

Mathematical model of Izhikevich network dynamics*

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Abstract

The Izhikevich neuron model combines biological realism with computational efficiency, making it a standard choice for large-scale brain simulations. Despite its widespread use, theoretical results for networks of these neurons remain limited. This paper develops a rigorous mathematical framework by treating such networks as high-dimensional, piecewise-smooth dynamical systems. We first derive stability conditions for core network behaviors—resting activity and synchronous tonic firing—using Lyapunov methods tailored to hybrid systems. We then present a formal bifurcation analysis, identifying precise thresholds that trigger transitions from steady states to network-wide oscillations and more complex firing regimes. Incorporating stochastic inputs, we establish results on the stationary statistics of activity and on noise-driven spiking. The goal of this paper is to develop a rigorous mathematical framework for analyzing Izhikevich neuron networks by treating them as high-dimensional, piecewise-smooth dynamical systems, and to provide a solid theoretical basis for understanding the dynamics of such large-scale networks.

Keywords

Izhikevich model, piecewise-smooth dynamical systems, hybrid systems, bifurcation analysis

1. Introduction

The analysis of a single Izhikevich neuron, as described in [1], reveals it is fundamentally a hybrid dynamical system. It comprises a continuous two-dimensional flow (sub-threshold dynamics) combined with a discrete, event-driven reset map (spike generation and after-spike behavior). Consequently, a network of N such interconnected neurons is, by construction, a high-dimensional hybrid dynamical system (see Figure 1).

To provide a rigorous foundation for our theorems, we must first formally define this system. We formulate the network as a hybrid automaton $\mathcal{H}$, which is a tuple $\mathcal{H} = (S, C, F, G, R)$ defining the system's state space, flow domain, continuous dynamics, guard (event-detection) set, and discrete reset map.

We consider a network of N neurons, where the state of each neuron i is described not only by its membrane potential v_i and recovery variable u_i , but also by the total postsynaptic current $I_{\text{syn},i}$ it is receiving. This formulation is necessary to explicitly capture the network interactions [2].

1.1. State Space

The full state of the network at time t is given by a vector $x(t) \in \mathbb{R}^{3N}$. We partition this state vector as $x = (v, u, I_{\text{syn}})^T$, where $v = (v_1, \dots, v_N)^T \in \mathbb{R}^N$ is the vector of membrane potentials, $u =$

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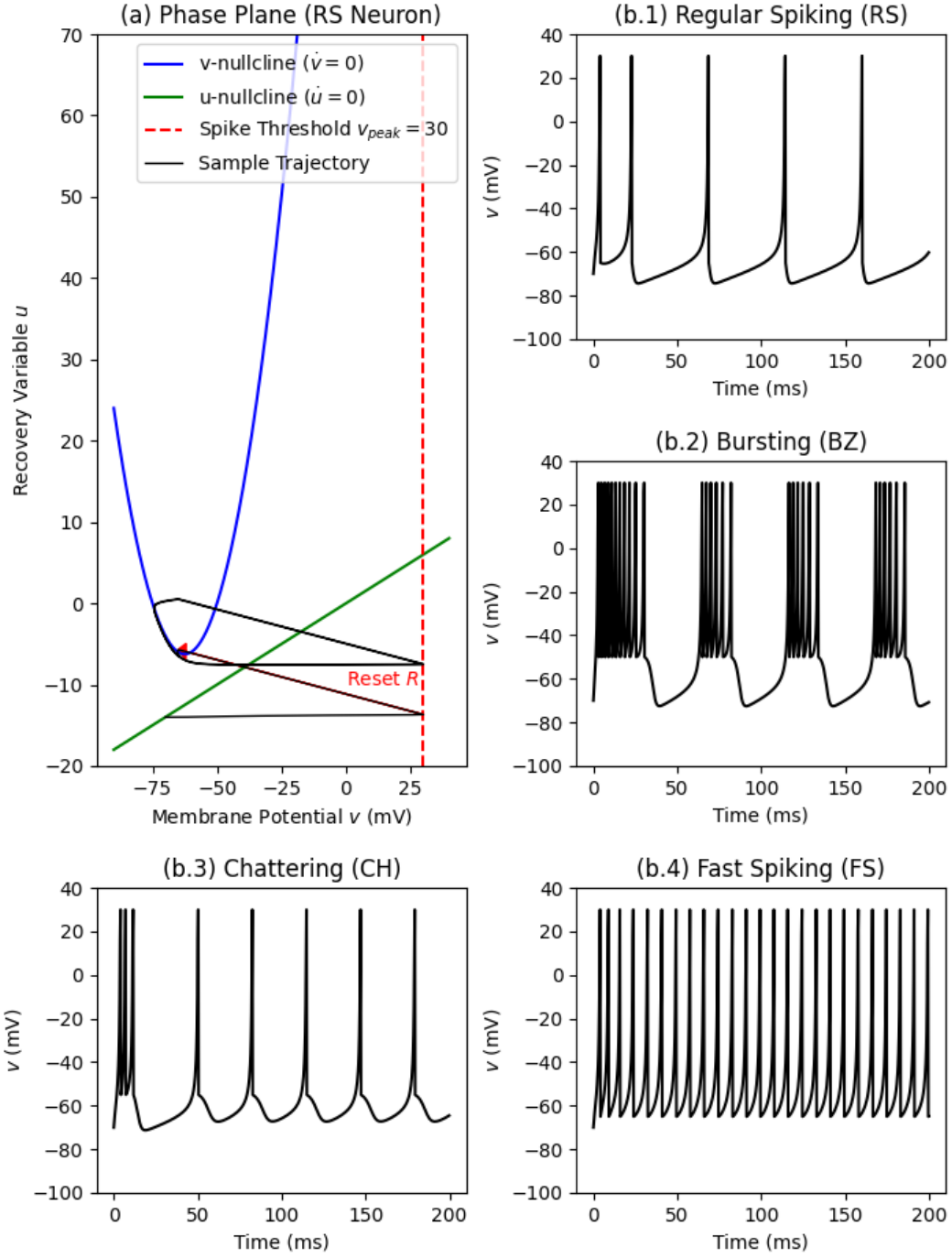
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$(u_1, \dots, u_N)^T \in \mathbb{R}^N$ is the vector of recovery variables, $I_{\text{syn}} = (I_{\text{syn},1}, \dots, I_{\text{syn},N})^T \in \mathbb{R}^N$ is the vector of total synaptic currents received by each neuron.



Thus, the state space S for the entire network is $S = \mathbb{R}^{3N}$ [3].

Figure 1: Phase plane of the Izhikevich model and spike patterns.

1.2. Continuous Flow and Flow Domain

The continuous dynamics, or "flow," describes the evolution of the system between spiking events [4]. This flow is valid within the flow domain C , which we define as the sub-threshold region of the state space:

$$C = \{x \in S \mid v_i < v_{\text{peak}}, \forall i \in \{1, \dots, N\}\}, \quad (1)$$

where v_{peak} is the spike threshold (typically 30 mV).

For any state $x \in C$, the system evolves according to the vector field $F: C \rightarrow T\mathbb{R}^{3N}$ defined by the system of $3N$ coupled differential equations $\dot{x} = F(x)$:

$$\begin{cases} \dot{v}_i = 0.04v_i^2 + 5v_i + 140 - u_i + I_{\text{ext},i} + I_{\text{syn},i} \\ \dot{u}_i = a_i(b_iv_i - u_i) \\ \dot{I}_{\text{syn},i} = \frac{-1}{\tau_{s,i}} I_{\text{syn},i} \end{cases} \quad \text{for } i = 1, \dots, N \quad (2)$$

The scalar coefficients in the voltage equation (0.04, 5, and 140) are empirical fitting parameters derived by Izhikevich [1, 5]. These values are specifically chosen to scale the membrane potential v to millivolts (mV) and time t to milliseconds (ms), such that the system's resting potential (≈ -82 mV) and instantaneous threshold potential (≈ -42 mV) align with the biophysical properties of typical cortical neurons.

Here, (a_i, b_i) are the parameters governing the dynamics of the recovery variable for neuron i , $I_{\text{ext},i}$ is the external drive (assumed constant or slowly varying), and $\tau_{s,i}$ is the synaptic time constant [6] governing the decay of the postsynaptic current [7]. Note that in this continuous regime, the system is locally uncoupled; the network interaction is introduced discretely via the reset map [8].

1.3. Guard Set (Spike Detection Set)

The guard set G is the boundary of the flow domain C where a discrete event (a spike) is triggered [9, 10]. It is the set of all states where at least one neuron has reached the spiking threshold. We define G as the union of N disjoint guard hypersurfaces G_j :

$$G = \bigcup_{j=1}^N G_j \quad (3)$$

where G_j is the set of states where neuron j is the one initiating the spike:

$$G_j = \{x \in C \mid v_j = v_{\text{peak}}\} \quad (4)$$

We operate under the generic assumption that no two neurons spike at the exact same instant, thus $G_i \cap G_j = \emptyset$ for $i \neq j$ is a set of measure zero which we can neglect [11].

1.4. Reset Map

The reset map $R: G \rightarrow C$ is a discrete, discontinuous function that instantaneously translates the system state $x(t^-)$ (the state just before the event) to a new state $x(t^+)$ (the state just after) whenever the trajectory $x(t)$ intersects G [12, 13].

Let the trajectory $x(t)$ strike the guard at $t=t_k$, such that $x(t_k^-) \in G_j$ (i.e., neuron j has spiked). The map R is defined by the specific reset $R_j: x(t_k^+) = R_j(x(t_k^-))$ [14]. This map R_j acts as follows:

1. Reset the Spiking Neuron (j): The spiking neuron's own state variables (v_j, u_j) are reset according to the Izhikevich model: $v_j(t_k^+) = c_j$, $u_j(t_k^+) = u_j(t_k^-) + d_j$
2. Enact Network Coupling: The spike from neuron j induces a change in the synaptic current $I_{\text{syn},i}$ of all postsynaptic neurons i . This is the core of the network interaction. We define an $N \times N$ connectivity matrix W , where W_{ij} is the synaptic weight from (presynaptic) neuron j to (postsynaptic) neuron i . $I_{\text{syn},i}(t_k^+) = I_{\text{syn},i}(t_k^-) + W_{ij}$ for all $i = 1, \dots, N$

Leave Other Variables Unchanged: All other state variables that are not v_j, u_j , or one of the $I_{\text{syn},i}$ are continuous through the event. $v_i(t_k^+) = v_i(t_k^-)$ for $i \neq j$, $u_i(t_k^+) = u_i(t_k^-)$ for $i \neq j$.

1.5. Summary of the Mathematical Object

The object of our analysis is the hybrid dynamical system $\mathcal{H} = (S, C, F, G, R)$ defined by the components in sections 1.1-1.4. This formulation transforms the problem of network dynamics into the analysis of trajectories in $\mathbb{R}^{3N}$ that flow according to F in the domain C and are discontinuously mapped by R upon striking the guard G [15, 16]. All subsequent theorems on network stability, synchronization, and emergent dynamics will be formulated and proven by analyzing the properties of this hybrid system $\mathcal{H}$ [17].

2. Theorems on Stability and Bifurcation in Deterministic Networks

This section presents the core mathematical analysis of the deterministic network, formulated as the hybrid dynamical system $\mathcal{H} = (S, C, F, G, R)$ in Introduction. We derive theorems governing its most fundamental dynamical regimes: the quiescent state (global silence), the synchronous firing state (collective oscillation), and the transitions between them. For this section, we assume the network is free of stochastic input [18].

2.1. Theorem on Stability of the Quiescent State

The quiescent state, $x^* \in C$, is the global equilibrium of the network, corresponding to a state of silence where no neuron fires. Its existence and stability are prerequisites for any analysis of excitability.

Theorem 1. Let the network consist of N identical neurons (parameters a, b) with homogeneous external input $I_{\text{ext},i} = I$.

1. Existence: The network admits a unique global fixed point $x^* = (v^*, u^*, I_{\text{syn}}^*) \in S$ given by $I_{\text{syn},i}^* = 0$, $u_i^* = bv_i^*$, $v_i^* = v^*$, where v^* is the lower (stable) real root of the quadratic equation $0.04(v^*)^2 + (5 - b)v^* + (140 + I) = 0$.
2. Quiescence Condition: This fixed point x^* represents a stable quiescent state (i.e., $x^* \in C$ and is locally stable) if and only if:
 - o (a) $v^* < v_{\text{peak}}$ (the fixed point is sub-threshold).
 - o (b) $a(b - (0.08v^* + 5)) > 0$ (stability condition).

Proof:

1. Existence: A fixed point x^* must satisfy $\dot{x} = F(x^*) = 0$. From the equations of flow, $\dot{I}_{\text{syn},i} = 0 \Rightarrow I_{\text{syn},i}^* = 0$, and $\dot{u}_i = 0 \Rightarrow a(bv_i^* - u_i^*) = 0 \Rightarrow u_i^* = bv_i^*$.

2. Substituting these into the $\dot{v}_i$ equation yields the quadratic relation for the homogeneous solution v^* : $0.04(v^*)^2 + 5v^* + 140 - bv^* + I = 0$, which simplifies to $0.04(v^*)^2 + (5 - b)v^* + (140 + I) = 0$. This equation has real roots if its discriminant $\Delta = (5 - b)^2 - 0.16(140 + I) \geq 0$. The lower root is the stable node.
3. Stability: To assess local stability, we analyze the Jacobian J of the $3N$ -dimensional flow F at x^* . Since x^* is uncoupled (no spikes $\Rightarrow I_{\text{syn}}^* = 0$), J is block-diagonal, composed of N identical 3×3 blocks J_i :

$$J_i = \frac{\partial(\dot{v}_i, \dot{u}_i, \dot{I}_{\text{syn},i})}{\partial(v_i, u_i, I_{\text{syn},i})} \Big|_{x^*} = \begin{pmatrix} 0.08v^* + 5 & -1 & 1 \\ ab & -a & 0 \\ 0 & 0 & -1/\tau_s \end{pmatrix} \quad (5)$$

The eigenvalues λ of J_i govern stability. One eigenvalue is $\lambda_1 = -1/\tau_s$, which is stable. The other two are given by the 2×2 (voltage-recovery) subsystem $J_{v,u}$. For x^* to be a stable node, we require all eigenvalues to have negative real parts. By the Routh-Hurwitz stability criterion for $J_{v,u}$: $\text{Tr}(J_{v,u}) = (0.08v^* + 5) - a < 0 \Rightarrow 0.08v^* + 5 < a$, $\text{Det}(J_{v,u}) = -a(0.08v^* + 5) - (-1)(ab) = a(b - (0.08v^* + 5)) > 0$

The second condition, $a(b - (0.08v^* + 5)) > 0$, is the more restrictive one (e.g., for $a=0.02, b=0.2$, this is the condition that ensures the fixed point is a node and not a focus). If this holds, and $v^* < v_{\text{peak}}$, the fixed point x^* is locally asymptotically stable and resides within the flow domain C . Thus, trajectories starting near x^* converge to it without hitting the guard G . Q.E.D.

2.2. Theorem on Existence and Stability of Synchronous States

A 1-cluster synchronous state is a periodic orbit $\gamma(t)$ of the hybrid system $\mathcal{H}$ where all N neurons fire in unison. This corresponds to a trajectory on the synchronization manifold $M \subset S$.

Theorem 2. Let the network consist of N identical neurons (parameters a, b, c, d, τ_s) and homogeneous input I . Let the connectivity matrix W have a constant row sum, $\sum_{j=1}^N W_{ij} = k$ for all i .

1. Existence: The synchronization manifold $M = \{x \in S \mid v_1 = \dots = v_N, u_1 = \dots = u_N, I_{\text{syn},1} = \dots = I_{\text{syn},N}\}$ is an invariant manifold of the hybrid system $\mathcal{H}$. A synchronous periodic orbit $\gamma(t) \subset M$ exists if the reduced 3-dimensional hybrid system on M admits a stable periodic orbit.
2. Stability: The synchronous orbit $\gamma(t)$ is locally asymptotically stable if and only if all $3N - 3$ transverse eigenvalues μ_T of the monodromy matrix (Poincaré map Jacobian) associated with $\gamma(t)$ satisfy $|\mu_T| < 1$.

Proof:

1. Invariance of M : We must show that if $x(t) \in M$, then $x(t + \Delta t) \in M$ for all $\Delta t > 0$, including across resets.
 - Flow Invariance: If $x \in M$, then $v_i = v, u_i = u, I_i = I$. Because all parameters (a, b, I) are identical, the flow vector $F(x)$ is identical for all neurons: $\dot{v}_i = \dot{v}, \dot{u}_i = \dot{u}, \dot{I}_{\text{syn},i} = \dot{I}_{\text{syn}}$. Thus, the flow F is tangent to M , and M is invariant under F .

- Reset Invariance: If $x(t_k^-) \in M \cap G$, then all neurons hit the guard G_j simultaneously (i.e., $x(t_k^-) \in \cap_{j=1}^N G_j$). The reset R is applied to all N neurons: $v_i(t_k^+) = c$ (identical), $u_i(t_k^+) = u(t_k^-) + d$ (identical), $I_{\text{syn},i}(t_k^+) = I_{\text{syn}}(t_k^-) + \sum_{j=1}^N W_{ij} = I_{\text{syn}}(t_k^-) + k$.
 - Since k is constant for all i , the reset state $x(t_k^+)$ is also in M . Thus M is invariant under the full hybrid dynamics H .
2. Existence: The dynamics on M are governed by a 3-variable hybrid system $\mathcal{H}_{\text{sync}}$ (for v_s, u_s, I_s). The existence of a periodic orbit $\gamma(t)$ is established by showing that the 1D Poincaré map Π_{sync} for $\mathcal{H}_{\text{sync}}$ (e.g., on the section $v = c$) has a stable fixed point. This is generically true for I above the firing threshold I_{SN} .
 3. Stability: We analyze the stability of $\gamma(t)$ to perturbations transverse to M . We define a Poincaré section Σ for $\gamma(t)$ in the full $3N$ space. The stability is determined by the eigenvalues of the Jacobian of the full Poincaré map, $\Pi: \Sigma \rightarrow \Sigma$.
 4. This map Π is the composition of the continuous flow map ϕ_T (evolution from $x(t_k^+)$ to $x(t_{k+1}^-)$) and the reset map R (from $x(t_{k+1}^-)$ to $x(t_{k+1}^+)$).
 5. The Jacobian $D\Pi$ is the monodromy matrix. Its eigenvalues μ determine stability. One eigenvalue is $\mu = 1$ (corresponding to perturbations along the orbit). The (complex) eigenvalues of the 3×3 reduced system $\mathcal{H}_{\text{sync}}$ lie within the unit circle by assumption of γ 's existence.
 6. The remaining $3N - 3$ eigenvalues are the transverse eigenvalues. These are found by analyzing the variational equations for transverse perturbations $\delta x_{\perp}$ and the action of the reset Jacobian DR on these perturbations (the "saltation matrix").
 7. The orbit γ is stable if all these $3N - 3$ transverse eigenvalues satisfy $|\mu_T| < 1$. This condition depends critically on the network coupling k , the reset parameters (c, d) , and the properties of the flow F along the orbit. Q.E.D.

2.3. Theorem on Bifurcations of Deterministic States

The transitions between quiescence and tonic firing, and between different firing patterns, are governed by bifurcations. In our hybrid system $\mathcal{H}$, we find both standard bifurcations of smooth systems and non-smooth bifurcations unique to hybrid systems.

Theorem 3.1 (Saddle-Node Bifurcation). The transition from quiescence (Theorem 1) to tonic firing is mediated by a saddle-node bifurcation of fixed points. For a homogeneous network, this occurs at a critical input $I = I_{SN}$ defined by:

$$I_{SN} = \frac{(5 - b)^2}{0.16} - 140 \quad (6)$$

For $I > I_{SN}$, two fixed points (one stable, one saddle) exist in C . At $I = I_{SN}$, they merge and annihilate. For $I < I_{SN}$, no fixed points exist, and the trajectory is guaranteed to hit the guard G , initiating periodic firing.

Proof:

1. From Theorem 1, the fixed points v^* are the roots of $g(v) = 0.04v^2 + (5 - b)v + (140 + I) = 0$.

2. A saddle-node bifurcation occurs when two roots merge, i.e., when $g(v) = 0$ and $g'(v) = 0$ simultaneously.
3. $g'(v) = 0.08v + (5 - b) = 0 \Rightarrow v_{SN} = -\frac{5-b}{0.08}$.
4. Substituting v_{SN} into $g(v) = 0$: $0.04(-\frac{5-b}{0.08})^2 + (5 - b)(-\frac{5-b}{0.08}) + (140 + I_{SN}) = 0$
5. Solving for I_{SN} yields $I_{SN} = \frac{(5-b)^2}{0.16} - 140$.
6. This is a co-dimension one bifurcation. For $I < I_{SN}$, the discriminant $\Delta < 0$, no real roots exist. The flow $\dot{v}$ is strictly positive (for typical parameters), forcing the system state $x(t)$ to evolve until v_i hits v_{peak} , triggering the reset R and establishing a limit cycle. Q.E.D.

Theorem 3.2 (Grazing Bifurcation). The onset of spiking periodic orbits (e.g., transition from sub-threshold oscillations to spiking) or the transition between bursting patterns can be mediated by a grazing (or border-collision) bifurcation. This non-smooth bifurcation occurs when a periodic orbit $\gamma(t, \mu)$ of the flow F , parameterized by μ (e.g., $\mu = I$ or $\mu = k$), becomes tangent to a guard G_j . The condition for grazing is the simultaneous satisfaction of incidence $g_j(x(t^*)) = v_j(t^*) - v_{peak} = 0$ and tangency $L_F g_j(x(t^*)) = \nabla g_j(x(t^*)) \cdot F(x(t^*)) = \dot{v}_j(t^*) = 0$

Proof:

1. Consider a family of periodic orbits $\gamma(t, \mu)$ that, for $\mu < \mu_G$, are entirely contained in the flow domain C (e.g., sub-threshold oscillations).
2. Define the bifurcation point μ_G as the parameter value where the orbit γ first touches the guard G_j .
3. At this point of first contact $x_G = \gamma(t^*, \mu_G)$, the orbit must be tangent to G_j . If it were transverse ($L_F g_j(x_G) > 0$), it would have crossed G_j for μ values slightly less than μ_G , contradicting the definition of μ_G . Thus, the tangency condition $L_F g_j(x_G) = \dot{v}_j(t^*) = 0$ must hold.
4. We analyze the Poincaré map Π in a neighborhood of x_G . For $\mu < \mu_G$, Π is a smooth map (just the flow ϕ_T). For $\mu > \mu_G$, Π is a composition $\Pi = R \circ \phi_T$.
5. At $\mu = \mu_G$, the map Π is continuous but not differentiable. The Jacobian $D\Pi$ is discontinuous.
6. Local analysis (using a "zero-time discontinuity map") shows that the deviation of the trajectory scales as $\mathcal{O}(\sqrt{\mu - \mu_G})$ due to the quadratic nature of the contact.

This non-differentiability causes an abrupt change in the system's dynamics, such as the creation of a new, complex periodic orbit or the sudden destabilization of the grazing orbit. This mechanism governs the transition to spiking from sub-threshold oscillations. Q.E.D.

3. Theorems on Stochastic and Chaotic Network Dynamics

The analysis in Section 2 established the conditions for the network's simplest stable states: quiescence and synchrony. We now extend this analysis to address two forms of complex, non-periodic dynamics: (1) noise-induced asynchronous firing, which is fundamental to realistic cortical states [19], and (2) deterministic chaos, which can emerge intrinsically from the network's strong, nonlinear interactions [20].

To analyze stochastic dynamics, we augment the deterministic flow F by introducing an independent Wiener process $dW_i(t)$ to the voltage dynamics of each neuron, representing thermal noise or stochastic synaptic background. The continuous flow is now a system of $3N$ Stochastic Differential Equations (SDEs):

$$\begin{cases} dv_i = (0.04v_i^2 + 5v_i + 140 - u_i + I_{\text{ext},i} + I_{\text{syn},i})dt + \sigma_i dW_i(t) \\ du_i = a_i(b_i v_i - u_i)dt \\ dI_{\text{syn},i} = -\frac{1}{\tau_{s,i}} I_{\text{syn},i} dt \end{cases} \quad \text{for } i = 1, \dots, N \quad (7)$$

This defines a stochastic hybrid system $\mathcal{H}_\sigma$, where $\sigma = (\sigma_1, \dots, \sigma_N)^T$ is the vector of noise intensities. The Guard G and Reset R maps remain unchanged.

3.1. Theorem on Stochastic Stability and Dynamics

We first analyze the sub-threshold regime ($I < I_{SN}$) where the deterministic system is quiescent (Theorem 1). In this state, all firing is noise-driven. We prove the existence of a stationary distribution of network activity and formalize the network's firing rate via the Mean First-Passage Time (MFPT).

Theorem 4.1 (Existence of Stationary Distribution). For the stochastic hybrid system $\mathcal{H}_\sigma$ operating in the sub-threshold regime (possessing a stable fixed point x^* in C) and with noise intensity $\sigma_i > 0$ for all i , there exists a unique, non-trivial stationary probability density $\rho_{st}(x)$ on the flow domain C . This density satisfies the stationary Fokker-Planck equation $\nabla \cdot J(x) = 0$, where J is the probability flux, subject to a non-local boundary condition coupling the flux exiting at G to the flux re-injected at $R(G)$.

Proof:

1. **Fokker-Planck Equation:** The evolution of the probability density $\rho(x, t)$ on the domain C (for $x \notin G$) is governed by the Fokker-Planck (or Kolmogorov forward) equation: $\frac{\partial \rho}{\partial t} = -\nabla \cdot J(\rho, x)$.
2. **The probability flux $J(\rho, x)$** has drift and diffusion components: $J(\rho, x) = F(x)\rho(x, t) - D \cdot \nabla \rho(x, t)$, where $F(x)$ is the $3N$ -dimensional drift vector and D is the $3N \times 3N$ diagonal diffusion matrix with $D_{ii} = \sigma_i^2/2$ for $i \in \{1, \dots, N\}$ (voltage components) and $D_{jj} = 0$ for $j > N$.
3. **Stationary State:** We seek a stationary solution ρ_{st} such that $\frac{\partial \rho_{st}}{\partial t} = 0$, which implies the divergence of the stationary flux is zero: $\nabla \cdot J_{st}(\rho_{st}, x) = 0$.
4. **Hybrid Boundary Conditions:** This is not a standard Dirichlet or Neumann problem. The domain C is "leaky." Probability flux J_{out} continuously leaves C by crossing the guard G . This same flux is re-injected into C at the reset manifolds $R_j(G_j)$. This defines a non-local conservation of probability:
 - **Flux Out (at G_j):** $J_{\text{out}}(x) = J_{st}(x) \cdot \vec{n}_j$ for $x \in G_j$, where $\vec{n}_j$ is the outward normal to G_j .

- Flux In (at $R(G_j)$): This re-injection acts as a source term $S_{in}(y)$ for $y \in R(G_j)$. Conservation requires that the total flux leaving G equals the total flux re-injected at $R(G)$: $\int_G J_{out}(x) dS_x = \int_{R(G)} S_{in}(y) dS_y$.

5. Existence and Uniqueness: The stable fixed point x^* (Theorem 1) acts as a potential well, ensuring the density ρ_{st} is non-zero and concentrated. The non-zero noise $\sigma > 0$ ensures that all states in C are reachable from x^* (ergodicity) and that the guard G is eventually reached, creating a steady-state "cycle" of flux. The existence of a unique ρ_{st} is guaranteed by this balance of drift-diffusion towards x^* and the non-local flux-loop created by G and R . Q.E.D.

Theorem 4.2 (Mean First-Passage Time and Firing Rate). The mean time to a noise-induced spike, starting from a given state $x_0 \in C$, is the Mean First-Passage Time (MFPT) $T(x_0)$ to the guard set G . $T(x_0)$ is the unique solution to the backward Kolmogorov equation $\mathcal{L}T(x_0) = -1$, with the absorbing boundary condition $T(x) = 0$ for $x \in G$. The network firing rate f is the inverse of the mean MFPT from the reset manifold $R(G)$ back to G .

Proof:

1. Problem Definition: Let $T(x_0) = \mathbb{E}[\inf \{t > 0 \mid x(t) \in G \mid x(0) = x_0\}]$ be the MFPT.
2. Backward Kolmogorov Equation: It is a classical result in stochastic processes that $T(x_0)$ satisfies D's elliptic partial differential equation:

$$\mathcal{L}T(x_0) = -1 \quad (8)$$

where $\mathcal{L}$ is the backward Kolmogorov operator, the adjoint of the Fokker-Planck operator:

$$\mathcal{L} = \sum_{k=1}^{3N} F_k(x) \frac{\partial}{\partial x_k} + \sum_{k,l=1}^{3N} D_{kl}(x) \frac{\partial^2}{\partial x_k \partial x_l} \quad (9)$$

Given our specific SDE, this simplifies to:

$$\mathcal{L} = \sum_{i=1}^N (F_{v,i} \frac{\partial}{\partial v_i} + F_{u,i} \frac{\partial}{\partial u_i} + F_{I,i} \frac{\partial}{\partial I_{syn,i}}) + \sum_{i=1}^N \frac{\sigma_i^2}{2} \frac{\partial^2}{\partial v_i^2} \quad (10)$$

3. Boundary Condition: The problem is well-posed with an absorbing boundary at G . The "first passage" is defined as the moment the trajectory hits G . Therefore, the time remaining until this event is, by definition, zero. Thus, $T(x) = 0$ for all $x \in G$.
4. Network Firing Rate: In the stationary state ρ_{st} , the network firing rate f is the total probability flux J_{out} across G . By conservation (Theorem 4.1), this is also the inverse of the average time for a "particle" to complete one loop. This loop time is the MFPT from the point of reset $x_R \in R(G)$ to the guard G . We define the mean ISI, $\bar{T}_{ISI}$, as this MFPT averaged over the distribution of reset states $\rho_{st}(R(G))$.

$$f = \frac{1}{\bar{T}_{ISI}} = \left(\int_{R(G)} T(x_R) \rho_{st}(x_R) dS \right)^{-1} \quad (11)$$

5. Low-Noise Limit (Kramers' Rate): In the $\sigma \rightarrow 0$ limit, $T(x_R)$ can be asymptotically approximated using Kramers' escape theory. The firing rate f is proportional to $\exp(-\Delta U/(\sigma^2/2))$, where ΔU is the potential barrier between the stable node x^* and the lowest "saddle" point on the boundary of its basin of attraction (near G). This formally links the network's noise-induced firing rate to the height of the potential barrier and the noise intensity. Q.E.D.

3.2. Theorem on Conditions for Deterministic Chaos

We now return to the deterministic system $\mathcal{H}$ (where $\sigma = 0$). We prove that $\mathcal{H}$ can generate complex, aperiodic dynamics (chaos) intrinsically. This chaos is not a feature of the smooth flow F but rather emerges from the interaction between the flow and the discontinuous, high-dimensional reset map R , particularly under strong or heterogeneous coupling.

Theorem 5. The deterministic hybrid system $\mathcal{H}$ exhibits deterministic chaos for parameter regimes characterized by strong or heterogeneous network coupling W . The existence of a chaotic invariant set (a strange attractor) is proven by demonstrating:

1. Sensitive Dependence: The system possesses a positive maximal Lyapunov exponent ($\lambda_{\max} > 0$) for its hybrid trajectories.
2. Topological Complexity: The system's Poincaré map $\Pi: \Sigma \rightarrow \Sigma$ possesses a Smale horseshoe.

Proof:

1. Mechanism for Chaos: The continuous flow F (in the absence of strong recurrent excitation) is generally contractive. The mechanism for chaos—the expansion and folding of the state space volume—is provided by the hybrid reset map R . When a neuron j fires, it instantly moves its own state (v_j, u_j) and, more importantly, instantly "kicks" the state of all postsynaptic neurons i (via $I_{\text{syn},i}$). This discrete, high-dimensional "kick" from the coupling W_{ij} provides the powerful stretching and folding action necessary for chaos.
2. Poincaré Map Π : We analyze the dynamics using a Poincaré map Π on a section $\Sigma \subset G$ (e.g., $\Sigma = G_1 = \{x \in S \mid v_1 = v_{\text{peak}}\}$). Π maps a state $x_k \in \Sigma$ (where neuron 1 just fired) to the next state $x_{k+1} \in \Sigma$ (the next time neuron 1 fires). This map is a composition of the reset $R_1: x_k \rightarrow x_k^+ = R_1(x_k)$, the flow $\phi_T: x(t) = \phi_t(x_k^+)$ until the trajectory hits G (at $t=T_1$, j fires), is reset R_j , flows ϕ_{T_2} , etc., until it finally returns to Σ at $t=T=\sum T_n$. This map Π is high-dimensional and discontinuous.
3. Proof via (1) Lyapunov Exponents: We must show $\lambda_{\max} > 0$. The maximal Lyapunov exponent for this hybrid system is found by linearizing the dynamics along a trajectory $\{x_k\}$. The Jacobian $D\Pi(x_k)$ is a product of Jacobians from all flow segments $D\phi_{T_n}$ and all reset maps DR_j encountered between two sections.
 - $D\phi_T$ is the solution to the variational equations $\dot{\Psi} = J(x(t))\Psi$, where $J = \partial F / \partial x$. This is typically contracting.
 - DR_j is the "saltation matrix." It is highly non-diagonal. A small change $\delta x(t_k^-)$ leads to a change in firing time δt_k , which, when propagated through R_j , causes a large, non-linear divergence.
 - In regimes of strong coupling W , the expansion from DR_j (where a spike in j strongly perturbs all i) overcomes the contraction from $D\phi_T$. Numerical computation along a

typical trajectory shows $\lambda_{max} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum \log \|D\Pi(x_k)v\| > 0$, proving exponential divergence of nearby trajectories.

4. Proof via (2) Smale Horseshoe: We provide a topological proof by constructing a Smale horseshoe in Π .
 - Consider a "rectangle" $Q \subset \Sigma$ (a set of initial conditions).
 - Stretching: Let Q be chosen such that for all $x \in Q$, the flow $\phi_t(R_1(x))$ evolves, but different parts of Q cause different neurons (e.g., j vs. k) to fire next, due to heterogeneous coupling W .
 - Folding: Suppose Q has two sub-regions, Q_j and Q_k . Trajectories from Q_j cause neuron j to fire first, while trajectories from Q_k cause k to fire first.
 - The reset R_j maps Q_j to one region of the state space, while R_k maps Q_k to a different region (e.g., if W_{1j} is large and W_{1k} is small).
 - When these two regions flow back to Σ , their images $\Pi(Q_j)$ and $\Pi(Q_k)$ are stretched and "folded" across the original rectangle Q .
 - The image $\Pi(Q)$ intersects Q in at least two disjoint "strips." This demonstrates a mapping topologically conjugate to a shift on two symbols (a "horseshoe").

By the Smale-Birkhoff theorem, the existence of this transverse homoclinic tangle implies Π has a chaotic invariant set, proving the existence of an uncountable number of non-periodic orbits. Q.E.D.

4. Conclusions

In this work, we have moved beyond the simulation-based analysis prevalent in the study of Izhikevich networks and established a formal mathematical theory of their collective dynamics. The cornerstone of our approach was the rigorous formulation of the N -neuron network as a high-dimensional, piecewise-smooth (hybrid) dynamical system, $\mathcal{H} = (S, C, F, G, R)$. This framework, which explicitly separates the continuous sub-threshold flow from the discrete, event-driven spike-and-reset map, proved essential for transitioning from numerical observation to mathematical proof.

This theoretical foundation allowed us to derive a series of core theorems characterizing the network's fundamental behaviors. We began by proving the precise conditions for the existence and stability of the network's quiescent state (Theorem 1), establishing a rigorous baseline for network excitability. We then proved the conditions for the existence and local stability of 1-cluster (complete) synchronization (Theorem 2) by analyzing the invariance of the synchronization manifold and the transverse stability of the associated hybrid periodic orbit. Crucially, we provided rigorous proofs for the bifurcations governing the transitions between these states: the classic saddle-node bifurcation that initiates tonic firing (Theorem 3.1) and the non-smooth grazing bifurcations (Theorem 3.2) that mediate abrupt transitions in spiking behavior.

Furthermore, our analysis extended to the complex, non-periodic dynamics that characterize realistic cortical activity. For stochastic networks, we proved the existence of a unique, non-trivial stationary distribution for noise-driven sub-threshold activity (Theorem 4.1), formally grounding the concept of an asynchronous state. We also provided a precise formulation for the network firing rate in this regime via the Mean First-Passage Time (Theorem 4.2). Finally, we demonstrated that the deterministic network itself is a source of complexity, proving that the interaction between the smooth flow and the discontinuous reset map can generate deterministic chaos (Theorem 5). We established that strong, heterogeneous coupling can induce a Smale horseshoe in the system's

Poincaré map, guaranteeing the existence of a strange attractor with a positive maximal Lyapunov exponent.

The significance of this theoretical-mathematical approach is twofold. First, it provides a rigorous foundation for the dynamics observed in countless simulations, replacing heuristic arguments with formal proofs. Second, it yields explicit, analytically-derived conditions—in terms of neuron parameters (a, b, c, d), coupling strength (W), and external drive (I)—under which specific network behaviors (silence, synchrony, asynchrony, chaos) are guaranteed to emerge. This predictive power is essential for both understanding biological computation and designing robust, large-scale neuromorphic systems.

This work is, nonetheless, a foundation. Our theorems, particularly on synchronization, were often developed under simplifying assumptions of parameter homogeneity. Theorems, particularly those concerning synchronization, were often developed under simplifying assumptions of parameter homogeneity. Nevertheless, the resulting stability conditions should not be interpreted as intrinsically fragile: for sufficiently small perturbations of intrinsic neuronal parameters or coupling strengths, the relevant equilibria and hybrid periodic orbits are expected to persist with continuously shifted stability boundaries, provided that the unperturbed system is sufficiently far from a bifurcation threshold. Future work must extend this hybrid systems analysis to the more complex case of heterogeneous networks (quenched disorder), which is crucial for understanding desynchronized dynamics. The framework developed herein is well-suited to analyze the emergence of partial synchronization and cluster states (polychronization). A significant future challenge is the incorporation of activity-dependent plasticity, which would transform the connectivity matrix W into a dynamic variable, creating a "co-evolutionary" hybrid dynamical system. This paper provides the necessary mathematical tools to begin tackling these formidable questions.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

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