

Skolem Forms and Equilibrium Semantics for Existential Datalog Rules

David Pearce^{1,†}, Agustín Valverde Ramos^{2,*,†}

¹Dept of Artificial Intelligence, Universidad Politécnica de Madrid, Campus de Montegancedo, 28660, Boadilla del Monte (Madrid), Spain

²Dept Matemática Aplicada, Universidad de Málaga, Bvd. Louis Pasteur, 29071, Málaga, Spain

Abstract

In the past few years different versions of stable model semantics have been proposed to interpret Datalog rules with existential quantifiers and nonmonotonic negation, so-called *normal tuple-generating dependencies* (NTGDs). The approach of [1] is based on a reduction to logic programs under Herbrand semantics, using Skolem functions. An alternative, non-reductive view is proposed in [2]. In this paper we examine these two approaches from the standpoint of equilibrium logic. We extend our previous results on Skolem forms in the logic of here-and-there to show how skolemisation preserves the equilibrium property of models. Then we consider how the first order equilibrium semantics relates to the two forms of stable model semantics from [1] and [2]. Lastly we consider how here-and-there and equilibrium logic might be applied in the investigation of logical relations between Datalog programs, for instance to study the property that two sets of NTGDs are strongly equivalent with respect to conjunctive queries.

Keywords

equilibrium logic, datalog, skolem form

1. Introduction

Equilibrium logic is a well known formalism for nonmonotonic reasoning. In the propositional case, it captures the stable model semantics underlying (ground) answer set programming (ASP). Already 20 years ago in [3], we studied a quantified version of equilibrium logic (QEL) as basis for defining and reasoning about stable models for first order theories and logic programs with variables. Since its syntax includes full first order logic, equilibrium logic can be applied to formulas and theories in a richer language than that associated with answer set programs. For example, besides universal formulas we can also deal with universal-existential or $\forall\exists$ -formulas, also known as Π_2^0 -formulas. These are formulas in prenex form all of whose universal quantifiers precede their existential quantifiers (if any).

$\forall\exists$ -formulas of a special kind arise in Datalog where they are known as tuple-generating dependencies, TGDs. Recently, in research on Datalog semantics, there have been some proposals to provide a stable model semantics for tuple-generating dependencies in which negation is allowed in rule bodies; these are so-called *normal* TGDs or NTGDs, [2, 1]. Given its generality and agreement with stable model semantics, it is clear that we can readily provide an equilibrium model semantics for NTGDs and for Datalog programs built from them.

In this note we describe this semantics and show how it relates to the two versions of stable model semantics for NTGDs proposed in [2] and [1]. Although conceptually related, these two approaches give quite different solutions and have very different computational properties. In both cases there is a version of equilibrium logic that coincides with them. Roughly speaking the differences arise depending on whether or not we make the unique names assumption (UNA) and employ Skolem forms to eliminate existential quantifiers. While the authors of [1] discuss the relation between equilibrium logic and their

6th International Workshop on the Resurgence of Datalog in Academia and Industry 2026

*Corresponding author.

✉ david.pearce@upm.es (D. Pearce); a_valverde@uma.es (A. Valverde Ramos)

🌐 <https://www.dpearce.net> (D. Pearce); <https://portaldeinvestigacion.uma.es/investigadores/1240651/detalle> (A. Valverde Ramos)

🆔 0000-0001-7407-326 (D. Pearce); 0000-0003-1441-7424 (A. Valverde Ramos)



© 2026 Copyright for this paper by its authors. Use permitted under Creative Commons License Attribution 4.0 International (CC BY 4.0).

own semantics, the connections are made indirectly via a comparison with the stable model semantics of [4]. A direct proof of equivalence with respect to equilibrium models may be simpler and may shed further light on the semantics of existential rules and NTGDs.

With this in mind our aim is the following. We present the essentials of the quantified logic of here-and-there, QHT, which forms the basis for quantified equilibrium logic, QEL. We then explore the behaviour of Skolem functions in QHT. In [3] we already observed that passing from a formula to its Skolem form preserves the property of satisfiability. In addition, we now show that also the equilibrium property is preserved under skolemisation. This means that if we apply skolemisation in the setting of Herbrand models, as used by [1], we obtain equilibrium models equivalent to the stable models of [1]. In the more general case of non-Herbrand models and where UNA is applied only to the dataset, we obtain a semantics equivalent to that of [2].

2. Equilibrium logic and the logic of here-and-there

For a description of equilibrium logic, the quantified logic of here-and-there, QHT, and their relation to answer set programming and stable model semantics, see [5]. Here is a brief summary. Let $L = \mathcal{L}(C, F, P)$ be a first order language with a set of constants C , functions F and predicates P ; we denote by $\mathcal{T}(C, F)$ the set of ground terms defined from C and F . The QHT semantics is a form of possible worlds semantics for first order intuitionistic logic [6] where all domains are equal and there are just two nodes or ‘worlds’, h and t , with $h \leq t$. We can represent a QHT interpretation for L as a triple $\langle (D, \sigma), H, T \rangle$, where D is the domain of objects, σ is an interpretation of the language L in the domain D . We denote by $L(D)$ the extension of L by constants for each element of D and we will identify σ with its extension to $L(D)$. Formally we have

- $\sigma : \mathcal{T}(C \cup D, F) \rightarrow D$ verifies $\sigma(d) = d$ if $d \in D$, and
- H, T are sets of atomic formulas in $L(D)$, such that $H \subseteq T$.

The *satisfaction* relation for a QHT-interpretation $\mathcal{M} = \langle (D, \sigma), H, T \rangle$, at a node $w \in \{h, t\}$, with $h \leq t$, is defined as follows:

- $\mathcal{M}, h \models p(t_1, \dots)$ if $p(\sigma(t_1), \dots) \in H$,
- $\mathcal{M}, t \models p(t_1, \dots)$ if $p(\sigma(t_1), \dots) \in T$,
- $\mathcal{M}, w \models \varphi \wedge \psi$ if $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models \varphi \vee \psi$ if $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models \varphi \rightarrow \psi$ if, for each node w' such that $w \leq w'$, $\mathcal{M}, w' \not\models \varphi$ or $\mathcal{M}, w' \models \psi$,
- $\mathcal{M}, w \models \neg \varphi$ if $\mathcal{M}, t \not\models \varphi$
- $\mathcal{M}, w \models \forall x \varphi(x)$ if $\mathcal{M}, w \models \varphi(d)$ for every $d \in D$.
- $\mathcal{M}, w \models \exists x \varphi(x)$ if $\mathcal{M}, w \models \varphi(d)$ for some $d \in D$.

One can easily check, for example with the help of [6], that these conditions form a special case of the possible worlds semantics for intuitionistic logic. The main departures are that we have only the two worlds, h and t , and that the same domain of elements is associated with each world. This allows us to simplify the satisfaction conditions for negation and universal quantification.

As usual a *literal* is an atomic formula or its negation. A sentence φ is *true* in an QHT-interpretation $\mathcal{M}$ (and $\mathcal{M}$ is a *model* of φ), in symbols $\mathcal{M} \models \varphi$, if $\mathcal{M}, h \models \varphi$. For any set Γ of sentences and any sentence φ , we write $\Gamma \models \varphi$ if every QHT-interpretation satisfying all formulas in Γ satisfies φ also.

Let $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ be a model of a theory Π . We say that: $\mathcal{M}$ is a PNA-model (parameter names assumption model) if the restriction of σ to $\mathcal{T}(C, F)$ is surjective; $\mathcal{M}$ is a UNA-model (unique names assumption model) if the restriction of σ to $\mathcal{T}(C, F)$ is injective; and we say that $\mathcal{M}$ is a SNA-model (standard names assumption model) if the restriction of σ to $\mathcal{T}(C, F)$ is a bijection.

To introduce the idea of equilibrium model we use the following notation. A QHT interpretation of the form $\mathcal{M} = \langle (D, \sigma), T, T \rangle$ is called a *total* interpretation. Of two interpretations $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ and $\mathcal{M}' = \langle (D, \sigma), H', T \rangle$, we write $\mathcal{M} \leq \mathcal{M}'$ if $H \subseteq H'$, and $\mathcal{M} < \mathcal{M}'$ if $H \subset H'$.

Definition 1 (Equilibrium). An equilibrium model of a theory Π is a total model of Π that is minimal under the ordering $\leq$, in other words if $\mathcal{M} = \langle (D, \sigma), T, T \rangle \models \Pi$ and $\langle (D, \sigma), H, T \rangle \not\models \Pi$ for every $H \subset T$.

It is also possible to consider Herbrand domains only. That is, the domains are sets of terms $\mathcal{T}(C, F)$ constructed from the constants and functions of the language, possibly expanding the set of constants. In this case, the interpretation σ will be the identity and therefore the resulting semantics will verify UNA.

For the *propositional logic of here-and-there*, HT, we take a propositional language L and regard an HT interpretation for L as a pair $\mathcal{M} = \langle H, T \rangle$ of sets of L -atoms, with $H \subseteq T$. Again we can think of this structure as comprising the sets of atoms true at two nodes h and t . And, if, as before, $w \in \{h, t\}$, the above satisfaction clauses for $\wedge, \vee, \rightarrow, \neg$ define the truth relation for any L -formula φ in $\mathcal{M}$. *Equilibrium models* for HT are defined in the same way as before, ie they are total models $\langle T, T \rangle$ of a theory Π such that there is no HT model $\langle H, T \rangle$ of Π such that $H \subset T$. The logics HT and QHT are determined by their respective *valid* formulas, ie those true in every HT or QHT interpretation. The inference relations of HT and QHT are denoted by $\vdash_{HT}$ and $\vdash_{QHT}$ respectively.

2.1. Some properties of QHT

We begin by recalling some standard features of QHT.

Theorem 1 ([3]). In QHT every formula is equivalent to a formula in prenex normal form, ie of the form $Q_1 x_1 \dots Q_n x_n A(x_1, \dots, x_n)$, where $Q_i \in \{\forall, \exists\}$ and $A(x_1, \dots, x_n)$ is a quantifier-free formula.

The prenex forms correspond to those familiar in classical first order logic. In particular the following equivalences hold in QHT for any formula C :

$$\begin{aligned} \exists x A(x) \rightarrow C &\equiv \forall x (A(x) \rightarrow C) & \forall x A(x) \rightarrow C &\equiv \exists x (A(x) \rightarrow C) \\ C \rightarrow \forall x A(x) &\equiv \forall x (C \rightarrow A(x)) & C \rightarrow \exists x A(x) &\equiv \exists x (C \rightarrow A(x)) \\ \neg \exists x A(x) &\equiv \forall x \neg A(x) & \neg \forall x A(x) &\equiv \exists x \neg A(x) \end{aligned}$$

Next we turn to satisfiability preservation under Skolemization and the associated Herbrand theorem. These results proceed by adding new constants and functions to the language, and for this reason we specify the signature of the logic: $QHT(C, F, P)$ denotes the logic over the language $\mathcal{L}(C, F, P)$.

2.2. Skolem forms and equilibrium

In the sequel, we will denote tuples with bold letters: $\mathbf{x} = x_1, x_2, \dots, x_n$

$$\begin{aligned} \forall \mathbf{x} A(\mathbf{x}) &= \forall x_1, \dots, \forall x_n A(x_1, \dots, x_n) \\ \exists \mathbf{x} A(\mathbf{x}) &= \exists x_1, \dots, \exists x_n A(x_1, \dots, x_n) \end{aligned}$$

If $\sigma : \mathcal{T}(C \cup D, F) \rightarrow D$, we will use the same letter to denote its extension to D^n :

$$\sigma(t_1, \dots, t_n) = (\sigma(t_1), \dots, \sigma(t_n))$$

The following results show that skolemisation preserves the equilibrium property. We state the results for quite general formulas of the form $\forall \mathbf{x} \exists y A(\mathbf{x}, y)$. We can then apply this to NTGDs.

Definition 2. 1. Let $\varphi = \exists x \alpha(x)$ be in $\mathcal{L}(C, F, P)$, we denote by φ_a the formula

$$\varphi_a = \alpha(a)$$

where $a \notin C$.

2. Let $\varphi = \forall \mathbf{x} \exists y \beta(\mathbf{x}, y)$ be a formula in $\mathcal{L}(C, F, P)$, we denote by φ_f the formula

$$\varphi_f = \forall \mathbf{x} \beta(\mathbf{x}, f(\mathbf{x}))$$

where $f \notin F$.

3. Let $\varphi = \mathcal{Q}_n x_n \dots \mathcal{Q}_1 x_1 \psi(x_1, \dots, x_n)$ be a prenex formula in $\mathcal{L}(C, F, P)$, we denote by φ^{sk} the formula obtained by applying the previous items to every existential quantifier of the prefix

$$\varphi^{sk} = \forall x_{i_m} \dots \forall x_{i_1} \psi(x_{i_1}, \dots, x_{i_m})$$

This formula belongs to the language $\mathcal{L}(C^{sk}, F^{sk}, P)$ that extends the original language with the fresh constants and functions introduced by the process described in the previous item. If Π is a set of formulas, we set $\Pi^{sk} = \{\varphi^{sk} : \varphi \in \Pi\}$.

In the following we show that all the models of a theory Π can be recovered from the theory's Skolem form. And this is true also for equilibrium models. We use the following notation: Let Π be a theory in $\mathcal{L}(C, F, P)$ and let $\mathcal{M} = \langle (D, \sigma), H, T \rangle$ be a QHT-model of Π ; that is, $\sigma : \mathcal{T}(C \cup D, F) \rightarrow D$. We denote by $\mathcal{M}^{sk} = \langle (D, \sigma^{sk}), H, T \rangle$ the extension of $\mathcal{M}$ with the assignment $\sigma^{sk} : \mathcal{T}(C^{sk} \cup D, F^{sk}) \rightarrow D$ defined as follows

1. If a is a Skolem constant introduced by $\exists x \alpha(x)$, then

$$\sigma^{sk}(a) = d, \quad \text{where} \quad \mathcal{M} \models \alpha(d)$$

2. If f is a Skolem function introduced by $\forall \mathbf{x} \exists y \beta(\mathbf{x}, y)$, then

$$\sigma^{sk}(f(\mathbf{d})) = d', \quad \text{where} \quad \mathcal{M} \models \beta(\mathbf{d}, d')$$

Theorem 2 (Herbrand's theorem for satisfiability [3]). *Let A be a formula in QHT, then there is an algorithm for converting A into a prenex formula A^* , with only universal quantifiers, such that A is satisfiable iff and only if A^* is satisfiable by an Herbrand model (of A^*).*

This theorem is consequence of the first part of the following lemma. The second item will allow us to show that skolemisation preserves the equilibrium property.

Lemma 1. *Let Π be a theory in $\mathcal{L}(C, F, P)$.*

1. *If $\mathcal{M}$ is a QHT model of Π^{sk} , then it is also a model of Π .*
2. *If $\mathcal{M}$ is a QHT model of Π , then $\mathcal{M}^{sk}$ is a model of Π^{sk}*

Proof.

1. Let us consider $\mathcal{M} = \langle (D, \sigma), H, T \rangle \models \Pi^{sk}$.

- If $\exists x \alpha(x) \in \Pi$, then $\alpha(a) \in \Pi^{sk}$ for some $a \in C^{sk}$:

$$\begin{aligned} \mathcal{M}, h \models \alpha(a) &\implies \mathcal{M}, h \models \alpha(d), \quad d \in D, d = \sigma(a) \\ &\implies \mathcal{M}, h \models \exists x \alpha(x) \end{aligned}$$

- If $\forall \mathbf{x} \exists y \beta(\mathbf{x}, y) \in \Pi$, then $\forall \mathbf{x} \beta(\mathbf{x}, f(\mathbf{x})) \in \Pi^{sk}$ for some $f \in F^{sk}$:

$$\begin{aligned} \mathcal{M}, h \models \forall \mathbf{x} \beta(\mathbf{x}, f(\mathbf{x})) &\implies \\ &\implies \mathcal{M}, h \models \beta(\mathbf{d}, d'), \quad \text{for all } \mathbf{d} \in D^n, \quad d' = \sigma(f(\mathbf{d})) \in D \\ &\implies \mathcal{M}, h \models \exists y \beta(\mathbf{d}, y), \quad \text{for all } \mathbf{d} \in D^n \\ &\implies \mathcal{M}, h \models \forall \mathbf{x} \exists y \beta(\mathbf{x}, y) \end{aligned}$$

2. This item is obvious by the definition of σ^{sk} :

- If a is a Skolem constant introduced by $\exists x \alpha(x)$, then

$$\mathcal{M} \models \alpha(d), \quad \sigma^{sk}(a) = d \implies \mathcal{M}^{sk} \models \alpha(a)$$

- If f is a Skolem function introduced by $\forall \mathbf{x} \exists y \beta(\mathbf{x}, y)$, then

$$\begin{aligned} \mathcal{M} \models \beta(\mathbf{d}, d'), \quad \sigma^{sk}(f(\mathbf{d})) = d' &\implies \mathcal{M}^{sk} \models \beta(\mathbf{d}, f(\mathbf{d})), \quad \mathbf{d} \in D^n \\ &\implies \mathcal{M}^{sk} \models \forall \mathbf{x} \beta(\mathbf{x}, f(\mathbf{x})) \end{aligned}$$

If we work with Herbrand models and skolemisation then we consider just the domain $D^{sk} = \mathcal{T}(C^{sk}, F^{sk})$ and models $\mathcal{M} = \langle (D^{sk}, id), H, T \rangle$. Then, Lemma 1 says

$$\langle (D^{sk}, id), H, T \rangle \models \Pi \quad \text{iff} \quad \langle (D^{sk}, id), H, T \rangle \models \Pi^{sk}$$

Theorem 3. *Let Π be a theory and $\mathcal{M} = \langle (D, \sigma), T, T \rangle$ a total model. $\mathcal{M}$ is an equilibrium model of Π if and only if $\mathcal{M}^{sk}$ is an equilibrium model of Π^{sk} .*

Proof.

- Assume that $\mathcal{M}$ is an equilibrium model of Π . Then $\mathcal{M}^{sk}$ is a model of Π^{sk} . Now consider another model $\mathcal{M}'$ of Π^{sk} such that $\mathcal{M}' \leq \mathcal{M}^{sk}$. Since the relation $\leq$ doesn't affect the initial interpretation of the language, we have that $\sigma' = \sigma^{sk}$. Then $(\mathcal{M}')^{sk} = \mathcal{M}'$ is a model of Π and by the equilibrium property $\mathcal{M}' = \mathcal{M}$ and thus $\mathcal{M}' = \mathcal{M}^{sk}$. That is $\mathcal{M}^{sk}$ is in equilibrium.
- Let us assume that $\mathcal{M}^{sk}$ is an equilibrium model of Π^{sk} . Then $\mathcal{M}$ is also a total model of Π . Let us consider another model $\mathcal{M}'$ of Π such that $\mathcal{M}' \leq \mathcal{M}$. For $\mathcal{M}$ and $\mathcal{M}'$ are both models of Π , we can assume that $(\sigma')^{sk} = \sigma^{sk}$ and thus $\mathcal{M}^{sk} \leq (\mathcal{M}')^{sk}$. By the equilibrium property $\mathcal{M}^{sk} = (\mathcal{M}')^{sk}$ and therefore $\mathcal{M} = \mathcal{M}'$. That is, $\mathcal{M}$ is in equilibrium.

Notice that the theorem shows that the equilibrium property is preserved in those models constructed by the extension defined in Lemma 2. In general it is not guaranteed that if $\langle (D, \tau), T, T \rangle$ is an equilibrium model of φ_f then $\langle (D, \tau_{|\mathcal{T}(C \cup D, F))}, T, T \rangle$ is an equilibrium model of φ .

Example

The following example is taken from [2]. Let σ be the conjunction of the following formulas:

1. $\forall x(\text{Person}(x) \rightarrow \exists y \text{IsFatherOf}(y, x))$
2. $\forall x \forall y(\text{IsFatherOf}(x, y) \rightarrow \text{SameAs}(x, x))$
3. $\forall x \forall y \forall z(\text{IsFatherOf}(x, y) \wedge \text{IsFatherOf}(z, y) \wedge \neg \text{SameAs}(x, z) \rightarrow \text{Abnormal}(y))$
4. $\text{Person}(\text{Alice})$

The prenex form of 1 is

$$\forall x \exists y(\text{Person}(x) \rightarrow \text{IsFatherOf}(y, x))$$

If we consider QHT interpretations of σ of the form $\langle (D, \sigma), H, T \rangle$, we can construct many equilibrium models, for example where

$$T = \{\text{Person}(\text{Alice}), \text{IsFatherOf}(d, \text{Alice}), \text{SameAs}(d, d)\}, \text{ for any } d \in D$$

However, all of these are ‘essentially’ the same model, all that changes is the ‘name’ that the father of Alice receives. Moreover, it is easy to check that, for example, the query $\exists x(\text{Person}(x) \wedge \neg \text{Abnormal}(x))$ is entailed by any of these models of the theory.

If we then consider the Skolem form of σ :

1. $\forall x(\text{Person}(x) \rightarrow \text{IsFatherOf}(f(x), x))$
2. $\forall x \forall y(\text{IsFatherOf}(x, y) \rightarrow \text{SameAs}(x, x))$
3. $\forall x \forall y \forall z(\text{IsFatherOf}(x, y) \wedge \text{IsFatherOf}(z, y) \wedge \neg \text{SameAs}(x, z) \rightarrow \text{Abnormal}(y))$
4. $\text{Person}(\text{Alice})$

In this case the QHT models have to fix a value for $f(Alice)$, that is to say, $\sigma(f(Alice)) = d$ for a unique element $d \in D$. Consequently the result is essentially the same as that obtained prior to skolemising. The following models are in equilibrium

$$T = \{\text{Person}(Alice), \text{IsFatherOf}(d, Alice), \text{SameAs}(d, d)\}$$

where $d = \sigma(f(Alice)) \in D$. Suppose, following, [2], that we add the following new fact to the database: $\text{Person}(Bob)$. If the witness for $(f(Alice))$ is identified with Bob, then there can be an equilibrium model where the query $\text{IsFatherOf}(Bob, Alice)$ is true. The situation changes if we consider Herbrand models with UNA, that is, $\sigma = id$, then we have just one equilibrium model:

$$T = \{\text{Person}(Alice), \text{IsFatherOf}(f(Alice), Alice), \text{SameAs}(f(Alice), f(Alice))\}$$

Now, query answering may be different; [2] uses this example as an illustration. In the case of the Herbrand semantics UNA applies to all the Herbrand terms and in particular we cannot have $\sigma(f(Alice)) = Bob$. It follows that the query $\neg \text{IsFatherOf}(Bob, Alice)$ must now be false.

2.3. Equilibrium and stable models

It is well-known that equilibrium logic captures the stable model or answer set semantics for logic programs. We recall some of the main properties. We assume the reader is familiar with the basic syntax of formulas or *rules* of programs in ASP and the standard stable model or answer set semantics. A first-order QHT theory Π in some signature $\mathcal{L}(C, F, P)$ is said to be *universal* if Π is QHT equivalent to a set of sentences in prenex form all of whose quantifiers are universal. Let Π be a universal theory (assumed to be presented in prenex form) in the signature $\mathcal{L}(C, F, P)$. Let $D \supseteq C$, we define the *grounding* of Π with respect to D as $g(\Pi, D) = \bigcup_{B \in \Pi} g(B, D)$ where

$$g(\forall x_1 \dots \forall x_n A(x_1, \dots, x_n), D) = \{A(t_1, \dots, t_n) : t_1, \dots, t_n \in \mathcal{T}(D, F)\}$$

Clearly, any grounding of a first-order theory Π can be represented as a (possibly infinite) theory in propositional HT logic and so the quantified version of equilibrium logic for universal theories can be related to the propositional equilibrium logic of their ground versions.

Lemma 2. *Let Π be a universal theory in $\mathcal{L}(C, F, P)$ and $D \supseteq C$. Then $\langle (D, id), H, T \rangle \models \Pi$ iff $\langle H, T \rangle$ is a propositional QHT-model of $g(\Pi, D)$.*

From this and the definition of equilibrium model we obtain:

Lemma 3. *Let Π be a universal theory in $\mathcal{L}(C, F, P)$ and let $\langle (D, id), T, T \rangle$ be a total model of Π , where $D \supseteq C$. Then $\langle (D, id), T, T \rangle$ is an equilibrium model of Π iff $\langle T, T \rangle$ is a propositional equilibrium model of $g(\Pi, D)$.*

The rules of any logic program are interpreted as holding for all values of the free variables in the Herbrand universe. Hence any program (disjunctive, nested, etc) for which answer set semantics is defined can be regarded as the universal closure of the translation of the program *rules* into QHT. Equilibrium models of a logic program can then be identified with the equilibrium models of the universal closure of Π , which is clearly a universal theory. In the case of logic programs it is very common to restrict attention to the Herbrand models of a program Π . Recall that by Theorem 2 every consistent universal theory has an Herbrand model. We obtain:

Theorem 4 ([5]). *Let Π be a theory in the signature $\mathcal{L}(C, F, P)$. A total Herbrand model $\langle (C, id), T, T \rangle$ of the universal closure of Π is an equilibrium model of Π iff T is an answer set (stable model) of Π .*

3. Datalog rules and stable models

3.1. Stable models according to [1]

Following the terminology of [1], given a language $\mathcal{L}$ comprising a finite set of predicate symbols (also called a *schema*), a *normal tuple-generating dependency* (NTGD) φ over $\mathcal{L}$ has the form

$$\forall \mathbf{x} \forall \mathbf{y} (\Phi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z} \Psi(\mathbf{x}, \mathbf{z})) \quad (1)$$

where $\Phi(\mathbf{x}, \mathbf{y})$ is a conjunction of $\mathcal{L}$ -literals and $\Psi(\mathbf{x}, \mathbf{y})$ is a conjunction of $\mathcal{L}$ -atoms with arguments from $\mathcal{V}$, the set of variables. It is assumed that NTGDs do not contain constants. It is also generally assumed in [1] that NTGDs are *guarded*, where an NTGD φ is said to be guarded if its body (Φ) has a positive literal, called a guard of φ , that contains all universally quantified variables of φ . It follows from Theorem 1 that in QHT we can equivalently write an NTGD of form (1) as

$$\forall \mathbf{x} \forall \mathbf{y} \exists \mathbf{z} (\Phi(\mathbf{x}, \mathbf{y}) \rightarrow \Psi(\mathbf{x}, \mathbf{z})) \quad (2)$$

The stable model semantics of [1] defines stable models for tuples (Δ, Σ) , where Δ is a database for a schema $\mathcal{L}$, that is a finite set of atoms formed from $\mathcal{L}$ -predicates and constants in the language; and Σ is a set of NTGDs over $\mathcal{L}$. Let us call a tuple of this kind a *knowledge base*. In the proposed semantics, existentially quantified variables in Ψ are replaced by Skolem functions. Given the syntax of an NTGD, we can apply (the more general properties expressed in) Lemma 1 and Theorem 3. The formula φ in Lemma 1 can in this case be considered as a formula in $\mathcal{L}(P)$ since there are no function and constant symbols in the initial NTGD.¹

If Σ is a set of formulas of form (2), we set $\Sigma^f = \{\varphi_f : \varphi \in \Sigma\}$. In [1] the *functional transformation* of an NTGD coincides with our transformation from φ to φ_f .² The stable model semantics of [1] is defined for a knowledge base (Δ, Σ) by taking the union $\Delta \cup \Sigma^f$ (considering Δ as a set of facts), treating this as a logic program in the Herbrand semantics and obtaining the stable models of the program in the usual way. Our first observation is therefore an immediate consequence of the definitions and Theorem 4.

Proposition 1. *Let $S = (\Delta, \Sigma)$ be a knowledge base with NTGDs. If we apply the Herbrand semantics together with skolemisation (ie we consider $\Delta \cup \Sigma^f$), then the equilibrium models of S coincide with the stable models of S according to [1].*

By taking Herbrand models of $\Delta \cup \Sigma^f$, we are just considering UNA models and therefore stable models of (Δ, Σ) according to the approach of [1]. Under the Herbrand semantics Lemma 1 tells us that there is a one-one correspondence between the equilibrium models of $\Delta \cup \Sigma$ and the equilibrium models of $\Delta \cup \Sigma^f$, in other words the stable models of (Δ, Σ) according to the approach of [1].

Example

Let us consider a second example discussed in [1]. Let Δ be the empty database, and let Σ contain the following NTGDs.

$$\begin{aligned} &\exists x \exists y H(x, y), \\ &\exists x H(x, x) \\ &\forall x \forall y (H(x, y) \rightarrow E(x, x)) \\ &\forall x \forall y (H(x, y) \rightarrow E(y, y)) \\ &\forall x \forall y (H(x, y) \wedge \neg E(x, y) \rightarrow \text{hello}). \end{aligned}$$

¹In Definition 2 part 2 we use just a single variable y for ease of readability. But the properties shown hold for tuples of variables and so will apply equally to NTGDs.

²Note that each existentially quantified variable in φ is assigned a new function symbol. Notice too that if the body Φ of the NTGD is empty then the existentially quantified variables are replaced by new constant symbols as in part 1 of Lemma 1.

The theory Σ^{sk} is

$$\{H(a, b), \quad H(c, c), \quad \forall x \forall y (H(x, y) \rightarrow E(x, x)), \\ \forall x \forall y (H(x, y) \rightarrow E(y, y)), \quad \forall x \forall y (H(x, y) \wedge \neg E(x, y) \rightarrow \text{hello})\}$$

And thus, if we consider Herbrand models, we have just one equilibrium model:

$$T = \{H(a, b), H(c, c), E(a, a), E(b, b), E(c, c), \text{hello}\}$$

This agrees with the logic programming approach as claimed in [1]. However, if general QHT models for Σ are considered, we can consider a domain with just one element, $D = \{d\}$, and the following equilibrium model:

$$T' = \{H(d, d), E(d, d)\}$$

3.2. Stable models according to [2]

The authors of [2] are critical of those semantical proposals, such as that of [1] just described, that are based on a logic programming approach using Herbrand models and skolemisation. They propose an alternative semantics that avoids skolemisation and does not rely on Herbrand models. It applies directly to a knowledge base (Δ, Σ) , where as before Σ is a collection of NTGDs of the form

$$\forall \mathbf{x} \forall \mathbf{y} (\Phi(\mathbf{x}, \mathbf{y}) \rightarrow \exists \mathbf{z} \Psi(\mathbf{x}, \mathbf{z}))$$

each of which is *safe* in that every variable in a negative literal occurs also in a positive literal in Φ . The semantics is given by taking the stable models of $\Delta \cup \Sigma$ viewed as a first order theory under the definition of [4]. These stable models are characterised by a condition expressed in second-order logic, but they are equivalent to equilibrium models, [4]. In view of this we obtain immediately:

Proposition 2. *Assume UNA for QHT interpretations. Then the stable models of a knowledge base (Δ, Σ) in the sense of [2] coincide with the equilibrium models of $\Delta \cup \Sigma$.³*

Although this property follows directly from the definitions and known results, it may be instructive to examine this second order characterisation of equilibrium models and to understand why it works. To this end we review briefly a known reduction of QHT to classical logic. This reduction allows us to encode the problem of validity in QHT by means of a second order condition in classical logic. It proceeds as follows. If p and q are predicate symbols of the same arity we let $p \leq q$ stand for

$$\forall \mathbf{x} (p(\mathbf{x}) \rightarrow q(\mathbf{x}))$$

and write $p < q$ if $p \leq q$ and not $q \leq p$. Similarly if $\mathbf{p}$ and $\mathbf{q}$ are tuples $p_1, \dots, p_n$ and $q_1, \dots, q_n$ of predicate symbols, then $\mathbf{p} \leq \mathbf{q}$ stands for

$$p_1 \leq q_1 \wedge \dots \wedge p_n \leq q_n$$

Let $\mathcal{L}$ be a language containing the predicate symbols $p_1, \dots, p_n$ and φ a formula in $\mathcal{L}$. Denote by $\mathcal{L}(\bar{\mathbf{p}})$ the expansion of $\mathcal{L}$ by adding a new predicate $\bar{p}_i$ of the same arity for each predicate symbol p_i of $\mathcal{L}$. Define φ^* inductively as follows.

- $p_i(t_1, \dots, t_m)^* = \bar{p}_i(t_1, \dots, t_m)$
- $(\neg \varphi)^* = \neg \varphi^* \wedge \neg \varphi$;
- $(\varphi \odot \psi)^* = \varphi^* \odot \psi^*$, where $\odot \in \{\wedge, \vee\}$;
- $(\varphi \rightarrow \psi)^* = (\varphi^* \rightarrow \psi^*) \wedge (\varphi \rightarrow \psi)$;
- $(Qx\varphi)^* = Qx\varphi^*$, where $Q \in \{\forall, \exists\}$.

³We add UNA because this is assumed in [2] and added as a condition to the second order definition.

Using this translation we can express validity in QHT as follows

Theorem 5 ([5]).

$$\models_{\text{QHT}} \varphi \Leftrightarrow \bar{\mathbf{p}} \leq \mathbf{p} \models \varphi^*$$

Here one can see that the condition on $\bar{\mathbf{p}}$ mimics the relation between the h -world and the t -world in a QHT model. A model of φ is in equilibrium if there is no ‘smaller’ model, that is, a model with the same ‘there’ part but with less atoms in the ‘here’ part. By the reduction technique the ‘here’ part can be reconstructed from the evaluation of the new atoms $\bar{\mathbf{p}}$ and thus, to obtain a smaller model, we need $\bar{\mathbf{p}} < \mathbf{p}$ and φ^* to be satisfiable. Therefore the equilibrium models of φ are those which cannot be extended to the atoms $\bar{\mathbf{p}}$ to obtain a model of $\bar{\mathbf{p}} < \mathbf{p} \wedge \varphi^*$. This property is expressed by a second order formula: the equilibrium models of φ are the classical models of

$$\varphi \wedge \neg \exists \bar{\mathbf{p}} (\bar{\mathbf{p}} < \mathbf{p} \wedge \varphi^*)$$

This formula is used by [4] to define stable models for arbitrary first-order formulas. Essentially the same condition is applied in [2] to yield a semantics for NTGDs. The only difference is that since [2] deals with NTGDs with their restricted syntax, there is no need to define the recursive translation φ^* for all formulas φ . This allows one to achieve the same result with a slightly simplified translation.

4. Equivalence concepts for knowledge bases

Again we use the terminology *knowledge base* (KB for short) for any pair (Δ, Σ) where Δ is a database and Σ is a set of NTGDs for a language or schema $\mathcal{L}$. In the study of KBs a key point of focus is on answering *conjunctive queries*. Both [2] and [1] study syntactic restrictions on KBs that allow the computation of conjunctive queries to be decidable. Given a database Δ and language (schema) $\mathcal{L}$ a *normal conjunctive query* q over $\mathcal{L}$ is a first order formula of the form

$$\exists \mathbf{y} \varphi(\mathbf{x}, \mathbf{y})$$

where φ is a conjunction of $\mathcal{L}$ -literals and each variable of $\mathbf{x}$ occurs in some positive literal in φ . *Solutions* to conjunctive queries over $\mathcal{L}$ given a KB (Δ, Σ) in $\mathcal{L}$ are tuples $\mathbf{t}$ such that $\exists \mathbf{y} \varphi(\mathbf{t}, \mathbf{y})$ is true in each stable model of (Δ, Σ) .⁴

A key feature of the logic QHT is that if two logic programs as sets of QHT formulas are equivalent in QHT, then they share the same equilibrium or stable models in any context [7]. This concept can be applied to KBs. For example, we can say that (Δ_1, Σ_1) and (Δ_2, Σ_2) are *strongly equivalent* if, for any Δ , Σ , $(\Delta_1 \cup \Delta, \Sigma_1 \cup \Sigma)$ and $(\Delta_2 \cup \Delta, \Sigma_2 \cup \Sigma)$ have the same stable models. For the second order semantics of [2] this property holds if and only if $(\Delta_1 \cup \Sigma_1)$ and $(\Delta_2 \cup \Sigma_2)$ are QHT equivalent, [7]. For the reductive semantics of [1] that converts the KB into a normal, ground logic program, we can take the reduced, Skolem form of the program and apply equivalence in propositional HT.

A natural property to consider is the equivalence of two sets of NTGDs over any database or the special case of strong equivalence where the additional set Σ is empty and only the databases Δ_1 and Δ_2 are extended. This property is known as *uniform equivalence*. For example we can say that two sets of NTGDs, Σ_1 and Σ_2 , over $\mathcal{L}$ are uniform equivalent if for any database Δ over $\mathcal{L}$, (Δ, Σ_1) and (Δ, Σ_2) have the same stable models. For ground programs this property is also characterised in the logic HT by considering both the HT models and HT countermodels of Σ_1 and Σ_2 , [8]. For first order programs, uniform equivalence has been studied in [9, 10], but in this case for programs without function symbols and without existential quantifiers in rule heads.

Recently, in [11], the concepts of strong and uniform equivalence have been extended to cover the case that programs or theories in equilibrium logic are equivalent with respect to their query answering. For example, considering the idea of cautious or skeptical consequence (truth in all equilibrium models), we can ask whether for any two theories that are not strongly equivalent we can find a context and a

⁴[1] studies other varieties including unions of conjunctive queries. For our purposes, however, this form will suffice.

formula that separates them, in other words that is true in one but not the other theory. For propositional theories [11] proves that this type of equivalence wrt consequences or query answering is equivalent to the usual type based on sameness of equilibrium models; and this is the case both for strong and for uniform equivalence.

Each answer to a conjunctive query q corresponds to a closed formula $q(\mathbf{t})$ true in each stable model. It follows that in the second order semantics, if two KBs are QHT equivalent in the above sense, they are equivalent for conjunctive query answering for any extension of the database or the set of NTGDs. The same will be true of the Herbrand semantics of [1] if we consider the logical equivalence of the Skolem forms, ie (Δ, Σ^f) , etc. In this way we obtain sufficient conditions in terms of a well-known logic to guarantee strong forms of equivalence on conjunctive queries.

It would be interesting to know if certain converse properties also hold. For example, suppose that two Datalog KBs are non-equivalent in having different stable models. Can they always be separated by means of conjunctive queries? To answer this we would have to extend the work of [11] in at least two directions: define the base equivalence concepts for the first order case (the easy part), and then examine whether some variants of conjunctive queries are strong enough to separate non-equivalent KBs. This last question remains to be answered even for the propositional case under skeptical consequence.

5. Conclusions

Since equilibrium logic embraces stable model semantics and is defined for first order languages it can be naturally applied to Datalog knowledge bases with NTGDs. As we have shown, in this way we can capture some of the different styles of stable model semantics proposed for Datalog with these existential rules. What is the value of this exercise? In the first place QHT provides a ‘cleaner’ and more elegant formalism. Since it is defined for a full first-order language there is no need to readjust the semantics for every new kind of formula of interest and no need to resort to artificial constructs such as reducts. Skolem forms and grounding are optional but not necessary techniques to interpret formulas. The second order semantics of [2] is directly derivable from equilibrium logic by the property that its underlying logic QHT can be embedded in classical first order logic with the use of auxiliary predicates. In addition, the base logic of equilibrium logic is a very natural choice because it expresses the difference between truth-by-default and provable or certain truth, a difference that arises very naturally in nonmonotonic reasoning and logic programming. Moreover like its propositional fragment, QHT is a well understood nonclassical logic with ‘good’ metalogical properties and whose proof and model theory have been extensively studied.

Both these logics have been successfully applied to yield tests for different concepts of equivalence for logic programs, not only in basic ASP but also in various extensions of the base language. The open research questions mentioned in the previous section are worthy of study in the future, and the logic of here-and-there should be a useful instrument for investigating them.

Acknowledgements

It is a pleasure to dedicate this paper to our friend and colleague, Gerd Wagner. For support we are grateful to the project LIANDA - BBVA Foundation Grants for Scientific Research Projects, by UPM project RP2410460010, by grant PID2023-148531NB-I00 of the Spanish Ministry MCIU, and by Junta de Andalucía (Spain) project TIC115. The contribution of the first-named author was made possible thanks to a research stay at the University of Potsdam. He is very grateful to Torsten Schaub for arranging and supporting this visit. We are grateful to referees for the Datalog 2.0 Workshop for helpful comments and suggestions.

Declaration on Generative AI

The authors have not employed any Generative AI tools.

References

- [1] G. Gottlob, A. Hernich, C. Kupke, T. Lukasiewicz, Stable model semantics for guarded existential rules and description logics: Decidability and complexity, *J. ACM* 68 (2021), 1–87.
- [2] M. Alviano, M. Morak, A. Pieris, Stable model semantics for tuple-generating dependencies revisited, in: *Proc. of the 36th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems (PODS '17)*, ACM Publications, pp. 377–388.
- [3] D. Pearce, A. Valverde, Towards a first order equilibrium logic for nonmonotonic reasoning, in: *Proc. of the 9th European Conf. on Logics in AI (JELIA'04)*, LNCS Volume 3229 2004, pp. 147–160.
- [4] P. Ferraris, J. Lee, V. Lifschitz, A new perspective on stable models, in: *Proc. of the 20th Intl. Joint Conference on Artificial Intelligence (IJCAI'07)*, 2007, pp. 372 - 379.
- [5] D. Pearce, A. Valverde, Quantified equilibrium logic and foundations for answer set programs, in: *Proc. of Logic Programming, 24th International Conference (ICLP'08)*, volume 5366 of *LNCS*, Springer, 2008, pp. 546–560.
- [6] D. van Dalen, *Logic and Structure*, Universitext, Springer, London, 2012.
- [7] V. Lifschitz, D. Pearce, A. Valverde, A characterization of strong equivalence for logic programs with variables, in: *Proc. of 9th International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'07)*, volume 4483 of *LNCS*, Springer, 2007, pp. 188–200.
- [8] M. Fink, Equivalences in answer-set programming by countermodels in the logic of here-and-there, in: *Proc. of Logic Programming, 24th International Conference, (ICLP'08)*, volume 5366 of *LNCS*, Springer, 2008, pp. 99–113.
- [9] T. Eiter, M. Fink, H. Tompits, S. Woltran, Strong and uniform equivalence in answer-set programming: Characterizations and complexity results for the non-ground case, in: *Proc. of The Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference*, AAAI Press / The MIT Press, 2005, pp. 695–700.
- [10] J. Oetsch, H. Tompits, Program correspondence under the answer-set semantics: The non-ground case, in: *Proc. of Logic Programming, 24th International Conference (ICLP'08)*, volume 5366 of *LNCS*, Springer, 2008, pp. 591–605.
- [11] J. Fandinno, D. Pearce, C. Vidal, S. Woltran, Comparing the reasoning capabilities of equilibrium theories and answer set programs, *Algorithms* 15(6):201 (2022),