(In)tractability of the maximum 2-clique problem for bipartite graphs

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Abstract

We investigate the complexity of a distance-based relaxation of the maximum clique problem, known as Max 2-Clique: A 2-*clique* in a graph $G = (V, E)$ is a vertex subset $S \subseteq V$ such that the distance between every pair of vertices in S is at most two in G. Given an n-vertex graph G, the objective of Max 2-Clique is to find a 2-clique of maximum cardinality in G. We show that (1) Max 2-Clique is solvable in polynomial time for chordal bipartite graphs; and (2) Max 2-Clique on bipartite graphs cannot be approximated in polynomial time within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$. This is the first clear separation, namely for $d = 2$, between the complexities of Max d-Clique and the closely related problem Max d-Club.

Keywords

2-clique, Bipartite graph, Chordal bipartite graph, Inapproximability

1. Introduction

Problem. As one of the central problems in graph theory and combinatorial optimization, Maximum Clique Problem (Max Clique) has attracted significant research attention (see e.g., [1]). In this paper, we study a distance-based relaxed variant of Max Clique, known as the Maximum Distance-d Clique Problem (Max d-Clique) for a positive integer d [2, 3, 4]: A *distance-d clique* (*d-clique* for short) in a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every pair of vertices $u, v \in S$, the distance between u and v is at most d **in the entire graph** G. Given a graph G, the goal of Max d-Clique is to find a d-clique of maximum cardinality in G. Another related problem is the Maximum Diameter-d Club Problem (Max d-Club), which resembles but is distinct from Max d-Clique: A *diameter-d club* (*d-club* for short) in a graph G is a subset $S' \subseteq V$ such that the subgraph of G induced by S' has diameter at most d, i.e., every pair of vertices has distance at most d **within the d-club itself**. Given a graph G, the goal of Max d-Club is to find a d-club of maximum cardinality in G.

Known Results. Max 1-Clique and Max 1-Club coincide with Max Clique. For $d \geq 2$, it is known that both Max d-Clique and Max d-Club cannot be approximated in polynomial time within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$ [5], where n is the number of vertices. On the positive side, polynomial-time $O(n^{1/2})$-approximation algorithms are known for both problems [5]. Since both problems are intractable on general graphs, several studies focus on restricted graph classes. Both Max d-Clique and Max d-Club are $\mathcal{NP}$-hard for graphs with degeneracy at most three when $d \geq 3$ [6]. For trees, both problems are solvable in polynomial time [7]; indeed, the two problems are equivalent on trees. Thus, their complexities coincide for the graph classes mentioned above. For other graph classes, the complexity of Max d-Club has been further studied. In particular, for bipartite graphs, Max 2-Club is polynomial-time solvable [8]. Additional results are known for, e.g., split and chordal graphs [8, 9].

Current Status. We study the complexities of Max d-Clique on graph classes for which Max d-Clique is less well understood than Max d-Club. For Max 2-Clique, we recently developed an $O(n^{1/3})$-approximation algorithm for chordal graphs and proved that Max 2-Clique on split graphs (which

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are also chordal) cannot be approximated within a factor of $n^{1/3-\varepsilon}$ for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$. We continue this line of research, focusing on bipartite graphs:

- For chordal bipartite graphs, Max 2-Clique can be solved in polynomial time (Section 2). This result is obtained by showing that every maximum 2-clique in a chordal bipartite graph is a 2-club, combined with the polynomial-time solvability of Max 2-Club on bipartite graphs [8].
- For bipartite graphs, Max 2-Clique cannot be approximated in polynomial time within a factor of $n^{1/2-\varepsilon}$ for any $\varepsilon > 0$ unless $\mathcal{P} = \mathcal{NP}$ (Section 3). This is the first result demonstrating a clear separation between the complexities of Max d-Club and Max d-Clique; in contrast to this inapproximability, Max 2-Club on bipartite graphs can be solved in polynomial time [8].

The following related questions remain open:

- Are Max d-Clique and Max d-Club with $d \geq 3$ on chordal bipartite graphs is solvable in polynomial time?
- Can Max d-Clique and Max d-Club with $d \geq 3$ on bipartite graphs be approximated in polynomial time within a factor of $o(n^{1/2})$?
- Another graph class for which every maximum 2-clique is a 2-club is a class of split graphs [10]. Are there any other graph classes with this property?

Notation. Let $G = (V, E)$ be a connected undirected graph, where V and E are the set of vertices and the set of edges, respectively. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote an edge with endpoints u and v by $\{u, v\}$. For a vertex v, the set of vertices adjacent to v in G is denoted by $N_G(v)$. A graph H is a subgraph of $G = (V, E)$ if $V(H) \subseteq V$ and $E(H) \subseteq E$. For $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by U, where the edge set of $G[U]$ includes all the edges of G connecting two vertices in U. A path P of length ℓ from a vertex v_0 to a vertex v_ℓ in G is represented as a sequence of vertices as $P = \langle v_0, v_1, \ldots, v_\ell \rangle$, where $v_i \in V(G)$ for $0 \leq i \leq \ell$ and $\{v_i, v_{i+1}\} \in E(G)$ for $0 \leq i \leq \ell - 1$. For a pair of vertices u and v in G, the length of a shortest path from u to v, i.e., the distance between u and v, is denoted by $dist_G(u, v)$. For a cycle $\langle v_0, v_1, \ldots, v_{\ell-1}, v_0 \rangle$, a *chord* is an edge connecting two vertices v_i and v_j such that $j \neq (i \pm 1) \mod \ell$. Consider a vertex subset $V' \subseteq V$. Then, $G[V']$ or V' is called a *clique* if $E(G[V']) = \{\{u, v\} \mid u \neq v,\ u, v \in V'\}$. When $E(G[V']) = \emptyset$, $G[V']$ or V' is called an *independent set*. A graph $G = (V, E)$ is *bipartite* if there is a partition of V into two disjoint independent sets U and W, where we write $G = (U, W, E)$. A bipartite graph is *chordal* if every cycle of length at least six has a chord. An algorithm ALG is called a σ-approximation algorithm if $OPT(G)/ALG(G) \leq \sigma$ holds for every input G, where $ALG(G)$ and $OPT(G)$ are the numbers of vertices in the subgraphs obtained by ALG and by an optimal algorithm, respectively. If no polynomial-time σ-approximation algorithm exists for a problem Π, we say that *Π cannot be approximated within a factor of σ unless $\mathcal{P} = \mathcal{NP}$.*

2. Polynomial-time solvability for chordal bipartite graphs

We show that every maximum 2-clique is a maximum 2-club and vice versa in a chordal bipartite graph, which yields polynomial-time solvability. We start with two properties of 2-cliques in bipartite graphs.

Lemma 1. *If $G[S]$ is a connected 2-clique in a bipartite graph G for $S \subseteq V(G)$, then $G[S]$ is a 2-club.*

Proof. Let $G = (U, W, E)$, and $H = G[S]$. To obtain a contradiction, assume that H is not a 2-club. Thus, there is a pair of distinct vertices x and y in S such that $dist_G(x, y) \leq 2$ (since H is a 2-clique) and $dist_H(x, y) \geq 3$ (since H is not a 2-club).

If $dist_G(x, y) = 1$, where $x \in U$ and $y \in W$, or $x \in W$ and $y \in U$, then there exists an edge $\{x, y\}$ in G, which is also included in H. This contradicts the assumption $dist_H(x, y) \geq 3$.

If $dist_G(x, y) = 2$, then there exists a vertex z such that $x, y \in N_G(z)$. If $z \in S$, it implies $dist_H(x, y) = 2$ contradicting the assumption $dist_H(x, y) \geq 3$. Thus, no such vertex z belongs to S.

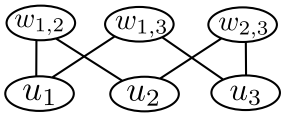


Figure 1: The case $|U'| = 3$ in the proof of Lemma 3.

Without loss of generality, assume that $x, y \in U \cap S$. Since H is connected, there is a path between x and y in H. Consider a shortest path P between x and y in H. The fact $x, y \in U$ implies that the length of P is even and at least 4, since $dist_H(x, y) \geq 3$. Let $w \in V(P)$ such that $dist_H(x, w) = 3$ (and thus $w \in W \cap S$). Since P is a shortest path between x and y in H, there is no edge between x and w in G. This implies that $dist_G(x, w) \geq 3$, since $x \in U \cap S$ and $w \in W \cap S$, which contradicts the assumption that both x and w are vertices of the 2-clique H. Therefore, H is a 2-club. $\square$

Lemma 2. *If $G[S]$ is a disconnected 2-clique in a bipartite graph $G = (U, W, E)$ for $S \subseteq U \cup W$, then either $S \subseteq U$ or $S \subseteq W$ holds.*

Proof. Let $H = G[S]$. To obtain a contradiction, suppose that $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$. Consider any pair of vertices $u \in S \cap U$ and $w \in S \cap W$. Since H is a 2-clique, $dist_G(u, w) \leq 2$. Then, since $u \in U$ and $w \in W$, $dist_G(u, w)$ is odd, and thus $dist_G(u, w) = 1$. The condition $dist_G(u, w) = 1$ holds only if the edge $\{u, w\}$ exists in G, and hence it also exists in H. Thus, for every pair of $u \in S \cap U$ and $w \in S \cap W$, there exists an edge $\{u, w\}$ in H, contradicting the assumption that H is disconnected. Therefore, $S \subseteq U$ or $S \subseteq W$ holds. $\square$

For disconnected 2-cliques, we obtain the following lemma and its corollary:

Lemma 3. *For a chordal bipartite graph $G = (U, W, E)$ and $U' \subseteq U$ with $|U'| \geq 2$, if U' is a 2-clique, then there exists a vertex $w \in W$ which is adjacent to every vertex of U' in G.*

Proof. Consider the case $|U'| = 2$. Let $U' = \{u_1, u_2\} \subseteq U$ and assume that U' is a 2-clique. Since U' is a 2-clique and $u_1, u_2 \in U$, $dist_G(u_1, u_2) = 2$, implying that a vertex $w \in W$ exists such that $\{u_1, w\}, \{u_2, w\} \in E$, i.e., w is adjacent to both u_1 and u_2.

The proof for $|U'| \geq 3$ is by induction on $|U'|$. As the base case, suppose that $|U'| = 3$. Let $U' = \{u_1, u_2, u_3\} \subseteq U$ and assume that U' is a 2-clique. Since U' is a 2-clique and $U' \subseteq U$, $dist_G(u_1, u_2) = 2$. Hence, a vertex $w_{1,2} \in W$ exists such that $\{u_1, w_{1,2}\}$ and $\{u_2, w_{1,2}\}$ exist in G. Similarly, since $dist_G(u_2, u_3) = dist_G(u_1, u_3) = 2$, vertices $w_{2,3}$ and $w_{1,3}$ exist in W such that $\{u_2, w_{2,3}\}$, $\{u_3, w_{2,3}\}$, $\{u_1, w_{1,3}\}$, and $\{u_3, w_{1,3}\}$ exist in G. See Fig. 1 for an illustration. If $w_{1,2} = w_{2,3}$, $w_{1,2} = w_{1,3}$, or $w_{1,3} = w_{2,3}$ holds, then the lemma follows. For example, if $w_{1,2} = w_{2,3}$, then $w_{1,2} (= w_{2,3})$ is adjacent to all of u_1, u_2, and u_3. Thus, assume that these three vertices $w_{1,2}$, $w_{2,3}$, and $w_{1,3}$ are different. By definition of u_1, u_2, u_3, $w_{1,2}$, $w_{1,3}$, and $w_{2,3}$, a cycle $\langle u_1, w_{1,2}, u_2, w_{2,3}, u_3, w_{1,3}, u_1 \rangle$ of length six exists in G. Since G is chordal bipartite, this cycle must have a chord, which necessarily joins a vertex in U and a vertex in W. If there exists a chord (edge) $\{u_1, w_{2,3}\}$ in G, $w_{2,3}$ is adjacent to all of u_1, u_2, and u_3. Analogously, if there exists a chord $\{u_2, w_{1,3}\}$ (or $\{u_3, w_{1,2}\}$), then $w_{1,3}$ (or $w_{1,2}$) is adjacent to all of u_1, u_2, and u_3. Therefore, if $|U'| = 3$, there exists a vertex $w \in W$ which is adjacent to every vertex of U'.

As the induction step, assume that for every $U_k = \{u_1, u_2, \ldots, u_k\} \subseteq U$ with $k \geq 3$ such that U_k is a 2-clique, there exists a vertex $w' \in W$ adjacent to every vertex in U_k. Suppose that $U_{k+1} = U_k \cup \{u_{k+1}\} \subseteq U$ and U_{k+1} is a 2-clique for some $u_{k+1} \in U \setminus U_k$. Consider three subsets of U_{k+1}: $X_1 = U_k$, $X_2 = U_{k+1} \setminus \{u_k\}$, and $X_3 = U_{k+1} \setminus \{u_1\}$. Observe that all of X_1, X_2, and X_3 are 2-cliques. By the induction hypothesis, there exist three vertices $w_1 (= w')$, w_2, and w_3 such that w_1 (w_2, w_3, resp.) is adjacent to every vertex in X_1 (X_2, X_3, resp.). See Fig. 2 for an illustration. If $w_1 = w_2$, $w_1 = w_3$, or $w_2 = w_3$ holds, then $w_1 (= w_2)$, $w_3 (= w_1)$, or $w_2 (= w_3)$ is adjacent to every vertex in U_{k+1}. Thus, assume that these three vertices w_1, w_2, and w_3 are different, implying that a cycle $\langle u_1, w_2, u_{k+1}, w_3, u_k, w_1, u_1 \rangle$ of length six exists. Then, since G is chordal bipartite, there must exist a chord $\{u_1, w_3\}$, $\{u_k, w_2\}$, or $\{u_{k+1}, w_1\}$, which respectively implies that either w_3, w_2, or w_1 is adjacent to every vertex of U_{k+1}. $\square$

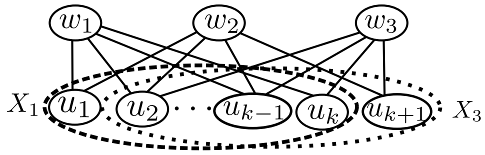


Figure 2: The case $|U'| = k+1$ in the proof of Lemma 3. The set X_2 consists of $u_1, u_2, \ldots, u_{k-1}, u_{k+1}$.

Corollary 4. *The maximum* 2*-clique in a chordal bipartite graph is connected.*

Proof. To obtain a contradiction, assume that a maximum 2-clique in a chordal bipartite graph $G = (U, W, E)$ is disconnected. From Lemma 2, without loss of generality, let $S \subseteq U$ be the vertex set of a maximum 2-clique such that $|S| \geq 2$. From Lemma 3, there exists a vertex w which is adjacent to every vertex of S. Thus, $dist_G(w, v) = 1$ for every $v \in S$. In addition, $dist_G(u, v) \leq 2$ for every pair $u, v \in S$ holds, since S is a 2-clique. This implies that the distance between any two vertices in $S \cup \{w\}$ is at most two; $S \cup \{w\}$ is also a 2-clique including more vertices than S. This contradicts that S is a maximum 2-clique. Therefore, the maximum 2-clique in a chordal bipartite graph is connected. □

Then, we obtain the following theorem.

Theorem 5. *In a chordal bipartite graph, a maximum* 2*-club is also a maximum* 2*-clique, and vice versa.*

Proof. Let H and H' be the vertex sets of a maximum 2-club and a maximum 2-clique in a chordal bipartite graph, respectively. From Lemma 1 and Corollary 4, the maximum 2-clique in a chordal bipartite graph is a 2-club. Thus, H' is also a 2-club, and hence $|H'| \leq |H|$ holds, since H is a maximum 2-club. It is clear that a 2-club is also a 2-clique, which implies that $|H| \leq |H'|$, since H' is a maximum 2-clique. Therefore $|H| = |H'|$ holds, i.e., H' (or H) is a maximum 2-club (or a maximum 2-clique). □

Thus, the polynomial-time solvability of Max 2-Club on bipartite graphs [8] yields:

Theorem 6. *Max* 2*-Clique on chordal bipartite graphs can be solved in polynomial time.*

3. Inapproximability for bipartite graphs

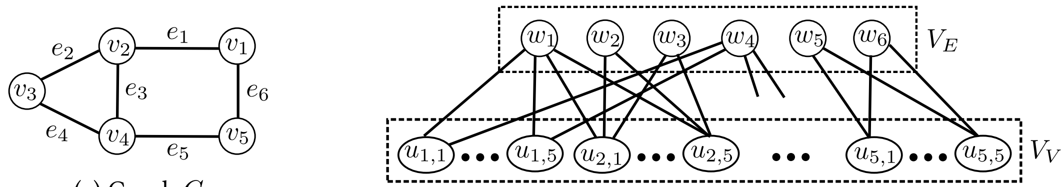
This section establishes an inapproximability result for Max 2-Clique on bipartite graphs. Recall that Max 2-Club on bipartite graphs can be solved in polynomial time. Thus, this inapproximability result demonstrates a clear separation between the complexities of Max 2-Clique and Max 2-Club.

Theorem 7. *Max* 2*-Clique on bipartite graphs cannot be approximated in polynomial time within a factor of* $n^{1/2-\varepsilon}$ *for any* $\varepsilon > 0$ *unless* $\mathcal{P} = \mathcal{NP}$.

Proof. We give a gap-preserving reduction [11] from Max Clique. From an input graph $G = (V, E)$ of Max Clique, we construct a bipartite graph $G' = (V', E')$ for Max 2-Clique. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. We construct two vertex sets: $V_E = \{w_i \mid 1 \leq i \leq m\}$ which mimics E, and $V_V = \{u_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq n\}$ which mimics V. The vertex set V' of G' is defined as $V' = V_E \cup V_V$. The edge set E' of G' connects V_E with V_V: $E' = \{\{w_\ell, u_{i,h}\}, \{w_\ell, u_{j,h}\} \mid e_\ell = \{v_i, v_j\} \in E,\ 1 \leq \ell \leq m,\ 1 \leq h \leq n\}$. See Fig. 3 for an illustration. Since both V_E and V_V are independent sets, G' is a bipartite graph. This reduction can be done in polynomial time, because $|V'| = |V_E| + |V_V| = m + n^2$ and $|E'| = 2nm$. The next claim follows immediately from the construction.

Claim 8. *For any* i*,* j*,* p*, and* q *such that* $i \neq p$*,* $dist_{G'}(u_{i,j}, u_{p,q}) = 2$ *if and only if* $dist_G(v_i, v_p) = 1$.

Consider a maximum 2-clique S' in G', and two distinct vertices $u_{i,j}$ and $u_{i,h}$ for some i and $j \neq h$. Then, $dist_{G'}(u_{i,j}, u_{i,h}) = 2$, $dist_{G'}(u_{i,h}, u_{p,q}) = dist_{G'}(u_{i,j}, u_{p,q})$ for any $p \neq i$ and q, and $dist_{G'}(u_{i,h}, w) = dist_{G'}(u_{i,j}, w)$ for any $w \in V_E$. Hence, if $u_{i,j} \in S'$ and $u_{i,h} \notin S'$, then $S' \cup \{u_{i,h}\}$



(a) Graph G

(b) Bipartite graph G' constructed from G

Figure 3: Reduction in the proof of Theorem 7.

is a 2-clique larger than S', which contradicts that S' is a maximum 2-clique. Thus we obtain the following claim.

Claim 9. *If $u_{i,j} \in S'$ for some j, then $u_{i,h} \in S'$ for all $1 \le h \le n$.*

Max Clique cannot be approximated in polynomial time within a factor of $n^{1-\varepsilon_1}$ for any $\varepsilon_1 > 0$ unless $\mathcal{P} = \mathcal{NP}$ [12]. We show that (I) if $OPT_1(G) \ge g(n)$, then $OPT_2(G') \ge n \cdot g(n)$; and (II) if $OPT_1(G) < g(n)/(n^{1-\varepsilon_1})$ for a constant $\varepsilon_1 > 0$, then $OPT_2(G') < n \cdot g(n)/(|V'|^{1/2-\varepsilon_2})$ for a constant $\varepsilon_2 > 0$, where $OPT_1(G)$ and $OPT_2(G')$ are the sizes of the maximum clique in G and the maximum 2-clique in G', respectively. These yield the inapproximability bound $|V'|^{1/2-\varepsilon_2}$.

(I) Let S be the maximum clique in G such taht $|S| \ge g(n)$. Consider a vertex set $S' = \{u_{i,j} \mid v_i \in S,\ 1 \le j \le n\}$ in G'. We claim that S' is a 2-clique. Let $u_{i,j}$ and $u_{p,q}$ be two distinct vertices in $V_V \cap S'$. If $i = p$, by the construction of G', there exist two edges $\{u_{i,j}, w_h\}$ and $\{u_{p,q}, w_h\}$ $(= \{u_{i,q}, w_h\})$ for some edge e_h incident with v_i in G. Consider the other case $i \ne p$. Since $v_i, v_p \in S$, there exists an edge $e_h = \{v_i, v_p\}$ in G. Thus, there exist two edges $\{u_{i,j}, w_h\}$ and $\{u_{p,q}, w_h\}$ in G', implying that $dist_{G'}(u_{i,j}, u_{p,q}) = 2$. Hence, for any two vertices $x, y \in S'$, $dist_{G'}(x, y) \le 2$ holds, i.e., S' is a 2-clique. Since $|S'| = n \cdot g(n)$, $OPT_2(G') \ge |S'| = n \cdot g(n)$.

(II) We prove the contraposition, i.e., if $OPT_2(G') \ge n \cdot g(n)/(|V'|^{1/2-\varepsilon_2})$ for a constant ε_2, then $OPT_1(G) \ge g(n)/(n^{1-\varepsilon_1})$ for a constant $\varepsilon_1 > 0$. Let $S' \subseteq V'$ be a maximum 2-clique in G' such that $|S'| \ge n \cdot g(n)/|V'|^{1/2-\varepsilon_2}$ for a constant $\varepsilon_2 > 0$. From Claims 8 and 9, $S = \{v_i \mid u_{i,j} \in S' \text{ for } 1 \le j \le n\}$ is a clique in G, and $|S| = |S' \cap V_V|/n$.

We observe $|V_E \cap S'| \le 2n$: Consider any vertex $w_i \in V_E \cap S'$ (if such a vertex exists). Since $dist_{G'}(w_i, w_j) = 2$ only if e_i and e_j share an endpoint (vertex) in G and a vertex in G is incident to at most n edges, the number of vertices in V_E whose distance from w_i is two is at most $2n$. Namely, the number of vertices from V_E that can be included in S' is at most $2n$. Therefore, $|S' \cap V_V| \ge n \cdot g(n)/(|V'|^{1/2-\varepsilon_2}) - 2n$, and hence $|S| = |S' \cap V_V|/n \ge g(n)/(|V'|^{1/2-\varepsilon_2}) - 2$.

Here we may assume $g(n) \ge n^{1-\varepsilon_1}$: if $g(n) < n^{1-\varepsilon_1}$, the condition $OPT_1(G) \le g(n)/(n^{1-\varepsilon_1})$ is false, since $OPT_1(G) \ge 1$ clearly holds, and hence (II) is obviously true. Also, since $|V'| = m+n^2$, there exists a constant ε_1 such that $0 < \varepsilon_1 < 2\varepsilon_2$ and $|V'|^{1/2-\varepsilon_2} \le n^{1-\varepsilon_1}/3$. Therefore, $OPT_1(G) \ge |S| \ge g(n)/(|V'|^{1/2-\varepsilon_2}) - 2 = (g(n) - 2|V'|^{1/2-\varepsilon_2})/(|V'|^{1/2-\varepsilon_2}) \ge g(n)/(3|V'|^{1/2-\varepsilon_2}) \ge g(n)/n^{1-\varepsilon_1}$. □

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Declaration on Generative AI

During the preparation of this work, the authors used Copilot in order to: Improve writing style, and Grammar and spelling check. After using the tool, the authors reviewed and edited the content as needed and take full responsibility for the publication's content.

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