripke's unification of normal modal logics, together with the standard attendant connections to model theory [9], decidability [10] and correspondence theory [11]. From this perspective, MTL, BL and GBL may be viewed as fuzzy counterparts of intuitionistic and superintuitionistic systems.

Many valued-algebras (hereon MV-algebras) provide the standard algebraic semantics for infinite-valued propositional Łukasiewicz logic. Monadic MV-algebras[1] add a unary quantifier [13] to the picture, providing an algebraic semantics for the one-variable monadic fragment of first-order infinite-valued Łukasiewicz logic [14]. Wajsberg algebras [15] are term-equivalent to MV-algebras but take implication and negation as primitives [16], thus providing an implication–negation presentation of the same monadic logic.[2]

The present paper[3] expands our programme studying fuzzy logics via relational semantics to include monadic Wajsberg logic, where to the best of our knowledge no such relational semantics has yet been devised. Similarly, it is a longstanding open problem to provide semantically motivated analytic tableau for fuzzy logics, including Łukasiewicz logic.

The semantic basis of our work is the duality results of Martinez and Figallo-Orellano for Monadic MV-algebras, containing both the involution g and the family of partial operations $\{\Phi_v\}_{v\in W}$ that arise in their representations. In the present semantics, implication is governed by the partial operation $(v, w) \longmapsto \Phi_v(w)$, defined when $v \leq g(w)$, with $\Phi_v(w)$ determining the point at which the consequent is tested. Furthermore, the present paper studies a monadic operator in the Wajsberg context. Figallo-

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[1]Wajsberg was also the first to connect the modal logic S5 with monadic Boolean algebras: See [12].

[2]Throughout the paper, "Wajsberg" refers to this logical signature, while "MV" is retained when discussing the established algebraic and duality-theoretic literature.

[3]The present paper is a truncated version of a much longer paper with detailed proofs and calculations given. The reader is encouraged to read that paper[17] for various details.

Orellano's [18] monadic extension of the Martínez original duality (MV-algebras) yields an equivalence relation R representing the monadic operator and interaction conditions with g and the partial maps Φ_v.

Our contribution extracts relational content from these dualities, providing a labelled tableau proof theory. Negation is controlled by an order-reversing involution g, implication by the partial operations Φ_v, and the monadic operator by R. The interaction conditions on these structures become structural tableau rules, while failures of implication and the monadic operator generate explicit witnesses. This produces a Kripke-style semantics and labelled tableau calculus adequate for monadic Wajsberg logic. The paper is organised as follows. Section 2 recalls Wajsberg and monadic MV-algebras. Section 3 introduces monadic Wajsberg frames, and Section 4 gives the forcing clauses and intuition behind Φ. Section 5 presents the tableau calculus and a worked monadic example. Sections 6 and 7 establish soundness and completeness, respectively. Finally, we obtain the finite relational countermodel property. We conclude with further directions.

2. Preliminaries on Wajsberg algebras and monadic MV-algebras

2.1. Wajsberg algebras

Definition 1. *A* Wajsberg algebra *is an algebra* $\mathbf{A} = \langle A, \to, \neg, 1\rangle$ *satisfying the following identities:*

$$\begin{aligned} &\text{(W1)} \quad 1 \to x = x, \\ &\text{(W2)} \quad (x \to y) \to ((y \to z) \to (x \to z)) = 1, \\ &\text{(W3)} \quad (x \to y) \to y = (y \to x) \to x, \\ &\text{(W4)} \quad (\neg y \to \neg x) \to (x \to y) = 1. \end{aligned}$$

The constant 0 is defined as $0 := \neg 1$. The lattice operations are definable by $x \vee y := (x \to y) \to y$, $x \wedge y := \neg(\neg x \vee \neg y)$. One can equivalently phrase the preceding as the Wajsberg algebra $\mathbf{A}$ as a many-valued (hereon MV)-algebra[4] [19] in the Wajsberg language, given the known term-equivalence between the two systems [15].

2.2. Monadic Wajsberg algebras

Definition 2. *A* monadic Wajsberg algebra*, equivalently a monadic MV-algebra*[5] *in the Wajsberg language, is an algebra* $\mathbf{A} = \langle A, \to, \neg, \forall, 1\rangle$ *such that* $\langle A, \to, \neg, 1\rangle$ *is a Wajsberg algebra and* $\forall$ *satisfies [18]*[6]*:*

$$\begin{aligned} &\text{(M1)} \quad \forall x \vee x = x, \\ &\text{(M2)} \quad \forall(\forall x \to y) = \forall x \to \forall y, \\ &\text{(M3)} \quad \forall(\neg x \to x) = \neg\forall x \to \forall x. \end{aligned}$$

The existential quantifier is defined by $\exists x := \neg\forall\neg x$.

We stated above that we can refer to the monadic Wajsberg algebra as the monadic MV-algebra in the Wajsberg language – below is a result of [20] that justifies this usage.

Proposition 1. *Monadic Wajsberg algebras and monadic MV-algebras are term-equivalent. The translations are given by:* $x \oplus y := \neg x \to y$*,* $0 := \neg 1$*,* $\exists x := \neg\forall\neg x$*, and, conversely,* $x \to y := \neg x \oplus y$*,* $1 := \neg 0$*,* $\forall x := \neg\exists\neg x$*.*

[4] An MV-algebra is an algebra $\langle A, \oplus, \neg, 0\rangle$ satisfying $x \oplus y = y \oplus x$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, $x \oplus 0 = x$, $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ [19].

[5] A *monadic MV-algebra* [14] is an MV-algebra $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$ equipped with a unary operation $\exists$ satisfying, for all $x, y \in A$, (E1) $x \vee \exists x = \exists x$; (E2) $\exists(x \vee y) = \exists x \vee \exists y$; (E3) $\exists\neg\exists x = \neg\exists x$; (E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$; (E5) $\exists(x \odot x) = \exists x \odot \exists x$; and (E6) $\exists(x \oplus x) = \exists x \oplus \exists x$, where $x \odot y := \neg(\neg x \oplus \neg y)$.

[6] This is exactly the formulation given by Figallo-Orellano, that is, using the language of monadic Wajsberg algebras, although he presents his results for Monadic MV.

In what follows, we make use of filters [21], so we fix a definition here for ease of reference:

Definition 3. *Let* $\mathbf{A}$ *be a Wajsberg algebra, and consider its bounded lattice reduct* $\langle A, \vee, \wedge, 0, 1\rangle$. *A subset* $P \subseteq A$ *is a* filter*[22] if*

1. $1 \in P$;
2. *if* $x, y \in P$, *then* $x \wedge y \in P$;
3. *if* $x \in P$ *and* $x \leq y$, *then* $y \in P$.

A lattice filter P *is* proper *if* $P \neq A$, *equivalently if* $0 \notin P$. *A proper lattice filter* P *is* prime *if* $x \vee y \in P$ *then* $x \in P$ *or* $y \in P$.[7]

We note in passing the stronger notion of implicative Wajsberg filter, which is also considered in [22]; however, the above is precisely required to obtain the Priestley duality upon which this paper is based.

3. Syntax and monadic Wajsberg frames

3.1. Language

Let Prop be a countable set of propositional variables. The set For of formulas is generated inductively by

$$\varphi ::= p \mid 1 \mid \neg\varphi \mid \varphi \to \psi \mid \forall\varphi.$$

where $p \in \mathrm{Prop}$. We define $\exists\varphi := \neg\forall\neg\varphi$, $0 := \neg 1$, $\varphi \vee \psi := (\varphi \to \psi) \to \psi$, $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$.

3.2. Frames and models

Definition 4 (Monadic Wajsberg-frame)**.** *A* monadic Wajsberg-frame *is a structure* $\mathfrak{F} = \langle W, \leq, g, \{\Phi_v\}_{v\in W}, R\rangle$ *satisfying the following conditions.*

1. $(W, \leq)$ *is a quasi-order.*
2. $g : W \to W$ *is an order-reversing involution:*
$$g(g(w)) = w, \qquad w \leq v \Longrightarrow g(v) \leq g(w).$$
3. *For each* $v \in W$, *put* $D_v := \{w \in W : v \leq g(w)\}$, *and let* $\Phi_v : D_v \to W$ *be defined exactly on* D_v. *For increasing* $U, V \subseteq W$[8], *write:*
$$U \to V := \bigcap_{p\in U} \left(D_p^c \cup \Phi_p^{-1}(V)\right), \qquad D_p^c := W \setminus D_p.$$
The family $\{\Phi_v\}_{v\in W}$ *satisfies:*
   (a) *each* Φ_v *is order-preserving on* D_v;
   (b) *if* $v \leq g(w)$, *then* $v \leq \Phi_v(w)$ *and* $w \leq \Phi_v(w)$ $\quad$ (Φ-*up*);
   (c) *if* $v \leq g(w)$, *then* $\Phi_v(w) = \Phi_w(v)$ $\quad$ (Φ-*sym*);
   (d) *whenever defined,* $\Phi_v(g(\Phi_v(w))) \leq g(w)$ $\quad$ (Φg);
   (e) *whenever defined,* $\Phi_v(\Phi_{v'}(w)) = \Phi_{v'}(\Phi_v(w))$ $\quad$ (Φ-*comm*);
   (f) *if* $U, V \subseteq W$ *are increasing and* $w \notin U \cup V$, *then there exists* $v \in W$ *such that* $v \leq g(w)$, $v \in U \to V$, *and* $\Phi_v(w) \notin V$ $\quad$ (Φ-*crit*).
4. R *is an equivalence relation. For* $U \subseteq W$, *put*
$$\forall_R(U) := \{w \in W : R(w) \subseteq U\}, \qquad R(w) := \{v \in W : wRv\},$$
and put $\neg U := \{w \in W : g(w) \notin U\} = W \setminus g(U)$.

[7] Unless we specify otherwise, when it is clear from the context, we will assume the lattice filters are proper.

[8] A subset $U \subseteq W$ is *increasing* (with respect to $\subseteq$) if $w \in U$ and $w \leq v$ imply $v \in U$. The same convention applies to V.

5. *R is compatible with g, i.e. $wRv \Longrightarrow g(w)Rg(v)$.* (Rg)
6. *R is compatible with the order, i.e. $x \le y$ & $yRz \Longrightarrow \exists t \in W(xRt$ & $t \le z)$.* $(R \le)$
7. *R is compatible with the Wajsberg operations:*

$$xRy,\ pRq,\ p \le g(x),\ q \le g(y) \Longrightarrow \Phi_p(x) R \Phi_q(y). \qquad (R\Phi)$$

8. *For all increasing $U, V \subseteq W$, the following Figallo-Orellano [18] conditions[9] hold:*
   (s5) *If xRy and there exists $p_0 \in \forall_R(U)$ such that $p_0 \le g(x)$ and $\Phi_{p_0}(x) \notin \forall_R(V)$, then there exists $p_1 \in \forall_R(U)$ such that $p_1 \le g(y)$ and $\Phi_{p_1}(y) \notin \forall_R(V)$.*
   (s6) *If $p \in \forall_R(U)$, $p \le g(z)$, and $\Phi_p(z) \notin \forall_R(V)$, then there exist $p_1, z_1 \in W$ such that*

$$p_1 \in \forall_R(U), \quad p_1 \le g(z_1), \quad \Phi_{p_1}(z_1) \notin V, \quad z_1 R z.$$

   (s7) *If xRy and there exists $p_0 \in \neg U$ such that $p_0 \le g(x), \Phi_{p_0}(x) \notin V$, then there exists $p_1 \in \neg\forall_R(U)$ such that $p_1 \le g(y), \Phi_{p_1}(y) \notin \forall_R(V)$.*
   (s8) *If $p \in \neg\forall_R(U)$, $p \le g(z)$, and $\Phi_p(z) \notin \forall_R(V)$, then there exist $p_1, z_1 \in W$ such that:*

$$p_1 \in \neg\forall_R(U), \quad p_1 \le g(z_1), \quad \Phi_{p_1}(z_1) \notin V, \quad z_1 R z.$$

Lemma 1. *Let $(W, \le, R)$ be a quasi-ordered set equipped with an equivalence relation R. The following are equivalent:*

1. *for every increasing set $U \subseteq W$, $\forall_R(U)$ is increasing;*
2. *when $x \le y$ and yRz, there exists $t \in W$ such that xRt and $t \le z$.*

Proof. See [17]. □

Remark 1 (Algebraic origin of the semantics). *Let $\mathbf{A}$ be a monadic MV-algebra and let $\mathfrak{F}_{\mathbf{A}}$ be its dual structure in the sense of Martínez and Figallo-Orellano. Under the dual representation, Wajsberg implication is represented by*

$$U \to V = \bigcap_{v \in U} \left(D_v^c \cup \Phi_v^{-1}(V) \right),$$

while the monadic operation is represented by

$$\forall_R(U) = \{w \in W \ : \ R(w) \subseteq U\}.$$

Hence the algebraic interpretations of $\neg$, $\to$, and $\forall$ agree with the corresponding relational clauses of the present semantics on the sets represented in the dual algebra.

Proof. This follows from the dual representation of Wajsberg implication established by Martínez and its monadic extension due to Figallo-Orellano. The present frames retain precisely the relational operations used in those representations. □

Definition 5 (Valuation). *Let $\mathfrak{F}$ be a monadic Wajsberg-frame. A valuation on $\mathfrak{F}$ is a map $V : \mathrm{Prop} \to \mathscr{P}^{\uparrow}(W)$, where $\mathscr{P}^{\uparrow}(W)$ is the family of all $\le$-increasing subsets of W.*

Definition 6 (Monadic Wajsberg-model). *A* monadic Wajsberg-model *is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a monadic Wajsberg-frame and V is a valuation on $\mathfrak{F}$.*

[9] We hereon refer to (s5)-(s8) as the "Figallo-Orellano conditions"; the full frame conditions we refer to as "Martinez-Figallo-Orellano conditions".

4. Forcing clauses

We now define the forcing relation for monadic Wajsberg logic. Negation is interpreted via the order-reversing involution g, Wajsberg implication is interpreted via the family of partial operations $\{\Phi_v\}_{v\in W}$, and the monadic quantifier is interpreted through the equivalence relation R. Let $\mathfrak{M} = \langle W, \leq, g, \{\Phi_v\}_{v\in W}, R, V\rangle$ be a monadic Wajsberg-model. The forcing relation is defined inductively as follows:

$$\begin{aligned}
\mathfrak{M}, w \Vdash p &\iff w \in V(p),\\
\mathfrak{M}, w \nVdash 0, &\qquad \mathfrak{M}, w \Vdash 1,\\
\mathfrak{M}, w \Vdash \neg\varphi &\iff \mathfrak{M}, g(w) \nVdash \varphi,\\
\mathfrak{M}, w \Vdash \varphi \to \psi &\iff \forall v \in W\big(\mathfrak{M}, v \Vdash \varphi \;\&\; v \leq g(w) \Rightarrow \mathfrak{M}, \Phi_v(w) \Vdash \psi\big),\\
\mathfrak{M}, w \Vdash \forall\varphi &\iff \forall v \in W\big(wRv \Rightarrow \mathfrak{M}, v \Vdash \varphi\big).
\end{aligned}$$

Recall connectives $\wedge$ and $\vee$ are abbreviations in the primitive language defined in Section 3.1, so the above is sufficient.For a formula φ, write $[\![\varphi]\!]_{\mathfrak{M}} = \{w \in W : \mathfrak{M}, w \Vdash \varphi\}$.

Lemma 2 (Semantic Persistence). *For every monadic Wajsberg-model $\mathfrak{M}$, every formula φ, and all $w, w' \in W$, if $w \leq w'$ and $\mathfrak{M}, w \Vdash \varphi$ then $\mathfrak{M}, w' \Vdash \varphi$. Hence $[\![\varphi]\!]_{\mathfrak{M}}$ is increasing.*

Proof. By induction on φ. For details, see [17]. □

Proposition 2 (Complex algebra of a frame). *Let $\mathfrak{F}$ be a monadic Wajsberg-frame. Then $\mathfrak{F}^+ = \langle \mathscr{P}^{\uparrow}(W), \to, \neg, \forall_R, W\rangle$ is a monadic Wajsberg algebra.*

Proof. See [17]. □

Definition 7 (Relational Validity). *Let $\mathfrak{M} = \langle W, \leq, g, \{\Phi_v\}_{v\in W}, R, V\rangle$ be a monadic Wajsberg-model. A formula φ is* valid *in $\mathfrak{M}$ if $\mathfrak{M}, w \Vdash \varphi$ for every $w \in W$. It is* valid *on a monadic Wajsberg-frame $\mathfrak{F}$ if it is valid in every model based on $\mathfrak{F}$, and* valid *in the class of monadic Wajsberg-frames if it is valid on every such frame.*

Definition 8. *(Algebraic valuation) Let $\mathbf{A} = \langle A, \to, \neg, \forall, 1\rangle$ be a monadic Wajsberg algebra. An* algebraic valuation *on $\mathbf{A}$ is a map $v\colon$ Prop $\to A$, extended recursively to all formulas by:*

$$\begin{aligned}
v(\varphi \to \psi) = v(\varphi) \to v(\psi), &\quad v(\neg\varphi) = \neg v(\varphi),\\
v(\forall\varphi) = \forall v(\varphi), &\quad v(1) = 1.
\end{aligned}$$

Definition 9 (Monadic Wajsberg tautology). *A formula φ is a* monadic Wajsberg tautology *if $\vDash_{\mathbf{MW}} \varphi$, that is, if $v(\varphi) = 1$ for every monadic Wajsberg algebra $\mathbf{A}$ and every algebraic valuation v on $\mathbf{A}$.*

Equivalently, a formula φ is a monadic Wajsberg tautology if and only if the identity $\varphi \approx 1$ holds in every monadic Wajsberg algebra.

Definition 10 (Consequence). *Let $\Gamma \cup \{\varphi\} \subseteq$ For. We write $\Gamma \vDash_{\mathbf{MW}} \varphi$ if, for every monadic Wajsberg algebra $\mathbf{A}$ and every algebraic valuation v on $\mathbf{A}$, $v(\gamma) = 1$ for every $\gamma \in \Gamma \Longrightarrow v(\varphi) = 1$. We write $\Gamma \vDash_{\mathrm{rel}} \varphi$ if, for every monadic Wajsberg-model $\mathfrak{M}$, $\mathfrak{M} \vDash \gamma$ for every $\gamma \in \Gamma \Longrightarrow \mathfrak{M} \vDash \varphi$, where $\mathfrak{M} \vDash \chi$ means $\mathfrak{M}, w \Vdash \chi$ for every world w of $\mathfrak{M}$.*

Remark 2. *Relational consequence is global: the premises must hold at every world of the model. This corresponds to the algebraic truth-preserving consequence relation, where premises take the designated value 1. This is known in the literature as the 1-assertional consequence relation determined by the algebra or variety [23] – in this case, (the variety) [24] of Monadic Wajsberg algebras. The tableau calculus developed below concerns validity, i.e. the empty-premise case of these consequence relations.*

4.1. Intuitive Explanation of the Semantics

One possible interpretation of the relational semantics is to regard the points of the dual space not simply as truth values, but as thresholds or *states of determination*. Since the points of the dual space are prime filters, each point records a threshold at which propositions are accepted. On this reading, the partial maps $\{\Phi_v\}_{v\in W}$ are not to be understood as degrees of determination, as one might be tempted to conclude (given Wajsberg is a fuzzy logic). Rather, the partial functions describe how these states of determination interact in the evaluation of implication.

Thus Wajsberg implication is not interpreted by simply quantifying over order extensions, quite unlike Intuitionistic implication. Instead, implication is represented through the family of partial operations $\Phi_v : D_v \to W$, $D_v = \{w \in W : v \leq g(w)\}$. Intuitively, $\Phi_v(w)$ is the point at which the consequent is tested when v witnesses the antecedent at the evaluation point w. So, implication does not merely inspect an accessibility relation: the pair (v, w) determines a further evaluation point. To illustrate: suppose $a \Vdash \varphi$, $a \leq g(b)$, and $\Phi_a(b) = c$. Then a is one of the points relevant to the evaluation of $\varphi \to \psi$ at b, and the consequent must therefore hold at the point $c = \Phi_a(b)$. Thus $c \nVdash \psi$ immediately gives $b \nVdash \varphi \to \psi$. Conversely, for $b \Vdash \varphi \to \psi$ to hold, every point a such that $a \Vdash \varphi$ and $a \leq g(b)$ must satisfy $\Phi_a(b) \Vdash \psi$. The rejection rule for implication reflects exactly this failure: from $b \nVdash \varphi \to \psi$ it introduces a point a satisfying the antecedent and the domain condition, together with a point $c = \Phi_a(b)$ at which the consequent fails. Hence implication does not merely preserve truth across accessible worlds, but combines or transforms graded states of determination, with Φ specifying the state at which the consequent must be evaluated.[10]

4.2. Summary of the Construction and Example

The relational construction can be summarised as follows. Given a monadic Wajsberg algebra $\mathbf{A}$, we proceed in the following steps.

1. Calculate the proper prime filters (see Definition 3) of $\mathbf{A}$ and order them by inclusion. This gives the ordered set $(W, \leq)$. We write P_w for the prime filter corresponding to $w \in W$.
2. Define the representation map $\sigma : A \to \mathrm{Up}(W)$ by $\sigma(x) = \{P \in W : x \in P\}$.
3. Define the order-reversing involution $g : W \to W$ by $g(P) = A \setminus \{\neg x : x \in P\}$.
4. For each $v \in W$, determine $D_v = \{w \in W : v \leq g(w)\}$. For $w \in D_v$, define $\Phi_v(w) = \bigcup_{x\in P_v}\{y \in A : x \to y \in P_w\}$.
5. Define the monadic relation R on W by PRQ iff $P \cap \forall A = Q \cap \forall A$, where $\forall A = \{\forall x : x \in A\}$.
6. Given a valuation V into $\mathrm{Up}(W)$, evaluate formulas using the forcing clauses. In particular, $w \Vdash \varphi \to \psi$ iff, for every $v \Vdash \varphi$ such that $v \leq g(w)$, $\Phi_v(w) \Vdash \psi$. The monadic operators are interpreted through R.

We now apply this construction to the two-element Wajsberg/MV-chains, further examples can be found in [17].

Example 1 (The two-element Wajsberg/MV-chain)**.** *Consider the two-element MV-chain* $\mathbf{L}_2 = \{0, 1\}$, *where* $\neg x = 1 - x$, $x \to y = \min\{1, 1 - x + y\}$, *and* $\forall x = x$.

Step 1. *The only proper prime filter is* $P_a = \{1\}$. *Thus* $W = \{a\}$, *with the trivial order.*

Step 2. *We calculate the representation map* $\sigma : A \to \mathrm{Up}(W)$ *by determining, for each algebra element, which prime filters contain it. Since* 0 *belongs to no proper prime filter (see step 1),* $\sigma(0) = \emptyset$. *Since* $1 \in P_a$, *we have* $\sigma(1) = \{a\} = W$. *Thus the two algebra elements are represented by the two increasing subsets of* W: $\emptyset$ *and* $\{a\}$.

Step 3. *We have* $g(a) = a$.

Step 4. *For* $v = a$, *the domain is* $D_a = \{w \in W : a \leq g(w)\}$. *Since* $g(a) = a$, *we have* $a \leq g(a)$, *and hence* $D_a = \{a\}$. *We now calculate* $\Phi_a(a)$. *By definition,* $\Phi_a(a) = \bigcup_{x\in P_a}\{y \in A : x \to y \in P_a\}$. *Since* $P_a = \{1\}$ *and* $1 \to y = y$, *this becomes* $\{y \in A : y \in P_a\} = P_a$. *Hence* $\Phi_a(a) = a$.

[10]We stress that this is a possible conceptual interpretation of the semantics. We do not find it proposed by Martínez or Figallo-Orellano.

Step 5. *Since* $\forall$ *is the identity,* $\forall A = A$*, and therefore* $R = \mathrm{Id}_W$*.*

Step 6. *Take the algebraic assignment* $e(p) = 1$ *and* $e(q) = 0$*. Feeding this assignment through* σ *gives* $V(p) = \sigma(1) = \{a\}$ *and* $V(q) = \sigma(0) = \varnothing$*.*

We now evaluate $p \to q$ *at* a*. First,* $a \Vdash p$*. Since* $g(a) = a$*, the point* a *also satisfies the domain condition* $a \leq g(a)$*. We must therefore inspect* $\Phi_a(a)$*. Since* $\Phi_a(a) = a$ *and* $a \notin V(q)$*, we have* $\Phi_a(a) \nVdash q$*. Hence* $\mathfrak{M}, a \nVdash p \to q$*. Therefore* $\{w \in W \,:\, \mathfrak{M}, w \Vdash p \to q\} = \varnothing = \sigma(0)$*, in agreement with the algebraic calculation* $1 \to 0 = 0$*.*

$$g(a) = a \quad a \qquad p,\ a \nVdash p \to q$$

$$R = \mathrm{Id}_W$$

5. Tableau for monadic Wajsberg logic

The tableau language contains expressions of the following forms:

$$w \Vdash \varphi, \qquad w \nVdash \varphi, \qquad w \leq v, \qquad wRv, \qquad u = \Phi_v(w).$$

A branch is *closed* if it contains both $w \Vdash \varphi$ and $w \nVdash \varphi$ for some w, φ, or if it contains $w \Vdash 0$ or $w \nVdash 1$. A branch is *open* if it is not closed. World terms are normalised with respect to the involution, so that $g(g(w)) \equiv w$. Functional atoms are handled by a global output-and-identification convention. For each ordered pair (v, w), at most one output label is retained for $\Phi_v(w)$. We reuse labels for already recorded values. Moreover, whenever a structural condition forces two previously introduced output labels to denote the same world (in particular through symmetry or commutation of Φ), the labels are identified by uniform substitution throughout the branch. A fresh output label is introduced only when no value for the required functional term has yet been recorded.

Logical and monadic rules.

$$(\neg\text{-acc.})\ \frac{w \Vdash \neg\varphi}{g(w) \nVdash \varphi} \qquad\qquad (\neg\text{-rej.})\ \frac{w \nVdash \neg\varphi}{g(w) \Vdash \varphi}$$

$$(\to\text{-acc.})\ \frac{\begin{array}{c} w \Vdash \varphi \to \psi, \quad v \Vdash \varphi, \\ v \leq g(w), \quad u = \Phi_v(w) \end{array}}{u \Vdash \psi} \qquad\qquad (\to\text{-rej.})\ \frac{w \nVdash \varphi \to \psi}{\begin{array}{c} v \Vdash \varphi, \quad v \leq g(w), \\ u = \Phi_v(w), \quad u \nVdash \psi \end{array}}$$

$$(\forall\text{-acc.})\ \frac{w \Vdash \forall\varphi, \quad wRv}{v \Vdash \varphi} \qquad\qquad (\forall\text{-rej.})\ \frac{w \nVdash \forall\varphi}{wRv, \quad v \nVdash \varphi}$$

In $(\to$ -rej.), v, u are fresh, and in ($\forall$-rej.), v is fresh.

Order and relational rules.

$$(\leq\text{-refl.})\ \frac{}{w \leq w} \qquad (\leq\text{-trans.})\ \frac{w \leq v,\ v \leq u}{w \leq u} \qquad (g\text{-ord.})\ \frac{w \leq v}{g(v) \leq g(w)}$$

$$(R\text{-refl.})\ \frac{}{wRw} \qquad (R\text{-sym.})\ \frac{wRv}{vRw} \qquad (R\text{-trans.})\ \frac{wRv,\ vRu}{wRu}$$

$$(Rg)\ \frac{wRv}{g(w)Rg(v)} \qquad (R\leq)\ \frac{x \leq y, \quad yRz}{xRt, \quad t \leq z}$$

Structural rules for Φ.

$$(\Phi\text{-intro.})\quad \frac{v \leq g(w)}{u = \Phi_v(w)} \qquad\qquad (\Phi\text{-dom.})\quad \frac{u = \Phi_v(w)}{v \leq g(w)}$$

$$(\Phi\text{-mon.})\quad \frac{w \leq w', \quad u = \Phi_v(w), \quad u' = \Phi_v(w')}{u \leq u'} \qquad\qquad (\Phi\text{-up.})\quad \frac{u = \Phi_v(w)}{v \leq u, \quad w \leq u}$$

$$(\Phi\text{-sym.})\quad \frac{u = \Phi_v(w)}{u = \Phi_w(v)} \qquad\qquad (\Phi g)\quad \frac{u = \Phi_v(w), \quad t = \Phi_v(g(u))}{t \leq g(w)}$$

$$(R\Phi)\quad \frac{\begin{array}{c} xRy, \quad pRq, \quad p \leq g(x), \quad q \leq g(y), \\ u = \Phi_p(x), \quad u' = \Phi_q(y) \end{array}}{uRu'} \qquad (\Phi\text{-comm.})\quad \frac{\begin{array}{c} u = \Phi_{v'}(w), \quad z = \Phi_v(u), \\ u' = \Phi_v(w) \end{array}}{z = \Phi_{v'}(u')}$$

For (Φ-intro.), an already recorded value of $\Phi_v(w)$ is reused; otherwise u is fresh.

Figallo–Orellano rules.

$$(s5)\quad \frac{\begin{array}{c} xRy, \quad p_0 \Vdash \forall\alpha, \quad p_0 \leq g(x), \\ u_0 = \Phi_{p_0}(x), \quad u_0 \nVdash \forall\beta \end{array}}{\begin{array}{c} p_1 \Vdash \forall\alpha, \quad p_1 \leq g(y), \\ u_1 = \Phi_{p_1}(y), \quad u_1 \nVdash \forall\beta \end{array}} \qquad (s6)\quad \frac{\begin{array}{c} p \Vdash \forall\alpha, \quad p \leq g(z), \\ u = \Phi_p(z), \quad u \nVdash \forall\beta \end{array}}{\begin{array}{c} p_1 \Vdash \forall\alpha, \quad p_1 \leq g(z_1), \\ u_1 = \Phi_{p_1}(z_1), \quad u_1 \nVdash \beta, \quad z_1 R z \end{array}}$$

$$(s7)\quad \frac{\begin{array}{c} xRy, \quad p_0 \Vdash \neg\alpha, \quad p_0 \leq g(x), \\ u_0 = \Phi_{p_0}(x), \quad u_0 \nVdash \beta \end{array}}{\begin{array}{c} p_1 \Vdash \neg\forall\alpha, \quad p_1 \leq g(y), \\ u_1 = \Phi_{p_1}(y), \quad u_1 \nVdash \forall\beta \end{array}} \qquad (s8)\quad \frac{\begin{array}{c} p \Vdash \neg\forall\alpha, \quad p \leq g(z), \\ u = \Phi_p(z), \quad u \nVdash \forall\beta \end{array}}{\begin{array}{c} p_1 \Vdash \neg\forall\alpha, \quad p_1 \leq g(z_1), \\ u_1 = \Phi_{p_1}(z_1), \quad u_1 \nVdash \beta, \quad z_1 R z \end{array}}$$

In ($s5$) and ($s7$), p_1, u_1 are fresh; in ($s6$) and ($s8$), p_1, z_1, u_1 are fresh.

$$(\Phi\text{-crit.})\quad \frac{w \nVdash \alpha, \quad w \nVdash \beta}{\begin{array}{c} v \Vdash \alpha \to \beta, \quad v \leq g(w), \\ u = \Phi_v(w), \quad u \nVdash \beta \end{array}}$$

Here v, u are fresh. This is the tableau counterpart of Martínez's condition IV.1(f) [22], providing the critical witness required for ($W3$).

Persistence and auxiliary rules.

$$(\text{Pers}^+)\quad \frac{w \leq v, \quad w \Vdash p}{v \Vdash p} \qquad (\text{Pers}^-)\quad \frac{w \leq v, \quad v \nVdash p}{w \nVdash p} \qquad (\text{dec.})\quad \frac{}{w \Vdash \psi \mid w \nVdash \psi}$$

Remark 3. *The rule* (dec.) *is an auxiliary saturation rule used in the completeness construction, not a decomposition rule for an object-language connective. Its soundness is metatheoretic: for every model, world, and formula ψ, exactly one of $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \nVdash \psi$ holds. In the full branch-extension construction the rule is scheduled over all formulas; in finite or goal-directed proof search it may be restricted to the formulas relevant to the task at hand.*

Remark 4 (Open branches and countermodels)**.** *An open completed tableau branch should be understood as a syntactic description of a counterexample rather than, in general, as the final relational countermodel itself. The labelled expressions*

$$w \Vdash \varphi, \qquad w \nVdash \varphi, \qquad w \leq v, \qquad wRv, \qquad u = \Phi_v(w)$$

record constraints that any faithful interpretation of the branch must satisfy. From a sufficiently saturated open branch we first obtain the monadic Wajsberg algebra of formula extensions. Finite embeddability and the dual representation then yield a genuine finite relational countermodel.

Thus an open branch still determines a countermodel. The countermodel-extraction map is simply not, in general, the identity map sending tableau labels directly to worlds of the final frame. In small examples the branch prestructure may already satisfy all frame conditions, in which case direct countermodel extraction remains possible.

Example 2 (The monadic implication axiom). *We illustrate the role of* $(s6)$ *by considering the critical branch arising in the tableau for* $\forall(\forall p \to q) \to (\forall p \to \forall q)$*:*

$$\begin{array}{cl}
z \Vdash \forall(\forall p \to q) & \\
z \nVdash \forall p \to \forall q & \\
\downarrow\ (\to\text{-}rej.) & \\
a \Vdash \forall p & \\
a \leq g(z) & \\
u = \Phi_a(z) & \\
u \nVdash \forall q & \\
\downarrow\ (s6) & \\
a_1 \Vdash \forall p & \\
a_1 \leq g(z_1) & \\
u_1 = \Phi_{a_1}(z_1) & \\
u_1 \nVdash q & \\
z_1 R z & \\
z R z_1 & (R\text{-}sym.) \\
z_1 \Vdash \forall p \to q & (\forall\text{-}acc.) \\
u_1 \Vdash q & (\to\text{-}acc.) \\
u_1 \nVdash q & \\
\times &
\end{array}$$

Thus the branch closes. The rule $(s6)$ *supplies the witness transfer needed to reach an R-related point where* $z \Vdash \forall(\forall p \to q)$ *can be exploited.*

6. Soundness

Definition 11 (Faithfulness). *Let* $\mathfrak{M} = \langle W, \leq, g, \{\Phi_v\}_{v \in W}, R, V\rangle$ *be a monadic Wajsberg-model and let* b *be a branch. We say that* $\mathfrak{M}$ *is* faithful *to* b *if there is a map* f *from the world terms occurring on* b *into* W *such that:*

1. *if* $w \Vdash \varphi$ *occurs on* b*, then* $\mathfrak{M}, f(w) \Vdash \varphi$*;*
2. *if* $w \nVdash \varphi$ *occurs on* b*, then* $\mathfrak{M}, f(w) \nVdash \varphi$*;*
3. *if* $w \leq v$ *occurs on* b*, then* $f(w) \leq f(v)$*;*
4. *if* wRv *occurs on* b*, then* $f(w)Rf(v)$*;*
5. $f(g(w)) = g(f(w))$*;*
6. *if* $u = \Phi_v(w)$ *occurs on* b*, then* $f(u) = \Phi_{f(v)}(f(w))$*.*

Lemma 3. *No model is faithful to a closed branch.*

Proof. See [17]. □

Lemma 4 (Soundness Lemma). *Let* b *be a branch and let* $\mathfrak{M}$ *be faithful to* b*. If a tableau rule is applied to* b*, then at least one successor branch* b' *is produced such that* $\mathfrak{M}$ *is faithful to* b'*.*

Proof. Given in [17]. □

Theorem 1 (Soundness). *If the tableau for* $w_0 \nVdash \varphi$ *closes, then* φ *is valid on every monadic Wajsberg-frame.*

Proof. We prove the contrapositive. Suppose φ is not valid. Then there is a monadic Wajsberg-model $\mathfrak{M}$ and a world a such that $\mathfrak{M}, a \not\Vdash \varphi$. Let the initial branch be $w_0 \not\Vdash \varphi$. Define $f(w_0) = a$. Then $\mathfrak{M}$ is faithful to the initial branch. By the Soundness Lemma, after each application of a tableau rule there is at least one successor branch to which $\mathfrak{M}$ remains faithful. Choosing such a successor at each stage gives a branch that is faithful to $\mathfrak{M}$ throughout the construction. A faithful branch cannot close. Hence the tableau has an open branch. Therefore, if every branch closes, φ is valid on every monadic Wajsberg-frame. □

7. Completeness

For the completeness proof, we work with open completed branches in the Hintikka sense: every rule whose premises occur on the branch has already been applied on that branch, and all relevant structural closure conditions have been saturated.

Definition 12 (Completed branch, or *completedness*). *A branch b is* completed *if it satisfies the following closure conditions:*

1. *all applicable propositional rules have been applied;*
2. *if $w \Vdash \varphi \to \psi$, $v \Vdash \varphi$, $v \leq g(w)$, and $u = \Phi_v(w)$ occur on b, then $u \Vdash \psi$ occurs on b;*
3. *if $w \not\Vdash \varphi \to \psi$ occurs on b, then for some terms v, u, so do: $v \Vdash \varphi$, $v \leq g(w)$, $u = \Phi_v(w)$,$u \not\Vdash \psi$;*
4. *if $w \Vdash \forall\varphi$ and wRv occur on b, then $v \Vdash \varphi$ occurs on b;*
5. *if $w \not\Vdash \forall\varphi$ occurs on b, then for some term v, wRv, $v \not\Vdash \varphi$ occur on b;*
6. *the structural rules for $\leq$, g, R, $(R \leq)$, Φ, $R\Phi$, (Φ-mon.), (Φ-up.), (Φ-sym.), (Φg), (Φ-comm.), and the Figallo rules (s5)–(s8) have been applied wherever possible; moreover, every applicable instance of (Φ-crit.) has been fulfilled. Whenever a rule requires an atom of the form $u = \Phi_v(w)$, the unique-output convention is observed: an already recorded output for the ordered pair (v, w) is reused, and a fresh output term is introduced only when no such output has previously been recorded.*

Definition 13 (Branch prestructure). *Let b be an open completed branch. Define $\mathfrak{S}_b = \langle W_b, \leq_b, g_b, \{\Phi^b_v\}_{v \in W_b}, R_b\rangle$ as follows:*

1. *W_b is the set of world terms occurring on b;*
2. *$w \leq_b v$ iff $w \leq v$ occurs on b;*
3. *wR_bv iff wRv occurs on b;*
4. *$g_b(w) = g(w)$;*
5. *For the partial functions: $\Phi^b_v(w) = u \iff u = \Phi_v(w)$occurs on b.*

By the unique-output convention, each Φ^b_v is a well-defined partial function.

Lemma 5 (Structural saturation). *If b is an open completed branch, then the branch prestructure $\mathfrak{S}_b$ has the following properties: $\leq_b$ is a quasi-order; g_b is an order-reversing involution; R_b is an equivalence relation satisfying (Rg) and $(R \leq)$; the partial operations have the exact domains*

$$\mathrm{dom}(\Phi^b_v) = \{w \in W_b \,:\, v \leq_b g_b(w)\},$$

and satisfy (Φ-mon.), (Φ-up.), (Φ-sym.), (Φg), (Φ-comm.), and $(R\Phi)$ whenever the displayed terms are defined.

Proof. Given in [17]. □

Definition 14 (Tableau-consistent branch). *A branch b is* tableau-consistent *if there is no finite closed tableau whose root branch is b.*

Lemma 6 (Consistent successor). *Let b be a tableau-consistent branch and suppose that a tableau rule is applicable to b. If the rule has a unique successor, then that successor is tableau-consistent. If the rule has finitely many successor branches, then at least one successor is tableau-consistent.*

Proof. See [17]. ☐

Definition 15 (Fully decided branch). *A branch b is* fully decided *if, for every world term w occurring on b and every formula $\varphi \in$ For, exactly one of $w \Vdash \varphi$, $w \nVdash \varphi$ occurs on b.*

Lemma 7 (Full Branch Extension Lemma). *Every tableau-consistent branch has a tableau-consistent open, completed, fully decided extension.*

Proof. The proof is given in [17]. ☐

Remark 5. *The auxiliary rule* (dec.) *and the notion of a fully decided branch play different roles. The rule* (dec.) *is the tableau mechanism used during the branch-extension construction to choose, for a given pair (w, φ), one of $w \Vdash \varphi$ or $w \nVdash \varphi$. Full decidedness is the global closure property obtained after this mechanism has been applied fairly to every world term occurring on the branch and every formula in* For. *This stronger property is required in the completeness construction because the algebra of formula extensions $|\varphi|_b = \{w \in W_b : w \Vdash \varphi \in b\}$ must determine both truth and failure of formulas at every world. In particular, full decidedness is used in the converse directions of the Formula-Extension Lemma, where the absence of $w \Vdash \varphi$ must entail the presence of $w \nVdash \varphi$.*

Definition 16 (Formula extension). *Let b be an open, completed, fully decided branch. For every formula φ, put $|\varphi|_b := \{w \in W_b : w \Vdash \varphi$ occurs on $b\}$.*

Lemma 8 (Formula-Extension Lemma). *Let b be open, completed and fully decided, and put $D^b_v := \{w \in W_b : v \leq_b g_b(w)\}$. Then, for all formulas φ, ψ,*

$$|\neg\varphi|_b = \{w \in W_b : g_b(w) \notin |\varphi|_b\} \qquad \textit{(FE1)}$$

$$|\varphi \to \psi|_b = \bigcap_{v \in |\varphi|_b} \Big((D^b_v)^c \cup (\Phi^b_v)^{-1}(|\psi|_b)\Big) \qquad \textit{(FE2)}$$

$$|\forall\varphi|_b = \{w \in W_b : R_b(w) \subseteq |\varphi|_b\} \qquad \textit{(FE3).}$$

Proof. Given in detail in [17]. ☐

Lemma 9 (Branch Persistence). *For every formula φ, the set $|\varphi|_b$ is $\leq_b$-increasing. Equivalently, $w \leq_b v$ and $w \Vdash \varphi \in b$ imply $v \Vdash \varphi \in b$.*

Proof. For details, see [17]. ☐

Definition 17 (Branch algebra). *Let b be an open, completed, fully decided branch. Put*

$$\mathscr{A}_b := \{|\varphi|_b : \varphi \in \mathsf{For}\}, \qquad \begin{array}{ll} \neg_b|\varphi|_b := |\neg\varphi|_b, & |\varphi|_b \Rightarrow_b |\psi|_b := |\varphi \to \psi|_b, \\ \forall_b|\varphi|_b := |\forall\varphi|_b, & 1_b := W_b. \end{array}$$

Lemma 10. *The operations $\neg_b$, $\Rightarrow_b$, and $\forall_b$ are well-defined on $\mathscr{A}_b$.*

Proof. See [17]. ☐

Lemma 11 (Critical witness on formula extensions). *Let $U = |\alpha|_b$, $V = |\beta|_b$. If $w \notin U \cup V$, then there exists $v \in W_b$ such that $v \leq_b g_b(w)$, $v \in U \Rightarrow_b V$, $\Phi^b_v(w) \notin V$.*

Proof. See [17]. ☐

Lemma 12 (Branch Algebra Lemma). *The algebra $\mathbf{A}_b = \langle \mathscr{A}_b, \Rightarrow_b, \neg_b, \forall_b, 1_b \rangle$ is a monadic Wajsberg algebra.*

Proof. For details, see [17]. ☐

Lemma 13 (Algebraic Countermodel Lemma)**.** *Let b be an open, completed, fully decided branch containing $w_0 \nVdash \varphi_0$. Then $|\varphi_0|_b \neq 1_b$ in the monadic Wajsberg algebra $\mathbf{A}_b$.*

Proof. Since $w_0 \nVdash \varphi_0$ occurs on the open branch b, the formula $w_0 \Vdash \varphi_0$ does not occur. Hence $w_0 \notin |\varphi_0|_b$. But $1_b = W_b$. Therefore $|\varphi_0|_b \neq W_b = 1_b$. □

Lemma 14 (Finite Algebraic Countermodel)**.** *If $|\varphi_0|_b \neq 1_b$ in $\mathbf{A}_b$, then φ_0 fails in a finite monadic Wajsberg algebra.*

Proof. Given in [17]. □

Lemma 15 (Finite Relational Countermodel)**.** *If a formula φ fails in a finite monadic Wajsberg algebra, then it fails in a finite monadic Wajsberg model of the relational semantics introduced above.*

Proof. For details, see [17]. □

Theorem 2 (Completeness)**.** *If the tableau for $w_0 \nVdash \varphi$ has an open branch (or equivalently: does not close), then φ is not valid on the class of monadic Wajsberg frames.*

Proof. Standard; given in [17]. □

Proposition 3 (Algebraic and relational validity)**.** *For every formula φ, the following are equivalent:*

1. *for every monadic Wajsberg algebra $\mathbf{A}$ and every valuation v into $\mathbf{A}$, $v(\varphi) = 1$;*
2. *φ is valid on every monadic Wajsberg-frame.*

Proof. The proof is given in [17]. □

Corollary 1 (Semantic and tableau correspondence)**.** *For every formula φ, the following are equivalent:*

1. *for every monadic Wajsberg algebra $\mathbf{A}$ and every valuation v into $\mathbf{A}$, $v(\varphi) = 1$;*
2. *φ is valid on every monadic Wajsberg-frame;*
3. *every tableau for $w_0 \nVdash \varphi$ closes.*

Proof. The equivalence of (1) and (2) follows from Proposition 3, while the implication from (2) to (3) is Soundness and the implication from (3) to (2) is Completeness. □

Corollary 2 (Finite relational countermodel property)**.** *Every formula that is not valid in monadic Wajsberg logic fails in a finite monadic Wajsberg model.*

Proof. If φ is not valid, Soundness tells us the tableau for $w_0 \nVdash \varphi$ cannot close. The completeness construction above then produces a finite monadic Wajsberg algebra and, by Lemma 14, a finite relational model in which φ fails. □

8. Conclusion

In the preceding, we extracted a Kripke-style relational semantics for monadic Wajsberg logic from the dual representations of Martínez and Figallo-Orellano. The resulting frames combine a quasi-order, an order-reversing involution, partial operations for Wajsberg implication, and an equivalence relation for the monadic operator. The labelled tableau calculus turns these semantic interaction conditions into explicit structural and witness rules. The tableau is sound and complete for the intended semantics. Finite embeddability and duality yield a finite relational countermodel. Future work will study the tense Łukasiewicz system (i.e. Łukasiewicz logic with tense operators), and compare the semantic structures eventuating from Slagter and Gallardo [25] with semantics for Tense Łukasiewicz logic derived from the poset products literature [26]. One can and should pursue game interpretations extracted from these relational semantics; one can compare these games with that of [27, 28].

Declaration on Generative AI

The author has not employed any Generative AI tools.

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