On the Convexity of Kantorovich Liftings of Fuzzy Relations

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Abstract

The Kantorovich-Rubinstein duality implies that the well known Kantorovich (also known as Wasserstein) metric on probability distributions admits two equivalent definitions expressed in terms of minimization and maximization problems, respectively. When applied to fuzzy relations (distances that do not necessarily satisfy the axioms of a metric) the two definitions are known to differ, with the minimization problem always yielding greater values than the maximization problem. In this work we provide characterizations of these two values using the notion of convex function from analysis.

Keywords

Kantorovich Lifting, Behavioural Distances, Convex Algebra, Convex Functions, Quantitative Algebra.

1. Introduction

In the field of program semantics, when considering the behavior of programs embodying quantitative information, such as probabilities, time or sensor values, the interest normally shifts from behavioral equivalences to behavioral distances. In particular, in the context of probabilistic programs, a *behavioral distance semantics* is based on some assignment of distances to probability distributions $\Delta_1, \Delta_2 \in D(X)$. If these distances are $[0,1]$-valued, for example, the assignment is a fuzzy relation on probability distributions $D(X)$, i.e., a function of type $e : D(X) \times D(X) \to [0,1]$.

Seminal works in the field of distance semantics (see, e.g., [1, 2, 3, 4, 5]) have focused on fuzzy relations e that are *pseudometrics*[1] and, most importantly, such that e is obtained by *lifting* some pseudometric $d : X \times X \to [0,1]$ on the underlying set X. Intuitively, a lifting is a recipe to map a distance $d : X \times X \to [0,1]$ on the base set to a *lifted distance* $L(d) : D(X) \times D(X) \to [0,1]$ on probability distributions. Technically, for such a lifting recipe to be well-behaved for program semantics constructions, it must be a lifting (in a standard categorical sense, see Definition 2.12) of the probability distribution **Set** monad $(D : \mathbf{Set} \to \mathbf{Set}, \eta, \mu)$. A prominent example of lifting of this type is often called *Kantorovich* (or *Wasserstein*) lifting and is defined as follows (see Section 2.5):

$$K_{\text{inf}}(d)(\Delta_1, \Delta_2) = \inf_{\gamma \in D(X \times X)} \Big\{ \sum_{x,y \in X} \gamma(x,y) \cdot d(x,y) \ \Big|\ \gamma \text{ is a coupling of } \Delta_1 \text{ and } \Delta_2 \Big\}.$$

Another lifting is given by the following definition (also, see Section 2.5):

$$K_{\text{sup}}(d)(\Delta_1, \Delta_2) = \sup_{\substack{f:(X,d) \to ([0,1], d_{\mathbb{R}}) \\ \text{nonexpansive}}} \sum_{x \in X} \Delta_1(x) \cdot f(x) - \sum_{x \in X} \Delta_2(x) \cdot f(x).$$

The Kantorovich-Rubinstein duality from optimal transport theory (see, e.g., [6]) implies that, when d is a pseudometric then $K_{\text{inf}}(d) = K_{\text{sup}}(d)$, i.e., the two liftings coincide. This identity is very

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[1]A fuzzy relation is a pseudometric if it is reflexive, symmetric and satisfies the triangular inequality (Definition 2.8).

valuable, especially in the context of *behavioral pseudometric semantics*. Indeed K_{inf} is useful for establishing upper bounds (see for example the quantitative algebra approach of [7, 8]) and K_{sup} for establishing lower bounds (e.g., when the nonexpansive functions $f\colon (X,d) \to ([0,1], d_{\mathbb{R}})$ are the semantic interpretation of real-valued modal logic formulas, as in [1, 9]).

However, when d is not a pseudometric, only $K_{\text{sup}}(d) \leq K_{\text{inf}}(d)$ holds true and therefore K_{inf} and K_{sup} are genuinely different types of liftings of fuzzy relations. The usefulness of *behavioral distance semantics* not based on pseudometrics but, more generally, on fuzzy relations, has emerged in the last years (see, e.g., semi-metrics in [10], quasi-metrics in [11], b-metrics in [12], diffuse metrics in [13] and generalized metrics in [14]). Therefore, the relation between K_{inf} and K_{sup}, as different liftings of fuzzy relations, is worthwhile exploring.

Contributions of this work. Our main conceptual contribution is to observe that the standard notion of *convex function*, from analysis (see sections 2.2 and 3), is a key tool for investigating the two different liftings K_{inf} and K_{sup}. We prove two main results:

1. Theorem 4.1. $K_{\text{inf}}(d) : D(X)^2 \to [0,1]$ is the greatest *convex fuzzy relation* such that the unit η_X (of the $\mathbf{Set}$ distribution monad D) is nonexpansive as a map $\eta_X : (X,d) \to (D(X), K_{\text{inf}}(d))$,

2. Theorem 4.7. $K_{\text{sup}}(d) : D(X)^2 \to [0,1]$ is the greatest *convex fuzzy relation*, that is also a *pseudometric*, such that the unit η_X is nonexpansive as a map $\eta_X : (X,d) \to (D(X), K_{\text{sup}}(d))$. Alternatively, $K_{\text{sup}}(d)$ is the largest *convex fuzzy relation*, that is a *pseudometric*, bounded above by $K_{\text{inf}}(d)$.

In particular, from Theorem 4.7, one can immediately deduce the (well known, see e.g. [2]) inequality:

$$K_{\text{sup}}(d)(\Delta_1, \Delta_2) \leq K_{\text{inf}}(d)(\Delta_1, \Delta_2)$$

that always holds between $K_{\text{sup}}(d)$ and $K_{\text{inf}}(d)$, and that becomes an equality exactly when d is a pseudometric (by the Kantorovich-Rubinstein duality).

From a technical point of view, our proofs are based on elementary and well known properties of convex functions and, most importantly, the standard formula for the *convex envelope* (see Proposition 2.3). An interesting aspect is that our proofs are purely algebraic (equational) and use the language of partially ordered convex algebras. Indeed, our research has originated from investigations in the field of quantitative algebra (see, e.g., [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]) where ordinary algebraic reasoning is combined with distance reasoning. As we discuss in Section 5, our results allow for significant simplifications of some proofs in this field. Furthermore this style of proofs, potentially, allows for the applicability of our results to generalizations of the liftings K_{inf} and K_{sup} to other functors (or monads) F other than the distribution functor (monad) D, as technically developed in [2].

Comparison with Rodary's master thesis [37]. The second part of the statement of Theorem 4.7 is the main result of the Master (M2) thesis of Apolline Rodary [37]. However, our proof diverges conceptually and technically: it relies on the separate statement and proof of Theorem 4.1, a result that is neither stated nor derivable from [37]. Our proof for Theorem 4.1 is distinctively compact, leveraging the standard formula for the convex envelope, a key tool absent from the work of [37]. Consequently, this approach not only yields a streamlined proof of the result of [37] but also establishes the first part of Theorem 4.7, which is likewise independent of [37]. Finally, the discussion of Section 5 is also novel and does not appear in [37].

Organization of this paper. This paper is organized as follows. In Section 2 we give the required definitions and basic facts regarding lattices, convex functions, fuzzy relations and liftings. In Section 3 we introduce the notion of convex fuzzy relation and in Section 4 we prove our main results. Finally, in Section 5, we relate our results with some recent advancements in quantitative algebra.

2. Technical Background

2.1. Basic Notions Regarding Lattices

Recall that a partially ordered set $(X, \leq)$ is a *complete lattice* if every subset $Y \subseteq X$ has a join (i.e., least upper bound) $\bigvee_X Y$. A *complete join-sublattice* of $(X, \leq)$ is a subset $Y \subseteq X$ that is closed under taking joins of arbitrary subsets, i.e., $\bigvee_X Z \in Y$ for all $Z \subseteq Y$.

A complete join-sublattice Y of $(X, \leq)$ determines an operator $\Diamond_Y : X \to X$ mapping $x \in X$ to the largest element in Y bounded above by x:

$$\Diamond_Y(x) = \bigvee\{y \in Y \mid y \leq x\}.$$

The operator $\Diamond_Y$ satisfies the following properties:

- (Idempotency) $\Diamond_Y \Diamond_Y x = \Diamond_Y x$,
- (Monotonicity) if $x \leq x'$ then $\Diamond_Y x \leq \Diamond_Y x'$.
- (Decreasing) $\Diamond_Y x \leq x$,
- (Fixed points) $x \in Y$ if and only if $x = \Diamond_Y x$.

The following are straightforward facts that will be used.

Proposition 2.1. *Let Y, Z be complete join-sublattices of $(X, \leq)$. Then the following hold:*

1. *$Y \cap Z$ is a complete join-sublattice of $(X, \leq)$.*
2. *if $\Diamond_Y(x) = \Diamond_Z \Diamond_Y(x)$ then $\Diamond_Y(x) = \Diamond_{Y \cap Z}(x)$,*
3. *if $Y \subseteq Z$ then $\Diamond_Y(x) \leq \Diamond_Z(x)$,*
4. *if $Y \subseteq Z$ then $\Diamond_Y(x) = \Diamond_Y \Diamond_Z(x)$.*

2.2. Basic Notions Regarding Convex Functions and Convex Envelopes

Recall the standard definition of convex function fron analysis (see, e.g., [38, Thm. 4.1 and 4.3]).

Definition 2.1 (Convex Function)**.** Let C be a convex subset of a real vector space. A function $f : C \to [0, 1]$ is *convex* if, for all $x, y \in C$ and $p \in (0, 1)$, it holds that:

$$f(p \cdot x + (1 - p) \cdot y) \leq p \cdot f(x) + (1 - p) \cdot f(y)$$

or equivalently, extending to n-ary convex combinations:

$$f(\sum_{i=1}^{n} p_i \cdot x_i) \leq \sum_{i=1}^{n} p_i \cdot f(x_i) \qquad \text{with } 0 \leq p_i \leq 1 \text{ and } \sum_{i=1}^{n} p_i = 1.$$

The following are well known[2] properties of convex functions.

Proposition 2.2 (see Thm. 5.5 of [38])**.** *The collection of convex functions $c : C \to [0, 1]$ is a complete join-sublattice of the complete lattice $[0, 1]^C$. This means that, for any $f \in [0, 1]^C$, there exists a largest convex function $c \leq f$, called the* convex envelope of f *and denoted by $c = \mathrm{conv}(f)$.*

Proposition 2.3. *For any $f \in [0, 1]^C$ the following identity holds:*

$$\mathrm{conv}(f)(x) = \inf\left\{ \sum_{i=1}^{n} p_i \cdot f(x_i) \;\middle|\; \sum_{i=1}^{n} p_i \cdot x_i = x, \text{ with } n \in \mathbb{N}, x_i \in C, \, p_i \geq 0, \, \sum_{i=1}^{n} p_i = 1 \right\}.$$

[2]Proposition 2.3 is most often equivalently stated as "the epigraph of $\mathrm{conv}(f)$ is the convex hull of the epigraph of f" (see, e.g., [38, §5, p. 36]).

2.3. Basic Notions Regarding Probability Distributions

Definition 2.2. Given a set X, a probability distribution on X is a function $\Delta\colon X \to [0,1]$ such that $\sum_{x\in X}\Delta(x) = 1$. The *support* of Δ is the set $supp(\Delta) = \{x \mid \Delta(x) > 0\}$. We say that Δ is *finitely supported* if $supp(\Delta)$ is a finite set. We denote with $D(X)$ the set of finitely supported probability distributions on X.

In the rest of this work, unless otherwise specified, a probability distribution is always assumed to have finite support. Distributions $\Delta \in D(X)$ are denoted by the corresponding formal convex sums $\sum_i p_i x_i$, with i implicitly ranging over a finite set. Probability distributions whose mass is concentrated at a single point $x \in X$, called *Dirac distributions*, are denoted either by $1x$ or by δ_x. The set $D(X)$ is convex with convex (as a subset of $\mathbb{R}^X$) with combinations defined as:

$$\big(p\Delta_1 + (1-p)\Delta_2\big)(x) = p\cdot\Delta_1(x) + (1-p)\cdot\Delta_2(x).$$

Definition 2.3 (D as a Functor and Monad on **Set**)**.** Given a function $f : X \to Y$, we define the function $D(f) : D(X) \to D(Y)$ as:

$$D(f)\big(\sum_i p_i x_i\big) = \sum_i p_i f(x_i).$$

This makes D into a functor $D : \mathbf{Set} \to \mathbf{Set}$. Furthermore, with the two natural transformations:

- (Unit) $\eta : \mathrm{Id}_{\mathbf{Set}} \Rightarrow D$, with components $\eta_X : X \to D(X)$ defined as $\eta_X(x) = \delta_x$,
- (Multiplication) $\mu : DD \Rightarrow D$, with components $\mu_X : DD(X) \to D(X)$ defined as:

$$\mu_X\big(\sum_i p_i(\sum_j q_{ij}x_{ij})\big) = \sum_{i,j}(p_i\cdot q_{ij})x_{ij}$$

the triple (D,η,μ) is a **Set** monad.

In the following definition we denote with $\pi_L : X\times Y \to X$ and $\pi_R : X\times Y \to Y$ the two projection operations, defined as $\pi_L(x,y) = x$ and $\pi_R(x,y) = y$. This gives functions $D(\pi_L) : D(X\times Y) \to D(X)$ and $D(\pi_R) : D(X\times Y) \to D(Y)$ mapping a probability distribution on a product space to its *marginals*.

Definition 2.4 (Couplings)**.** Given $\Delta_1, \Delta_2 \in D(X)$, a *coupling of* Δ_1 *and* Δ_2 is a probability distribution $\gamma \in D(X\times X)$ on the product space $X \times X$ whose marginals are Δ_1 and Δ_2:

- $D(\pi_L)(\gamma) = \Delta_1$, or equivalently:

$$\sum_{y\in\mathrm{Supp}(\Delta_2)}\gamma(x,y) = \Delta_1(x) \qquad \text{for all } x\in\mathrm{Supp}(\Delta_1),$$

- $D(\pi_R)(\gamma) = \Delta_2$, or equivalently:

$$\sum_{x\in\mathrm{Supp}(\Delta_1)}\gamma(x,y) = \Delta_2(y) \qquad \text{for all } y\in\mathrm{Supp}(\Delta_2).$$

Note that $\mathrm{Supp}(\gamma) \subseteq \big(\mathrm{Supp}(\Delta_1)\cup\mathrm{Supp}(\Delta_2)\big)^2$ is indeed a finite set. We denote with $\Gamma(\Delta_1,\Delta_2)$ the collection of all couplings of Δ_1 and Δ_2. This is nonempty because the independent product of Δ_1 and Δ_2 is always a coupling.

Definition 2.5 (Expected Value)**.** Given a function $f : X \to [0,1]$ and a probability distribution $\sum_i p_i x_i$ in $D(X)$, the *expected value of* f *under* $\sum_i p_i x_i$ is denoted by:

$$E(f, \sum_i p_i x_i) = \sum_i p_i\cdot f(x_i).$$

Remark 2.6. Note, in the right-hand side of the above definition, the use of the scalar product $(\cdot)$ in the expression $\sum_i p_i\cdot f(x_i)$ denoting a real number, which distinguishes it from the expression $\sum_i p_i f(x_i)$ denoting a probability distribution in $D([0,1])$.

2.4. Basic Notions Regarding Fuzzy Relations and Pseudometrics

Definition 2.7 (Fuzzy Relations and Nonexpansive Maps)**.** Given a set X, a function $d\colon X \times X \to [0,1]$ is called a *fuzzy relation* on X. We also refer to the pair (X,d) as a fuzzy relation. Given two fuzzy relations (X,d_X) and (Y,d_Y), a function $f : X \to Y$ is *nonexpansive* (or also *1-Lipschitz*) if $d_Y\big(f(x),f(x')\big) \leq d_X(x,x')$, for all $x,x' \in X$.

We will denote with $\mathbf{FRel}$ the category of fuzzy relations and nonexpansive maps. We denote with $U : \mathbf{FRel} \to \mathbf{Set}$ the forgetful functor defined on objects as $U(X,d) = X$ and on morphisms as $U(f) = f$.

Note that the collection $(X \times X \to [0,1])$ of all fuzzy relations on X is a complete lattice with the pointwise order $(\leq)$. We will consider the following subsets of $X \times X \to [0,1]$.

Definition 2.8. Fix a set X. We define the following subsets of $X \times X \to [0,1]$:

- Reflexive fuzzy relations:

$$R = \big\{d : X \times X \to [0,1] \mid d(x,x) = 0, \text{for all } x \in X\big\}$$

- Symmetric fuzzy relations:

$$S = \big\{d : X \times X \to [0,1] \mid d(x,y) = d(y,x), \text{for all } x,y \in X\big\}$$

- Fuzzy relations satisfying the triangular inequality:

$$T = \big\{d : X \times X \to [0,1] \mid d(x,y) \leq d(x,z) + d(z,y), \text{for all } x,y,z \in X\big\}$$

- Fuzzy relations satisfying the identity of indiscernibles property:

$$I = \big\{d : X \times X \to [0,1] \mid \text{if } d(x,y) = 0 \text{ then } x = y, \text{for all } x,y \in X\big\}$$

Furthermore we denote with $P = R \cap S \cap T$ and $M = P \cap I$ the fuzzy relations that are *pseudometrics* and *metrics*, respectively

Proposition 2.4. *The sets R, S and T are complete join-sublattice of $X \times X \to [0,1]$, and therefore also $P = R \cap S \cap T$ is a complete join-sublattice. Furthermore, the corresponding operators satisfy:*

$$\Diamond_{S\cap R} = \Diamond_S \circ \Diamond_R = \Diamond_R \circ \Diamond_S \qquad \Diamond_P = \Diamond_T \circ \Diamond_{S\cap R}.$$

Remark 2.9. Note that the least element of P is the constant $(x,x') \mapsto 0$ pseudometric. Importantly, this is not a metric in M, because it does not satisfy the identity of indiscernibles. Indeed I and M are not a complete join-sublattice because they lack a least element.

We denote with $\mathbf{PMet}$ the full subcategory of $\mathbf{FRel}$ whose objects (X,d) are pseudometrics (i.e., $d \in P$). The restrictions of the forgetful functor $U : \mathbf{FRel} \to \mathbf{Set}$ to $\mathbf{PMet}$ is also denoted by U.

The assignment $(X,d) \mapsto (X, \Diamond_P d)$, that maps a fuzzy relation to a pseudometric space, can be turned into a functor.

Proposition 2.5 (Functor $\Diamond_P$)**.** *The following assignments define a functor $\Diamond_P : \mathbf{FRel} \to \mathbf{PMet}$:*

- *on objects (X,d) of $\mathbf{FRel}$, define $\Diamond_P(X,d) = (X, \Diamond_P(d))$,*
- *on morphisms $f : (X,d) \to (Y,e)$ of $\mathbf{FRel}$, define $\Diamond_P(f) = f$.*

Proof. Fix $f : (X,d) \to (Y,e)$ nonexpansive. We need to prove that $\Diamond_P(f) : (X, \Diamond_P(d)) \to (Y, \Diamond_P(e))$ is also nonexpansive. Consider the fuzzy relation c on X defined as $c(x,x') = \Diamond_P(e)(f(x), f(x'))$. Note that c is a pseudometric, $c \leq d$ and $f : (X,c) \to (Y, \Diamond_P(e))$ is nonexpansive. Since $c \leq d$, we have $c \leq \Diamond_P d$. Therefore $f : (X, \Diamond_P(d)) \to (Y, \Diamond_P(e))$ is nonexpansive. $\square$

Remark 2.10. It is well-known that $\Diamond_P$ is left adjoint to the inclusion functor $\mathbf{PMet} \hookrightarrow \mathbf{FRel}$.

An important property of pseudometrics is that they are closed under taking restrictions to subsets.

Proposition 2.6. *Let (X, d) be a fuzzy relation, and let $Y \subseteq X$ be nonempty. The* restriction of d to Y, *denoted by $d|_Y : Y^2 \to [0,1]$, is defined as $d|_Y(y, y') = d(y, y')$, for all $y, y' \in Y$. If d is a pseudometric then $d|_Y$ is a pseudometric.*

2.5. Liftings of D to $\mathbf{FRel}$

In this section we introduce three well-known *liftings* methods for lifting a fuzzy relations on X to fuzzy relations on $D(X)$, in the standard sense of *monad lifting*, defined below.

Definition 2.11 (Functor lifting)**.** A functor $L : \mathbf{FRel} \to \mathbf{FRel}$ is a *lifting* of the functor $D : \mathbf{Set} \to \mathbf{Set}$ if $UL = DU$, where $U : \mathbf{FRel} \to \mathbf{Set}$ is the forgetful functor.

Definition 2.12 (Monad lifting)**.** A $\mathbf{FRel}$ monad (M, η^M, μ^M) lifts the $\mathbf{Set}$ monad (D, η, μ) if M lifts D as a functor (see Definition 2.11) and $U(\eta^M) = \eta$ and $U(\mu^M) = \mu$.

Equivalently, recalling that $U(X, d) = X$ and $U(f) = f$, a functor $L : \mathbf{FRel} \to \mathbf{FRel}$ lifts D if its action of morphisms is $L(f) = D(f)$ and its action of objects is $(X, d) \mapsto (D(X), L(d))$ for some fuzzy relation denoted by $L(d) : D(X) \times D(X) \to [0,1]$. Furthermore, L is a lifting of the monad (D, η, μ) if, for every $(X, d) \in \mathbf{FRel}$, the components of the unit η and the multiplication μ are nonexpansive as maps:

$$\begin{array}{cc} \eta_X : (X, d) \to \big(D(X), L(d)\big) & \qquad \mu_X : \big(D(D(X)), L(L(d))\big) \to \big(D(X), L(d)\big) \\ x \mapsto \delta_x & \qquad \sum_i p_i(\sum_j q_{ij} x_{ij}) \mapsto \sum_{i,j} (p_i \cdot q_{ij}) x_{ij}. \end{array}$$

Definition 2.13 ($L_\top$)**.** Let (X, d) be a fuzzy relation. The fuzzy relation $L_\top(d) : D(X)^2 \to [0,1]$ is defined as follows:

$$L_\top(d)(\Delta_1, \Delta_2) = \begin{cases} d(x, y) & \text{if } \Delta_1 = \delta_x \text{ and } \Delta_2 = \delta_y \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.14 (K_{inf})**.** Let (X, d) be a fuzzy relation. The fuzzy relation $K_{\text{inf}}(d) : D(X)^2 \to [0,1]$ is defined as follows:

$$K_{\text{inf}}(d)(\Delta_1, \Delta_2) = \inf_{\gamma \in \Gamma(\Delta_1, \Delta_2)} E(d, \gamma).$$

Definition 2.15 (K_{sup})**.** Let (X, d) be a fuzzy relation. The fuzzy relation $K_{\text{sup}}(d) : D(X)^2 \to [0,1]$ is defined as follows:

$$K_{\text{sup}}(d)(\Delta_1, \Delta_2) = \sup_{\substack{f : (X,d) \to ([0,1], d_{\mathbb{R}}) \\ \text{nonexpansive}}} d_{\mathbb{R}}\big(E(f, \Delta_1), E(f, \Delta_2)\big)$$

where $d_{\mathbb{R}}$ is the Euclidean metric (and thus, fuzzy relation) on the reals defined as $d_{\mathbb{R}}(r, s) = |r - s|$.

All three assignments give rise to functor and monad lifting of D.

Proposition 2.7. *For $L \in \{L_\top, K_{\text{inf}}, K_{\text{sup}}\}$, the assignments:*

- $(X, d) \mapsto (D(X), L(d))$*, on objects* $(X, d) \in \mathbf{FRel}$*,*
- $f \mapsto D(f)$*, on morphisms* $f : (X, d) \to (X', d') \in \mathbf{FRel}$*,*

define a functor, also denoted by $L : \mathbf{FRel} \to \mathbf{FRel}$. This is a functor and monad lifting of (D, η, μ).

The lifting $L_\top(d)$ is the *pushforward* of d along the unit $\eta_X : X \to D(X)$ of the monad D, i.e., it is the greatest fuzzy relation e on $D(X)$ making the map $x \mapsto \delta_x$ nonexpansive as a map $(X, d) \to (D(X), e)$. We state this as a proposition.

Proposition 2.8. *Let (X, d) be a fuzzy relation. For any fuzzy relation $e : D(X)^2 \to [0, 1]$ on $D(X)$, the following are equivalent:*

1. $e \leq L_\top(d)$,
2. e *makes the unit* $\eta_X : X \to D(X)$ *nonexpansive as a map* $\eta_X : (X, d) \to (D(X), e)$.

In [2], the liftings K_{inf} and K_{sup} are referred to as *Wasserstein lifting* and *Kantorovich lifting*, respectively. In what follows we will just refer to them using the K_{inf} and K_{sup} notation which, of course, refers to the shape of their definitions.

Remark 2.16. Often the definition of K_{sup} involves nonexpansive functions $g : (X, d) \to (\mathbb{R}, d_\mathbb{R})$. This does not affect the value of the definition, because any such function g necessarily has a diameter of at most 1 and therefore can be shifted by a constant to have range in $[0, 1]$ while preserving the expected value, because E is linear.

Remark 2.17. In the definition of $K_{\text{sup}}(\Delta_1, \Delta_2)$, for a fixed nonexpansive $f : (X, d) \to ([0, 1], d_\mathbb{R})$, the values of $E(f, \Delta_1)$ and $E(f, \Delta_2)$ are determined entirely by the values of f on the finite subset $Y = \text{Supp}(\Delta_1) \cup \text{Supp}(\Delta_2)$ of X. In other words, we have that $E(f, \Delta_i) = E(f|_Y, \Delta_i)$ for the nonexpansive restriction $f|_Y : (Y, d|_{Y\times Y}) \to ([0, 1], d_E)$ of f. However, note that in general:

$$K_{\text{sup}}(d)(\Delta_1, \Delta_2) < \sup_{\substack{g:(Y,d|_{Y\times Y})\to([0,1],d_E) \\ \text{nonexpansive}}} d_\mathbb{R}\big(E(g, \Delta_1), E(g, \Delta_2)\big), \qquad Y = \text{Supp}(\Delta_1) \cup \text{Supp}(\Delta_2) \tag{$\dagger$}$$

This is because a nonexpansive $g : Y \to [0, 1]$ does not admit, in general, a compatible nonexpansive extension to the larger (X, d). For example, consider the fuzzy relation d on $X = \{0, 1, \star\}$ defined to be reflexive, symmetric and with assignments $d(0, 1) = 1$, $d(1, \star) = 0.1$ and $d(0, \star) = 0.1$. Let $Y = \{0, 1\}$ and consider the function $g : Y \to [0, 1]$ defined by $g(0) = 0$ and $g(1) = 1$. Then the function $g : (Y, d|_{Y\times Y}) \to ([0, 1], d_\mathbb{R})$ is nonexpansive, but it does not admit any nonexpansive extension of type $(X, d) \to ([0, 1], d_\mathbb{R})$. It can be shown that if the fuzzy relation d satisfies the triangular inequality (which indeed is not the case of the previous example) then the extension is guaranteed to exist and the inequality (†) becomes an equality.

Remark 2.18. For fixed Δ_1, Δ_2, the value $K_{\text{inf}}(\Delta_1, \Delta_2)$ can clearly be formulated as the solution of a Linear Programming problem and computed efficiently. Indeed the solution set $\gamma \in \Gamma(\Delta_1, \Delta_2)$ is a convex polytope and the objective $E(d, \_)$ is linear. On the other hand, the value $K_{\text{sup}}(\Delta_1, \Delta_2)$ can be directly formulated as the solution of a LP problem only if X is finite or, in light of Remark 2.17, if the fuzzy relation d satisfies the triangular inequality. The problem of establisihing the complexity of computing $K_{\text{sup}}(\Delta_1, \Delta_2)$, in full generality, is left for future work.

The following is the content of the (finite dimensional version of) Kantorovich-Rubinstein duality.

Theorem 2.9 (Kantorovich-Rubinstein Duality). *If (X, d) is a pseudometric then $K_{\text{inf}}(d) = K_{\text{sup}}(d)$ and both are pseudometrics i.e.:*

$$d = \Diamond_P(d) \qquad \Rightarrow \qquad \Diamond_P K_{\text{inf}}(d) = K_{\text{inf}}(d) = K_{\text{sup}}(d) = \Diamond_P K_{\text{sup}}(d).$$

3. Convex Fuzzy Relations on $D(X)$

Note that $D(X) \subseteq \mathbb{R}^X$ is a convex subset of $\mathbb{R}^X$. Similarly, $D(X) \times D(X) \subseteq \mathbb{R}^X \times \mathbb{R}^X$ is a convex subspace of the vector space $\mathbb{R}^X \times \mathbb{R}^X$ with operations defined componentwise as:

$$p\langle\Delta_1, \Delta_2\rangle + (1 - p)\langle\Delta'_1, \Delta'_2\rangle = \big\langle p\Delta_1 + (1 - p)\Delta'_1 \ , \ p\Delta_2 + (1 - p)\Delta'_2\big\rangle.$$

Therefore, the following definition is a special case of Definition 2.1, for $C = D(X) \times D(X)$.

Definition 3.1 (Convex Fuzzy Relation)**.** A fuzzy relation $c : D(X) \times D(X) \to [0,1]$ on $D(X)$ is *convex* if, for all $\Delta_1, \Delta_1', \Delta_2, \Delta_2' \in D(X)$ and $p \in (0,1)$, it holds that:

$$c\big(p\Delta_1 + (1-p)\Delta_1' \ , \ p\Delta_2 + (1-p)\Delta_2'\big) \le p \cdot c(\Delta_1, \Delta_2) + (1-p) \cdot c(\Delta_1', \Delta_2').$$

or, using n-ary notation:

$$c\Big(\sum_{i=1}^{n} p_i \Delta_1^i, \sum_{i=1}^{n} p_i \Delta_2^i\Big) \le \sum_{i=1}^{n} p_i \cdot c(\Delta_1^i, \Delta_2^i) \qquad \text{with } 0 \le p_i \le 1 \text{ and } \sum_{i=1}^{n} p_i = 1.$$

The definition of convex fuzzy relation is relevant in this work because both $K_{\text{inf}}(d)$ and $K_{\text{sup}}(d)$ are convex. Furthermore, $K_{\text{sup}}(d)$ is always a pseudometric for any fuzzy relation d. These facts can easily be proved directly. They also follow from our stronger results (Theorem 4.1 for K_{inf} and Theorem 4.7 for K_{sup}) so we state the following Proposition without a proof.

Proposition 3.1. *Let (X, d) be an arbitrary fuzzy relation. Then:*

1. $K_{\text{inf}}(d) \colon D(X) \times D(X) \to [0,1]$ *is a convex fuzzy relation,*
2. $K_{\text{sup}}(d) \colon D(X) \times D(X) \to [0,1]$ *is a convex fuzzy relation and it is a pseudometric.*

Recall, from Proposition 2.2, that the collection of convex fuzzy relations is a complete join-sublattice of $D(X) \times D(X) \to [0,1]$. We denote with $\Diamond_C$ the corresponding operator:

$$\Diamond_C(e) = \bigvee \{c \le e \mid c : D(X) \times D(X) \to [0,1] \text{ is convex}\} \qquad\qquad e : D(X) \times D(X) \to [0,1]$$

and we refer to $\Diamond_C(e)$ as the *convex envelope* of e.

In general, the operation of pseudometrification does not preserve convexity and, in particular, it is not true that $K_{\text{sup}}(d)$ is the pseudometrification of $K_{\text{inf}}(d)$. The following example witnesses this fact.

Example 3.2. Consider $X = \{a, b\}$ with the fuzzy relation $d : X^2 \to [0,1]$ assigning value 1 to all four pairs. Let $e = K_{\text{inf}}(d) : D(X) \times D(X) \to [0,1]$. One calculates that e is also constant to 1, and therefore e is convex. Furthermore, it is immediate to verify that $\Diamond_P(e)$ is the discrete metric on $D(\{a, b\})$:

$$\Diamond_P(e)(\Delta_1, \Delta_2) = \begin{cases} 0 & \text{if } \Delta_1 = \Delta_2 \\ 1 & \text{otherwise.} \end{cases}$$

and this is not convex because the convexity inequality is violated for $p \in (0,1)$:

$$\Diamond_P e\Big(p\delta_a + (1-p)\delta_a \ , \ p\delta_b + (1-p)\delta_a\Big) = 1$$

$$\not\le$$

$$p \cdot \Diamond_P e(\delta_a, \delta_b) + (1-p) \cdot \Diamond_P e(\delta_a, \delta_a) = p.$$

From the previous example we derive the following important corollary, which states that (in general) $K_{\text{sup}}(d)$ is not just the "pseudometrization" of $\Diamond_P K_{\text{inf}}(d)$.

Corollary 3.2. *In general, $\Diamond_P K_{\text{inf}}(d) \neq K_{\text{sup}}(d)$.*

Proof. Let d be defined as in the previous example, where it was calculated that

$$\Diamond_P K_{\text{inf}}(d)\Big(p\delta_a + (1-p)\delta_a \ , \ p\delta_b + (1-p)\delta_a\Big) = 1.$$

From the definition of K_{sup} one calculates that:

$$K_{\text{sup}}(d)\Big(p\delta_a + (1-p)\delta_a \ , \ p\delta_b + (1-p)\delta_a\Big) = p$$

and therefore $\Diamond_P(K_{\text{inf}}(d)) \neq K_{\text{sup}}(d)$. $\square$

4. Main Results

Our first result, relates the liftings $L_\top$ and $K_{\inf}$.

Theorem 4.1. *Let (X, d) be a fuzzy relation. Then $K_{\inf}(d)$ is the convex envelope of $L_\top(d)$, i.e.,*

$$\Diamond_C\big(L_\top(d)\big) = K_{\inf}(d).$$

Equivalently, $K_{\inf}(d)$ is the greatest fuzzy relation $e : D(X)^2 \to [0,1]$ on $D(X)$ such that:

1. *e makes the unit η_X nonexpansive as a map $\eta_X : (X, d) \to (D(X), e)$, and*
2. *e is convex.*

Proof. By definition, $\Diamond_C\big(L_\top(d)\big)$ is the greatest convex fuzzy relation $e : D(X) \times D(X) \to [0,1]$ below $L_\top(d)$. We can then use Proposition 2.3 to express $\Diamond_C\big(L_\top(d)\big)$ as:

$$\Diamond_C\big(L_\top(d)\big)\big(\Delta_1, \Delta_2\big) = \inf\big\{\sum p_i \cdot L_\top(d)(\Delta_1^i, \Delta_2^i) \mid \langle \Delta_1, \Delta_2\rangle = \sum p_i \cdot \langle \Delta_1^i, \Delta_2^i\rangle\big\}.$$

By definition, $L_\top(d)(\Delta_1^i, \Delta_2^i) = d(x, y)$ if $\Delta_1^i = \delta_{x_i}$ and $\Delta_2^i = \delta_{y_i}$, for $x_i, y_i \in X$, and takes value 1 otherwise. Hence, since every $\langle \Delta_1, \Delta_2\rangle$ can be decomposed as a convex combination of pairs of Diracs, and since we are taking an infimum, we can simplify the right-hand-side as follows:

$$\Diamond_C\big(L_\top(d)\big)\big(\Delta_1, \Delta_2\big) = \inf\big\{\sum p_i \cdot d(x, y) \mid \langle \Delta_1, \Delta_2\rangle = \sum p_i \cdot \langle \delta_{x_i}, \delta_{y_i}\rangle\big\}.$$

By definition, the collection of decompositions $\sum p_i \cdot \langle \delta_{x_i}, \delta_{y_i}\rangle = \langle \Delta_1, \Delta_2\rangle$ coincides with $\Gamma(\Delta_1, \Delta_2)$, the collection of couplings of Δ_1, Δ_2. Therefore we have:

$$\Diamond_C\big(L_\top(d)\big)\big(\Delta_1, \Delta_2\big) = \inf_{\gamma \in \Gamma(\Delta_1, \Delta_2)} \big\{\sum p_i \cdot d(x, y) \mid \gamma = \sum p_i \cdot \langle \delta_{x_i}, \delta_{y_i}\rangle\big\}.$$

Lastly, for $\gamma = \sum p_i \cdot \langle \delta_{x_i}, \delta_{y_i}\rangle$, the expression $\sum p_i \cdot d(x, y)$ is the definition of $E(d, \gamma)$. We conclude, by Definition 2.14, that:

$$\Diamond_C\big(L_\top(d)\big)\big(\Delta_1, \Delta_2\big) = \inf_{\gamma \in \Gamma(\Delta_1, \Delta_2)} E(d, \gamma) = K_{\inf}(d)\big(\Delta_1, \Delta_2\big).$$

□

We now proceed with a series of useful lemmas required in the proof of our next Theorem, relating $K_{\inf}$ and $K_{\sup}$.

Proposition 4.2. *Let $e, f : D(X) \times D(X) \to [0,1]$ be fuzzy relations. Suppose e is convex and f is reflexive on Dirac distributions (i.e., $f(\delta_x, \delta_x) = 0$ for all $x \in X$). If $e \le f$, then $e \le \Diamond_R(f)$.*

Proof. First, observe that the operator $\Diamond_R$ admits the following explicit definition:

$$(\Diamond_R g)(\Delta_1, \Delta_2) = \begin{cases} 0 & \text{if } \Delta_1 = \Delta_2 \\ g(\Delta_1, \Delta_2) & \text{otherwise.} \end{cases}$$

Now, fix arbitrary $\Delta_1, \Delta_2 \in D(X)$. We distinguish two cases:

(Case: $\Delta_1 = \Delta_2 = \Delta$). Since e is convex, for any representation $\Delta = \sum_i p_i \delta_{x_i}$, we have:

$$e(\Delta, \Delta) \le \sum_i p_i e(\delta_{x_i}, \delta_{x_i}).$$

From the assumptions, $e \le f$ and $f(\delta_{x_i}, \delta_{x_i}) = 0$, we deduce that $e(\Delta, \Delta) = 0$. By definition, $(\Diamond_R f)(\Delta, \Delta) = 0$. Thus, $e(\Delta, \Delta) \le (\Diamond_R f)(\Delta, \Delta)$.

(Case: $\Delta_1 \ne \Delta_2$). By the hypothesis $e(\Delta_1, \Delta_2) \le f(\Delta_1, \Delta_2)$ and, since $\Delta_1 \ne \Delta_2$, we have that $(\Diamond_R f)(\Delta_1, \Delta_2) = f(\Delta_1, \Delta_2)$. Therefore, $f(\Delta_1, \Delta_2) \le (\Diamond_R f)(\Delta_1, \Delta_2)$. □

Corollary 4.3. *Let $d : X \times X \to [0,1]$ be a fuzzy relation. If d is reflexive (i.e., $d = \Diamond_R d$), then the following holds:*

$$\Diamond_C L_\top(d) = \Diamond_C \Diamond_R L_\top(d).$$

Proof. By definition of $L_\top$ and assumption on d we have $L_\top(d)(\delta_x, \delta_x) = d(x,x) = 0$. Hence, by the previous Proposition, $\Diamond_C L_\top(d)$ which is the greatest convex fuzzy relation bounded above by $L_\top(d)$ is also the greatest convex fuzzy relation bounded above by $\Diamond_R L_\top(d)$. □

Lemma 4.4. *Let (X, d) be a fuzzy relation. Then:*

$$\Diamond_P(L_\top(d)) = \Diamond_R L_\top(\Diamond_P(d)).$$

Proof. First, consider $\big(D(X), L_\top(\Diamond_P(d))\big)$. By definition of $L_\top$, it is a copy of $\Diamond_P(d)$ on Diracs and defined as 1 on all other pairs, i.e.,

$$L_\top(\Diamond_P(d))(\Delta_1, \Delta_2) = \begin{cases} \Diamond_P(d)(x,y) & \text{if } \Delta_1 = \delta_x \text{ and } \Delta_2 = \delta_y \\ 0 & \text{if } \Delta_1 = \Delta_2 = \delta_x\text{, a special case of the above, since } \Diamond_P(d) \text{ is reflexive} \\ 1 & \text{otherwise.} \end{cases}$$

Hence it is symmetric and satisfies the triangle inequality. Note that $L_\top(\Diamond_P(d))$ is not a pseudometric because reflexivity fails on non-Dirac pairs. We then calculate that:

$$\Diamond_R L_\top(\Diamond_P(d))(\Delta_1, \Delta_2) = \begin{cases} \Diamond_P(d)(x,y) & \text{if } \Delta_1 = \delta_x \text{ and } \Delta_2 = \delta_y \\ 0 & \text{if } \Delta_1 = \Delta_2 \\ 1 & \text{otherwise.} \end{cases}$$

is a pseudometric and, furthermore, it is the greatest pseudometric below $L_\top(\Diamond_P(d))$ and it is a copy of $\Diamond_P(d)$ on Diracs.

Now consider $\Diamond_P(L_\top(d))$ which is the greatest pseudometric below $L_\top(d)$. This means, in particular, that it is greater than $\Diamond_R L_\top(\Diamond_P(d))$, because the latter is a pseudometric and it is below $L_\top(\Diamond_P(d))$, and $L_\top(\Diamond_P(d)) \le L_\top(d)$. Therefore $\Diamond_P(L_\top(d))$ has the following shape:

$$\Diamond_P(L_\top(d))(\Delta_1, \Delta_2) = \begin{cases} r_{x,y} & \text{if } \Delta_1 = \delta_x \text{ and } \Delta_2 = \delta_y \\ 0 & \text{if } \Delta_1 = \Delta_2 \\ 1 & \text{otherwise.} \end{cases}$$

for some values, for $x, y \in X$, in between the following bounds:

$$L_\top(d)(\delta_x, \delta_y) \ge r_{x,y} \ge \Diamond_P(d)(x,y).$$

Now consider the subset $\{\delta_x \mid x \in X\}$ of $D(X)$ consisting of Diracs, which is in one-to-one correspondence ($\delta_x \leftrightarrow x$) with X. By Proposition 2.6, the restriction of $\Diamond_P(L_\top(d))$ to $\{\delta_x \mid x \in X\}$ is also a pseudometric space. Therefore, the values $r_{x,y}$ give a fuzzy relation $r : X^2 \to [0,1]$ defined as $r(x,y) = r_{x,y}$ such that:

1. r is a pseudometric on X,
2. $r \le d$, because $r(x,y) = r_{x,y} \le L_\top(d)(\delta_x, \delta_y) = d(x,y)$,
3. $r \ge \Diamond_P(d)$

Since $\Diamond_P(d)$ is the greatest such pseudometric below d, we deduce that $r = \Diamond_P(d)$.

This implies that $\Diamond_P(L_\top(d)) = \Diamond_R L_\top(\Diamond_P(d))$ holds. □

Recall, from Subsection 2.1, that $\Diamond_{P\cap C}(d)$ denotes the largest fuzzy relation that is both *convex and pseudometric* bounded above by d.

Lemma 4.5. *Let (X, d) be a fuzzy relation. Then the following equalities hold:*

$$K_{\text{inf}}(\Diamond_P(d)) = \Diamond_C \Diamond_P L_\top(d) = \Diamond_{P\cap C} L_\top(d) = \Diamond_{P\cap C} K_{\text{inf}}(d).$$

Proof. The first equality is derived as follows:

$$\begin{array}{rcll} K_{\text{inf}}(\Diamond_P(d)) & = & \Diamond_C L_\top(\Diamond_P(d)) & \text{(Theorem 4.1)} \\ & = & \Diamond_C \Diamond_R L_\top(\Diamond_P(d)) & \text{(Corollary 4.3)} \\ & = & \Diamond_C \Diamond_P L_\top(d) & \text{(Lemma 4.4)} \end{array}$$

For the second inequality, by Theorem 2.9, we know that $K_{\text{inf}}(\Diamond_P(d))$ is a pseudometric. Therefore:

$$\begin{array}{rcll} \Diamond_C \Diamond_P L_\top(d) & = & \Diamond_P \Diamond_C \Diamond_P L_\top(d) & \text{(Pseudometric)} \\ & = & \Diamond_{P\cap C} \Diamond_P L_\top(d) & \text{(Proposition 2.1(2))} \\ & = & \Diamond_{P\cap C} L_\top(d) & \text{(Proposition 2.1(4))} \end{array}$$

For the third inequality,

$$\begin{array}{rcll} \Diamond_{P\cap C} L_\top(d) & = & \Diamond_{P\cap C} \Diamond_C L_\top(d) & \text{(Proposition 2.1(4))} \\ & = & \Diamond_{P\cap C} K_{\text{inf}}(d) & \text{(Theorem 4.1.)} \end{array}$$

□

Lemma 4.6. *Let (X, d) be a fuzzy relation. Then $K_{\text{sup}}(d) = K_{\text{sup}}(\Diamond_P(d))$.*

Proof. Recall, from Definition 2.15, that $K_{\text{sup}}(d)$ is defined by means of a supremum over:

$$\text{Hom}_{\mathbf{FRel}}\Big((X, d), ([0,1], d_{\mathbb{R}})\Big) = \{f \mid f \text{ is nonexp. as a map } f : (X, d) \to ([0,1], d_{\mathbb{R}})\}$$

and, similarly, $K_{\text{sup}}(\Diamond_P(d))$ is defined by a supremum over:

$$\text{Hom}_{\mathbf{FRel}}\Big((X, \Diamond_P(d)), ([0,1], d_{\mathbb{R}})\Big) = \{f \mid f \text{ is nonexp. as a map } f : (X, \Diamond_P(d)) \to ([0,1], d_{\mathbb{R}})\}.$$

It then suffices to show that the two sets are the same. Since $\Diamond_P : \mathbf{FRel} \to \mathbf{PMet}$ is left adjoint to $U : \mathbf{PMet} \to \mathbf{FRel}$ (see Remark 2.10) and $d_{\mathbb{R}}$ is a pseudometric, the hom-sets coincide.

□

We derive, as a direct corollary of the previous lemmas, the following result.

Theorem 4.7. *Let (X, d) be a fuzzy relation. Then $K_{\text{sup}}(d)$ is the largest fuzzy relation on $D(X)$ that is both a convex and a pseudometric and that it makes the unit $\eta_X : X \to D(X)$ nonexpansive.*

$$K_{\text{sup}}(d) = \Diamond_{P\cap C} L_\top(d).$$

Equivalently, $K_{\text{sup}}(d)$ is the greatest fuzzy relation $e : D(X)^2 \to [0,1]$ on $D(X)$ such that:

1. *e makes the unit η_X nonexpansive as a map $\eta_X : (X, d) \to (D(X), e)$, and*
2. *e is both convex and a pseudometric.*

Furthermore, $K_{\text{sup}}(d)$ can also be described as the greatest fuzzy relation $e : D(X)^2 \to [0,1]$ on $D(X)$ such that:

1. *e is bounded above by $K_{\text{inf}}(d)$ and e is both convex and a pseudometric.*

Proof. By the following chain of equalities:

$$\begin{array}{rcll} K_{\text{sup}}(d) & = & K_{\text{sup}}(\Diamond_P(d)) & \text{(by Lemma 4.6)} \\ & = & K_{\text{inf}}(\Diamond_P(d)) & \text{(by Theorem 2.9)} \\ & = & \Diamond_{P\cap C} L_\top(d) & \text{(by Lemma 4.5).} \end{array}$$

This equality means exactly that $K_{\text{sup}}(d)$ satisfies the first characterization. For the second, we know from Lemma 4.5 that $\Diamond_{P\cap C} L_\top(d) = \Diamond_{P\cap C} K_{\text{inf}}(d)$. Hence $K_{\text{sup}}(d) = \Diamond_{P\cap C} K_{\text{inf}}(d)$ and this corresponds to the second characterization. □

5. Comparison with Results in Quantitative Algebra

Quantitative algebra is a line of work (see, e.g., [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]) originated from the seminal paper [7], that studies algebraic structures (like monoids, groups, *etc*) that are also metric spaces or, more generally, fuzzy relations. In the contex of the recent [8], a *quantitative convex algebra* is a structure $(X, d, \{+_p\}_{p\in(0,1)})$ such that: (X, d) is a fuzzy relation and $(X, \{+_p\}_{p\in(0,1)})$ is a convex algebra (see, e.g., [39, 40, 41] for references), i.e., a set X equipped with binary operations $+_p : X^2 \to X$, one for each $p \in (0, 1)$, satisfying:

$$\begin{array}{lll} \text{Idempotency:} & x +_p x = x & \forall x \in X, \forall p \in (0,1), \\ \text{Skew commutativity:} & x +_p y = y +_{1-p} x & \forall x, y \in X, \forall p \in (0,1), \\ \text{Skew associativity:} & (x +_p y) +_q z = x +_{pq} (y +_{\frac{(1-p)q}{1-pq}} z) & \forall x, y, z \in X, \forall p, q \in (0,1). \end{array}$$

Homomorphisms of quantitative convex algebras are homomorphisms (i.e., preserving all $+_p$ operations) that are also nonexpansive. Denote with $\mathbf{QCA}$ the category of quantitative convex algebras and their homomorphisms. We have a forgetful functor $U : \mathbf{QCA} \to \mathbf{FRel}$ that forgets the $+_p$ operations.

Example 5.1. The set $D(X)$ of finitely supported probability distributions on some set X is a convex algebra with operations defined as $\Delta_1 +_p \Delta_2 = p\Delta_1 + (1 - p)\Delta_2$. Hence, for any fuzzy relation $e : D(X) \times D(X) \to [0, 1]$, we have that $(D(X), e, \{+_p\}_{p\in(0,1)})$ is a quantitative convex algebra.

In [8], a quantitative convex algebra $(X, d, \{+_p\}_{p\in(0,1)})$ is said to satisfy the *interpolative barycentric axioms* (ICA) if, for all $p, q \in (0, 1)$ and $\epsilon, \delta \in [0, 1]$, it satisfies the implication:

$$\forall x, x', y, y'. \qquad \Big(d(x, y) \le \epsilon \ \wedge \ d(x', y') \le \delta\Big) \Rightarrow d(x +_p x', y +_p y') \le p \cdot \epsilon + (1 - p) \cdot \delta.$$

Remark 5.2. We note that for the quantitative algebras of Example 5.1, satisfying the ICA axioms[3] is equivalent to $e\colon D(X) \times D(X) \to [0, 1]$ being convex in the sense of Definition 3.1 (see, e.g., Theorem 4.2 of [38]). To the best of our knowledge, this observation, while straightforward has never been made in the literature. It actually constituted the starting point of this work.

Denote with $\mathbf{QCA_{ICA}}$ the full subcategory of $\mathbf{QCA}$ consisting of quantitative convex algebras satisfying ICA. The restriction of U ot $\mathbf{QCA_{ICA}}$ is also denoted by $U : \mathbf{QCA_{ICA}} \to \mathbf{FRel}$. The following is a result from [8] and it generalizes previous results from [7].

Theorem 5.1 (from [8]). *The functor $U : \mathbf{QCA_{ICA}} \to \mathbf{FRel}$ has a left adjoint $F : \mathbf{FRel} \to \mathbf{QCA_{ICA}}$ and the monad induced by $F \dashv U$ is the $K_{\inf} : \mathbf{FRel} \to \mathbf{FRel}$ monad (see Definition 2.3). This implies that $F(X, d) = (D(X), K_{\inf}(d), \{+_p\}_{p\in(0,1)})$.*

The fact that $(D(X), K_{\inf}(d), \{+_p\})$ is a free object can be seen as a maximality principle and indeed one can derive our Theorem 4.1 from this. Indeed, let $e : D(X)^2 \to [0, 1]$ be any convex fuzzy relation making the unit $\eta_X : (X, d) \to (D(X), e)$ nonexpansive. This implies that the identity map $\mathrm{id} : D(X) \to D(X)$ is a convex algebra homomorphism between $\mathbf{QCA_{ICA}}$ objects:

$$\mathrm{id} : (D(X), K_{\inf}(d), \{+_p\}_{p\in(0,1)}) \to (D(X), e, \{+_p\}_{p\in(0,1)})$$

and this means, in particular, that id is nonexpansive. This implies that $K_{\inf}(d) \ge e$. In fact, our Theorem 4.1, which we proved in a few lines using standard facts from the theory of convex functions, can be used to give a very short proof of Theorem 5.1 (from [8]). Our other main result, Theorem 4.7, is instead novel, to the best of our knowledge.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

[3]The quantified x, x', y, y' variables in the ICA implication range over the carrier, which in this case is $D(X)$, and thus over $\Delta_1, \Delta_1', \Delta_2, \Delta_2' \in D(X)$.

References

[1] J. Desharnais, V. Gupta, R. Jagadeesan, P. Panangaden, Metrics for labelled Markov processes, Theor. Comput. Sci. 318 (2004) 323–354. URL: https://doi.org/10.1016/j.tcs.2003.09.013. doi:10.1016/j.tcs.2003.09.013.

[2] P. Baldan, F. Bonchi, H. Kerstan, B. König, Coalgebraic behavioral metrics, Log. Methods Comput. Sci. 14 (2018). URL: https://doi.org/10.23638/LMCS-14(3:20)2018. doi:10.23638/LMCS-14(3:20)2018.

[3] F. van Breugel, J. Worrell, A behavioural pseudometric for probabilistic transition systems, Theor. Comput. Sci. 331 (2005) 115–142. URL: https://doi.org/10.1016/j.tcs.2004.09.035. doi:10.1016/j.tcs.2004.09.035.

[4] L. de Alfaro, M. Faella, M. Stoelinga, Linear and branching system metrics, IEEE Trans. Software Eng. 35 (2009) 258–273. URL: https://doi.org/10.1109/TSE.2008.106. doi:10.1109/TSE.2008.106.

[5] V. Castiglioni, D. Gebler, S. Tini, Logical characterization of bisimulation metrics, in: M. Tribastone, H. Wiklicky (Eds.), Quantitative Aspects of Programming Languages and Systems, QAPL 2016, volume 227 of *EPTCS*, 2016, pp. 44–62. URL: https://doi.org/10.4204/EPTCS.227. doi:10.4204/EPTCS.227.

[6] C. Villani, Optimal Transport: Old and New, Springer, 2009.

[7] R. Mardare, P. Panangaden, G. D. Plotkin, Quantitative algebraic reasoning, in: M. Grohe, E. Koskinen, N. Shankar (Eds.), Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS ’16, New York, NY, USA, July 5-8, 2016, ACM, 2016, pp. 700–709. URL: https://doi.org/10.1145/2933575.2934518. doi:10.1145/2933575.2934518.

[8] M. Mio, Compact quantitative theories of convex algebras, in: C. Kupke, S. Milius (Eds.), Proceedings of the 41st Conference on the Mathematical Foundations of Programming Semantics, MFPS XLI, University of Strathclyde, Glasgow, UK, June 16-21, 2025, Electronic Notes in Theoretical Informatics and Computer Science, EpiSciences, 2025. URL: https://doi.org/10.46298/entics.16876. doi:10.46298/ENTICS.16876.

[9] P. Panangaden, Labelled Markov Processes, Imperial College Press, 2009. URL: http://www.worldscibooks.com/compsci/p595.html.

[10] W. A. Wilson, On semi-metric spaces, American Journal of Mathematics 53 (1931) 361–373. URL: http://www.jstor.org/stable/2370790.

[11] W. A. Wilson, On quasi-metric spaces, American Journal of Mathematics 53 (1931) 675–684. URL: http://www.jstor.org/stable/2371174.

[12] H. Kunzi, H. Pajoohesh, b-metrics, Topology and its Applications 309 (2022) 107905. URL: https://www.sciencedirect.com/science/article/pii/S0166864121003205. doi:https://doi.org/10.1016/j.topol.2021.107905, remembering Hans-Peter Kunzi (1955-2020).

[13] P. S. Castro, T. Kastner, P. Panangaden, M. Rowland, Mico: Improved representations via sampling-based state similarity for markov decision processes, in: M. Ranzato, A. Beygelzimer, Y. N. Dauphin, P. Liang, J. W. Vaughan (Eds.), Advances in Neural Information Processing Systems 34: Annual Conference on Neural Information Processing Systems 2021, NeurIPS 2021, December 6-14, 2021, virtual, 2021, pp. 30113–30126. URL: https://proceedings.neurips.cc/paper/2021/hash/fd06b8ea02fe5b1c2496fe1700e9d16c-Abstract.html.

[14] M. Bonsangue, F. van Breugel, J. Rutten, Generalized metric spaces: Completion, topology, and powerdomains via the yoneda embedding, Theoretical Computer Science 193 (1998) 1–51. URL: https://www.sciencedirect.com/science/article/pii/S030439759700042X. doi:https://doi.org/10.1016/S0304-3975(97)00042-X.

[15] J. Adámek, Varieties of quantitative algebras and their monads, in: 37th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS’22), ACM, 2022. doi:10.1145/3531130.3532405.

[16] J. Adámek, M. Dostál, J. Velebil, Strongly finitary monads for varieties of quantitative algebras, in: 10th Conference on Algebra and Coalgebra in Computer Science (CALCO’23), volume 270 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2023, pp. 10:1–10:14. doi:10.4230/LIPICS.CALCO.2023.10.

[17] G. Bacci, G. Bacci, K. G. Larsen, R. Mardare, A complete quantitative deduction system for the bisimilarity distance on markov chains, Log. Methods Comput. Sci. 14 (2018). URL: https://doi.org/10.23638/LMCS-14(4:15)2018. doi:10.23638/LMCS-14(4:15)2018.
[18] G. Bacci, R. Mardare, P. Panangaden, G. Plotkin, An algebraic theory of markov processes, in: 33rd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'18), ACM, 2018, pp. 679–688. doi:10.1145/3209108.3209177.
[19] G. Bacci, R. Mardare, P. Panangaden, G. D. Plotkin, Propositional logics for the lawvere quantale, in: M. Kerjean, P. B. Levy (Eds.), Proceedings of the 39th Conference on the Mathematical Foundations of Programming Semantics, MFPS XXXIX, Indiana University, Bloomington, IN, USA, June 21-23, 2023, volume 3 of *EPTICS*, EpiSciences, 2023. URL: https://doi.org/10.46298/entics.12292. doi:10.46298/ENTICS.12292.
[20] G. Bacci, R. Mardare, P. Panangaden, G. D. Plotkin, Sum and tensor of quantitative effects, Log. Methods Comput. Sci. 20 (2024). URL: https://doi.org/10.46298/lmcs-20(4:9)2024. doi:10.46298/LMCS-20(4:9)2024.
[21] F. Gavazzo, Quantitative behavioural reasoning for higher-order effectful programs: Applicative distances, in: Logic in Computer Science, LICS 2018, ACM, 2018, pp. 452–461. doi:10.1145/3209108.3209149.
[22] F. Gavazzo, C. D. Florio, Elements of quantitative rewriting, Proc. ACM Program. Lang. 7 (2023) 1832–1863. URL: https://doi.org/10.1145/3571256. doi:10.1145/3571256.
[23] R. Mardare, P. Panangaden, G. D. Plotkin, On the axiomatizability of quantitative algebras, in: 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20-23, 2017, IEEE Computer Society, 2017, pp. 1–12. URL: https://doi.org/10.1109/LICS.2017.8005102. doi:10.1109/LICS.2017.8005102.
[24] R. Sarkis, Lifting Algebraic Reasoning to Generalized Metric Spaces, Phd thesis, ENS de Lyon, 2024. Available at https://doi.org/10.5281/zenodo.14001076.
[25] M. Mio, R. Sarkis, V. Vignudelli, Universal quantitative algebra for fuzzy relations and generalised metric spaces, Log. Methods Comput. Sci. 20 (2024) 19:1–19:56.
[26] M. Mio, R. Sarkis, V. Vignudelli, Beyond nonexpansive operations in quantitative algebraic reasoning, in: Proc. 37th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS'22), ACM, 2022, pp. 52:1–52:13. doi:10.1145/3531130.3533366.
[27] M. Mio, R. Sarkis, V. Vignudelli, Combining nondeterminism, probability, and termination: Equational and metric reasoning, in: 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29 - July 2, 2021, IEEE, 2021, pp. 1–14. URL: https://doi.org/10.1109/LICS52264.2021.9470717. doi:10.1109/LICS52264.2021.9470717.
[28] M. Mio, V. Vignudelli, Monads and quantitative equational theories for nondeterminism and probability, in: I. Konnov, L. Kovács (Eds.), 31st International Conference on Concurrency Theory, CONCUR 2020, September 1-4, 2020, Vienna, Austria (Virtual Conference), volume 171 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, pp. 28:1–28:18. URL: https://doi.org/10.4230/LIPIcs.CONCUR.2020.28. doi:10.4230/LIPICS.CONCUR.2020.28.
[29] U. D. Lago, F. Honsell, M. Lenisa, P. Pistone, On quantitative algebraic higher-order theories, in: A. P. Felty (Ed.), 7th International Conference on Formal Structures for Computation and Deduction, FSCD 2022, August 2-5, 2022, Haifa, Israel, volume 228 of *LIPIcs*, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, pp. 4:1–4:18. URL: https://doi.org/10.4230/LIPIcs.FSCD.2022.4. doi:10.4230/LIPICS.FSCD.2022.4.
[30] J. Jurka, S. Milius, H. Urbat, Algebraic reasoning over relational structures, in: V. De Paiva, A. Simpson (Eds.), Proc. 40th Conference on Mathematical Foundations of Programming Semantics (MFPS), volume 4 of *ENTICS*, 2024, pp. 13:1–13:20.
[31] S. Milius, H. Urbat, Equational axiomatization of algebras with structure, in: M. Bojańczyk, A. Simpson (Eds.), Proc. Foundations of Software Science and Computation Structures (FoSSaCS), volume 11425 of *Lecture Notes Comput. Sci. (ARCoSS)*, Springer, 2019, pp. 400–417.
[32] P. Wild, L. Schröder, Characteristic Logics for Behavioural Metrics via Fuzzy Lax Extensions, in: I. Konnov, L. Kovács (Eds.), 31st International Conference on Concurrency Theory, CONCUR

2020, September 1-4, 2020, Vienna, Austria (Virtual Conference), volume 171 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020, pp. 27:1–27:23. URL: https://doi.org/10.4230/LIPIcs.CONCUR.2020.27. doi:10.4230/LIPIcs.CONCUR.2020.27. arXiv:2007.01033.

[33] P. Wild, L. Schröder, A quantified coalgebraic van benthem theorem, in: S. Kiefer, C. Tasson (Eds.), Foundations of Software Science and Computation Structures - 24th International Conference, FOSSACS 2021, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2021, Luxembourg City, Luxembourg, March 27 - April 1, 2021, Proceedings, volume 12650 of *LNCS*, Springer, 2021, pp. 551–571. URL: https://doi.org/10.1007/978-3-030-71995-1_28. doi:10.1007/978-3-030-71995-1_28.

[34] S. Goncharov, D. Hofmann, P. Nora, L. Schröder, P. Wild, Kantorovich functors and characteristic logics for behavioural distances, in: O. Kupferman, P. Sobocinski (Eds.), Foundations of Software Science and Computation Structures – 26th International Conference, FoSSaCS 2023, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2023, Paris, France, April 22-27, 2023, Proceedings, volume 13992 of *LNCS*, Springer, 2023, pp. 46–67. URL: https://doi.org/10.1007/978-3-031-30829-1_3. doi:10.1007/978-3-031-30829-1_3.

[35] K. D'Angelo, S. Gurke, J. M. Kirss, B. König, M. Najafi, W. Rozowski, P. Wild, Behavioural metrics: Compositionality of the kantorovich lifting and an application to up-to techniques, in: R. Majumdar, A. Silva (Eds.), 35th International Conference on Concurrency Theory, CONCUR 2024, September 9-13, 2024, Calgary, Canada, volume 311 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024, pp. 20:1–20:19. URL: https://doi.org/10.4230/LIPIcs.CONCUR.2024.20. doi:10.4230/LIPICS.CONCUR.2024.20.

[36] J. Forster, L. Schröder, P. Wild, H. Beohar, S. Gurke, B. König, K. Messing, Quantitative graded semantics and spectra of behavioural metrics, in: J. Endrullis, S. Schmitz (Eds.), 33rd EACSL Annual Conference on Computer Science Logic, CSL 2025, February 10-14, 2025, Amsterdam, Netherlands, volume 326 of *LIPIcs*, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2025, pp. 33:1–33:21. URL: https://doi.org/10.4230/LIPIcs.CSL.2025.33. doi:10.4230/LIPICS.CSL.2025.33.

[37] A. Rodary, Kantorovich liftings for fuzzy relations, Master's thesis, ENS de Lyon, July 2025.

[38] R. T. Rockafellar, Convex Analysis, Princeton Mathematical Series, Princeton University Press, 1970.

[39] A. Sokolova, H. Woracek, Termination in convex sets of distributions, Logical Methods in Computer Science Volume 14, Issue 4 (2018). URL: https://lmcs.episciences.org/4036. doi:10.23638/LMCS-14(4:17)2018.

[40] T. Fritz, Convex spaces i: Definition and examples, 2015. arXiv:0903.5522.

[41] B. Jacobs, Convexity, duality and effects, in: C. S. Calude, V. Sassone (Eds.), Theoretical Computer Science, Springer Berlin Heidelberg, Berlin, Heidelberg, 2010, pp. 1–19.