New computational results for the degree/diameter problem for 4-valent Cayley graphs

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Abstract

The degree/diameter problem for Cayley graphs is the problem of finding, for any pair $\Delta \geq 3$ and $D \geq 2$ of integers, the largest possible order $N_{Cay}(\Delta, D)$ of a Cayley graph of degree Δ and diameter D. We present results of our exhaustive computer-aided search for 4-valent Cayley graphs constructed from groups in the Small Groups Library in GAP. We show that the existing lower bounds on $N_{Cay}(4, D)$ are tight for $D = 5, 6$ and present new lower bounds for $D = 7, 8$. We also present a new lower bound for $N_{Cay}(4, 12)$.

Keywords

Degree/diameter problem, Cayley graphs, GAP

1. Introduction

Let Δ and D be positive integers. A graph Γ is a (Δ, D)-*graph* if it is Δ-regular and has diameter D. The *degree/diameter problem* calls for determining the largest possible order $N(\Delta, D)$ of a (Δ, D)-graph and finding all the graphs of order $N(\Delta, D)$. To avoid trivial examples of odd cycles and complete graphs, it is assumed that $\Delta \geq 3$ and $D \geq 2$.

The *Moore bound* states that for any Δ and D we have $N(\Delta, D) \leq M(\Delta, D)$, where

$$M(\Delta, D) = 1 + \Delta(1 + (\Delta - 1) + (\Delta - 1)^2 + \cdots + (\Delta - 1)^{D-1}).$$

The graphs attaining the Moore bound, the so-called *Moore graphs*, are extremely rare. In fact, they exist only if D equals 2 and Δ equals 3, 7 and possibly 57 [3, 10, 15]. Surprisingly, the only general upper bound for the remaining cases is just $M(\Delta, D) - 1$ or $M(\Delta, D) - 2$, depending only on whether Δ is even or odd. Lower bounds on $N(\Delta, D)$ are usually obtained by constructing a (Δ, D)-graph of a given order. Historically, these bounds and corresponding graphs are called records and record holders. As it is generally hard to determine whether a record is the actual extremal value, the list of records and record graphs is kept for small values of Δ and D, usually for $D \leq 10$. Besides the general degree/diameter problem, related versions for various subfamilies of graphs are studied. As noted in [21], the most obvious candidates here are vertex-transitive and Cayley graphs, suitable because of their quick computer generation as well as from the point of view of diameter checking. The degree/diameter problem, both for general graphs and for various families of graphs, is covered by the Dynamic Survey by M. Miller and J. Širáň [21] and by the online project Combinatorics Wiki [16].

In the degree/diameter problem for Cayley graphs, the corresponding extremal value is denoted by $N_{Cay}(\Delta, D)$. Besides the observation that no Moore graph is Cayley, the only improvement to the upper bound is the asymptotic result derived for vertex-transitive graphs by G. Exoo, R. Jajcay, M. Mačaj, and J. Širáň [13]. There are two main methods used to obtain new lower bounds on $N_{Cay}(\Delta, D)$. The first method relies on a clever construction (usually of an infinite family of graphs) while the second method relies on computer-aided search for graphs with given Δ and D; for example, see [25] and [11]. The increasing power of computers and the computational systems for algebra and discrete

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mathematics, such as GAP [9] and Magma [5], enable us to significantly extend the range of results obtainable by the second method. A prominent example of the use of Magma in the second method is the result of P. Potočnik, P. Spiga and G. Verret, who used their census of all cubic Cayley graphs of order up to 1280 to completely determine the values $N_{Cay}(3, D)$ for $D \leq 8$ and to obtain new lower bounds for $N_{Cay}(3, 9)$ and $N_{Cay}(3, 10)$ [22].

In the present paper, we focus on the degree/diameter problem for $(4, D)$-Cayley graphs for $D \leq 8$. The status of the problem at the start of our project is depicted in the following table, see [17, 21]. The numbers in bold denote the exact values of $N_{Cay}(\Delta, D)$, while the remaining numbers are just the best known lower bounds.

$\Delta \backslash D$	2	3	4	5	6	7	8	9	10
3	**8**	**14**	**24**	**60**	**72**	**168**	**300**	506	882
4	**13**	**30**	**84**	216	513	1155	3080	7550	17608

During the preparation of the paper, several new results appeared. R. J. Evans and P. Potočnik [12] published a census of 4-valent Cayley graphs of orders up to 1025 and of order 1536, however, without discussing the implications for the value $N_{Cay}(4, D)$. We use the census to obtain a (partial) independent verification of our results and to search for $(4, 7)$-Cayley graphs of order 1536. Moreover, M. Conder [7] published an update of the results presented on the Combinatorics Wiki [16] and F. Comellas [6] published a complete list of extremal (Δ, D)-Cayley graphs for pairs (Δ, D), for which the exact value of $N_{Cay}(\Delta, D)$ is known. In both cases, their results for $(\Delta, D) = (4, 5)$ agree with our computations.

2. Preliminaries

In our computations, we used GAP [9] with packages GRAPE [26] and Digraphs [8]. Our computations rely on the Small Groups Library in GAP, which provides access to several important families of groups. More details about the library can be found in [4]. We worked mostly in GRAPE and only used Digraphs to work with the census constructed by R. J. Evans and P. Potočnik [12].

In what follows, we only use well-known results concerning groups and Cayley graphs and refer the reader interested in more details to [24] and [1].

There are several (equivalent) definitions of Cayley graphs in the literature. We chose to use the one used in GRAPE: Let G be a group and let S be a subset of G. The *Cayley digraph* $Cay(G, S)$ for G with *connection* (or Cayley) *set* S has the vertex set equal to G and there is a dart, that is, an oriented edge, from x and y if and only if $y = sx$ for some $s \in S$. To obtain a simple graph (in which each edge consists of a pair of opposite darts), it is necessary (and sufficient) that $1_G \notin S$ and $S = S^{-1}$, and, from now on, we assume that these two conditions are satisfied.

3. Details of implementation

Modern mathematical software such as GAP and Magma provide efficient tools to work with a wide variety of algebraic and combinatorial objects. Moreover, the naming convention in GAP usually provides functions with names that clearly state what the given function does. Therefore, the following text, which might appear to be just a pseudocode, is in fact a working function in GAP within the package GRAPE.

```
Has4DCayleyGraph:=function( G, D)
local S;
for S in Combinations( Elements(G), 4) do
   if ForAll( S ,x -> x <> One( G) and x^-1 in S ) then
       if Diameter( CayleyGraph( G, c))=D then
            return true;
```

```
        fi;
    fi;
od;
return false;
end;
```

Although the above function works correctly, its use is feasible only for very small groups. Despite that, the function we actually ended up using required only a relatively small number of modifications. The modification whose goal was to quickly predict a negative answer, as well as functions used to solve specific cases, are based on mathematical results, and we will describe them in the next section. In this section, we describe modifications the aim of which was to remove redundant computations. One important property of connection sets is the number of involutions they contain. It turned out that some parts of the code heavily depended on this number. In fact, we had three versions of the code and ran three separate computations, one for each possible number.

The main source of redundant computations is the well-known fact that for any set $S \subseteq G$ and any automorphism φ of G, the Cayley graphs for S and S^φ are isomorphic, so they have the same diameter. To obtain all Cayley graphs of G (of given valency), one wants to check at least one set S from each orbit of $Aut(G)$ with as few repetitions as possible. (As graphs may be Cayley for connection sets from different orbits of the automorphism group or even for different groups, it is still necessary to test all the obtained graphs of given order for isomorphism. Thus, it may be convenient to allow a small number of repetitions.) The general method for realizing this goal was introduced by Read in [23], and in the context of the generation of Cayley graphs, it was described in [12]. Therefore, we shall explain our approach only for one type of connection sets.

Let us assume that, for a group G with a fixed ordering of elements, we want to examine, with as few repetitions as possible, all Cayley graphs with connection set S of the form $\{a, a^{-1}, b, c\}$ where b and c are involutions and a is an element of order greater than two. First, we compute the orbits of $Aut(G)$ on pairs $\{a, a^{-1}\}$ and select a minimal pair from each orbit. For each minimal pair $\{a, a^{-1}\}$ we compute the set-wise stabilizer H of $\{a, a^{-1}\}$ in $Aut(G)$. Then we compute minimal elements of orbits of H on involutory elements of G. Further, we compute minimal elements of orbits of H_b and throw out those which are not larger than b. For each remaining element c we examine $Cay(G, \{a, a^{-1}, b, c\})$ only in the case when there is no automorphism $\varphi \in H$, which maps c onto b and maps b onto an element smaller than c. In this way, all repetitions that arise from the action of $Aut(G)$ are avoided.

The function `CayleyGraph` in GRAPE creates some auxiliary data to speed up computations. In principle, this function starts by creating $Cay(G, \emptyset)$. As we generated many Cayley graphs with the same group, it makes sense to create the auxiliary data just once and reuse it for all graphs. For small groups, it even makes sense to pre-compute some more data, which would otherwise be computed repeatedly for each connection set. We modified our function accordingly.

What we really want to know in this kind of computation is not the diameter of a given Cayley graph but whether the diameter equals D; therefore, we want to obtain value `false` as soon as it is clear that some vertices of the graph (i.e., elements of the group) are at a distance greater than D from 1_G. In our modifications of the function `Diameter` of GRAPE, we also simplified some of its parts to work more efficiently for Cayley graphs. However, the modified version may not work for graphs that are not Cayley.

4. Shortcuts and special cases

There are several cases where some elementary mathematical observations enabled us to significantly speed up our computations. Here, we present these observations together with their applications.

Although its proof is very simple, The first of these observations has been, in fact, used in several places.

Lemma 4.1. *Let $\Gamma = Cay(G, S)$ be a connected Cayley graph with diameter D and let N be a normal subgroup of G, then $Cay(G/N, S/N)$ has diameter at most D.*

Note that the connection set $S/N = \{sN,\ s \in S\}$ in $Cay(G/N, S/N)$ can be interpreted in two ways. First, as a proper set with removed repetitions of cosets of N and occurrences of N stemming from elements from $S \cap N$, and second, as a multi-set of the same size as S. In the first case, we always obtain a simple graph of valency which may be smaller than $|S|$, and in the second case, we always obtain a graph of the same valency as the valency of Γ, but with possible parallel edges, loops, and semi-edges. As the lemma obviously holds in both cases, we will not fix either interpretation of S/N in our paper, which allows us to use the more convenient one when needed.

Lemma 4.1 enabled us to easily identify many groups for which there exists no connected 4-valent Cayley graph. Recall that the derived subgroup G' of a group G generated by all the commutators $[a, b] = a^{-1}b^{-1}ab$ is normal (even characteristic) in G, and the factor group G/G' is always abelian. From the classification of finite abelian groups, we immediately obtain:

Lemma 4.2. *Let G be a finite abelian group. Then*

- *G is generated at most by two elements if and only if it is isomorphic to $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_{nm}$ for positive integers n, m;*
- *G is generated by at most three elements, at most two of them involutions if and only if it is isomorphic to $\mathbb{Z}_n$ or $\mathbb{Z}_2^k \times \mathbb{Z}_{2n}$ for positive integer n and $k \in \{0, 1, 2\}$*
- *G is generated by at most four involutions if and only if it is isomorphic to $\mathbb{Z}_2^k$ for some $k \in \{0, 1, \ldots, 4\}$.*

In GAP, the structure of G/G' in the form above can be computed using the function `AbelianInvariants` which works without explicitly computing G'. The application of Lemma 4.1 and Lemma 4.2 was sufficient to quickly finish computations for all admissible orders, except for order 1536, and connection sets with 0 or 2 involutions and also for all graphs with 4 involutory elements in the connection set and diameter at most 6.

In what follows we use the phrase "our general functions" to refer to functions from the previous section which are modified to use Lemma 4.1 and Lemma 4.2.

In order to deal with generating sets with 4 involutions for diameters 7 and 8 in a reasonable amount of time, we used some other ideas.

There are 11858 integers smaller than 13121 for which the full list of groups of given size is available, and only for 56 of them do there exist more than 2500 groups. From those, 44 are a multiple of 128 and 38 are of the form $128p$ or $256p$ for some odd prime p. From the Sylow theorem we obtain strong structural properties of groups of these sizes.

Lemma 4.3. *Let p be an odd prime and let G be a group of order $128p$ or $256p$ if $p \notin \{3, 5, 7, 17, 31, 127\}$. Then, the Sylow p-subgroup of G is normal in G.*

For $p \in 3, 5, 7, 17, 31$, the built-in functions in GAP were used to find all groups G of orders $128p$ or $256p$ between 1200 and 13120 for which the Sylow p-subgroup is not normal:

Lemma 4.4. *The number of groups of order $128p$ or $256p$ which admit a connected 4-valent Cayley graph with generating set consisting of 4 involutions and for which the Sylow p-subgroup is not normal is given in the following table.*

p	5	7	17	31
$128p$	–	–	0	0
$256p$	55	3	0	0

The importance of Lemma 4.3 and Lemma 4.4 is given by the following elementary fact.

Lemma 4.5. *Let p be an odd prime and let G be a group of order $2^k p$ in which the Sylow p-subgroup is normal in G and let H be a Sylow 2-subgroup of G. Then $G = \mathbb{Z}_p \rtimes H$.*

Proof . Each group G which contains a normal subgroup N, and a subgroup H such that $|N \cap H| = 1$ and $|N| \cdot |H| = |G|$ is isomorphic to their semidirect product. As $p^2 \not| |G|$, the Sylow subgroup of G is isomorphic to $\mathbb{Z}_p$.□

Lemma 4.5 and Lemma 4.4 enable us to use an alternative approach for groups of order $128p$ or $256p$ – we first computed the list of all possible Cayley graphs $Cay(H, T)$ on 128 and 256 vertices with connection set T consisting of $k \leq 4$ involutions and with diameter at most 8, then for each group H appearing in the list we computed all semidirect products $G = \mathbb{Z}_p \rtimes H$ and for each T we examine all sets S consisting of 4 involutions in G such that $S/\mathbb{Z}_p = T$. By Lemma 4.1, only those sets may give rise to Cayley graphs of diameter 7 or 8. The groups from Lemma 4.4 were dealt with using our general functions.

Furthermore, we used a similar approach for groups of order $1920 = 15 \cdot 128$. We used the above list of groups of order 128 to examine all semidirect products $\mathbb{Z}_{15} \rtimes H$ and examined the small number of groups of order 1920 that are not of this form with our general functions.

The dihedral group D_n of order $2n$ with n or $n + 1$ involutions and $n\varphi(n)$ automorphisms (for $n > 2$) provides a hard case for our general functions. Fortunately, we can solved this case quickly with the help of the following result.

Lemma 4.6. *Let $n > 2$ and let $S = \{a, b, c, d\}$ be a set of elements of order 2 in the dihedral group D_n such that $Cay(D_n, S)$ has diameter D and let C_n be the cyclic subgroup of D_n of order n. Then*

1. *if $S \cap C_n = \emptyset$, then $Cay(C_n, \{ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc\})$ has diameter at most $\lfloor D/2 \rfloor$;*
2. *if $S \cap C_n = \{d\}$, then $Cay(C_n, \{ab, ba, ac, ca, bc, cb, d\})$ has diameter at most $\lceil D/2 \rceil$.*

Proof . By the assumption, each element of C_n can be expressed as a product of a sequence of elements from S of length at most D.

If n is odd or n is even and S does not contain the central involution of D_n, then any such sequence has even length $l \leq D$, and can be interpreted as a sequence of symbols from $\{ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc\} \subseteq C_n$ of length $l/2$.

If n is even and d is the central involution, then we can assume that our sequence contains at most one copy of d. If there is no d in the sequence, then the length l of the sequence is even, and it can be interpreted as a sequence of symbols from $\{ab, ba, ac, ca, bc, cb\} \subseteq C_n$ of length $l/2$. If d occurs, then l is odd, and the sequence can be interpreted as a sequence of symbols from $\{ab, ba, ac, ca, bc, cb, d\} \subseteq C_n$ of length $(l + 1)/2$. □

For diameters 7 and 8, the corresponding circulants have diameter at most 4 and valency at most 10. According to [14, 18], the largest order of such a circulant is 457. Therefore, for n greater than 914 each Cayley graph $Cay(D_n, S)$ with S consisting of 4 involutions has diameter at most 6.

In order to determine $N_{Cay}(4, 6)$, one has to determine whether there exist $(4, 6)$-Cayley graphs of order 1024. Although it is known that there are 49487365422 groups of order 1024, their complete list is not available in GAP. To solve this case, we therefore used the method described by P. Potočnik, P. Spiga, and G. Verret to construct all cubic Cayley graphs up to order 1280 cf. [22]. We used essentially the same computations; the only difference is that we were looking for all $(4, 6)$-Cayley graphs instead of all cubic Cayley graphs, and that we worked in GAP instead of `Magma`.

5. The results

In this section, we present the results of our computations.

Theorem 5.1. *The largest order of a* $(4,5)$*-Cayley graph is* 216*. Up to isomorphism, there exist exactly five extremal* $(4,5)$*-Cayley graphs.*

Theorem 5.2. *The largest order of a* $(4,6)$*-Cayley graph is* 513*. Up to isomorphism, there exists unique extremal* $(4,6)$*-Cayley graph.*

Theorem 5.3. *With the possible exception of order* 1536*, the largest order of a* $(4,7)$*-Cayley graph of a group contained in the Small Groups Library in GAP is* 1200*. Up to isomorphism, there exists a unique* $(4,7)$*-Cayley graph of order* 1200*.*

Theorem 5.4. *The largest order of a* $(4,8)$*-Cayley graph of a group contained in the Small Groups Library in GAP is* 3125*. Up to isomorphism, there exists a unique* $(4,8)$*-Cayley graph of order* 3125*.*

Besides groups of small orders, GAP can be made to construct groups from other families of groups, in particular, simple groups. Applying our modified functions to these these groups, we also obtained the following result.

Lema 5.1. *There exists a* $(4,12)$*-Cayley graph of order* 102660*.*

The census [12] of 4-valent Cayley graphs up to order 1025 and of order 1536 was published after we finished our computations for diameters 5 and 6, therefore we used it just to obtain an independent verification of our results for those diameters. However, we used the list of graphs of order 1536 to complete our computations for diameter 7. In order to minimize the dependencies, we used the package Digraphs instead of Grape to examine graphs from the census. We found no discrepancies between our results and the census for graphs with diameter at most 6. For the diameter 7, the result of our computations is as follows.

Lema 5.2. *There are no* $(4,7)$*-Cayley graphs of order* 1536*.*

The groups and connection sets of Cayley graphs referred to in the the above results are presented in the following table (note that Combinatorics Wiki [19] only contains the first graph):

Group	Generators	Involutions
$[216,87]$	$g_1, g_2, g_3 * g_4, g_1 * g_3 * g_5 * g_6^2$	4
$[216,90]$	$g_1, g_2 * g_3, g_1 * g_4 * g_5 * g_6$	2
$[216,96]$	$g_1, g_2 * g_5, g_1 * g_3 * g_6^2$	2
$[216,157]$	$g_1 * g_2 * g_3, g_1 * g_5, g_4 * g_6$	2
$[216,158]$	$g_1, g_2, g_3 * g_4, g_1 * g_3 * g_4 * g_5 * g_6$	4
$[513,3]$	$g_1, g_1 * g_2^2 * g_4$	0
$[1200,946]$	$g_1 * g_4 * g_5, g_4 * g_6, g_2 * g_4 * g_5 * g_6^3 * g_7^4$	2
$[3125,38]$	$g_1^3 * g_2, g_1^2 * g_2^2$	0
$PSL(2,59)$	$g_1, g_1 * g_2 * g_1^{-2} * g_2^2 * g_1 * g_2^{-1} * g_1^2$	0

In order to obtain the above groups in GAP, it suffices to write `g:=SmallGroup(n,ind)` or `g:=PSL(2,59)`. To obtain the generators, it suffices to write the words from the table with g_i replaced by $g.i$, i.e., $g_1 * g_5$ replaced by $g.1 * g.5$. Note that the identifiers of the groups are the same in `Magma`.

After determining the largest 4-valent Cayley graphs (for groups from the Small Groups Library) with given diameter, we extended our computations to find all 4-valent Cayley graphs with diameter at most 6 (in the following tables the total number of 4-valent Cayley graphs of given order is obtained from [12]).

Diameter	Nr of graphs	The largest order	Nr of extremal graphs
1	1	5	1
2	13	13	3
3	74	30	1
4	329	84	2
5	2020	216	5
6	10928	513	1

For diameters 4, 5 and 6, we also present the 10 largest orders admitting a Cayley graph with given diameter. For each diameter, the second column contains the number of all 4-valent Cayley graphs and the third column contains the number of $(4, D)$-Cayley graphs of given order.

	$D = 4$			$D = 5$			$D = 6$	
Order	All	Diam D	Order	All	Diam D	Order	All	Diam D
56	126	2	156	1158	10	406	2310	2
57	22	1	160	2828	7	408	9838	2
60	291	15	162	876	12	420	12132	5
63	33	3	165	90	2	432	42654	7
64	367	2	168	2469	9	441	240	1
66	101	1	171	84	2	448	26130	1
68	122	1	192	12594	8	465	246	1
72	541	4	203	69	2	480	77759	3
81	51	2	205	69	1	504	26514	2
84	426	2	216	5454	5	513	300	1

6. Graphs with bounded diameter-by-girth ratio

In 2022, G. Arzhantseva, and A. Biswas introduced several infinite families of Cayley graphs with bounded diameter-by-girth ratio [2]. In particular, they proved the following result.

Lema 6.1. *[2, Corollary 3.8] Let $a, b \in \mathbb{Z} \setminus \{0, 1\}$. The diameter-by-girth ratio of the sequence of Cayley graphs of $G = SL_2(\mathbb{F}_p)$ with respect to $S = \{A^{\pm 1}, B^{\pm 1}\} \mod p$ as $p \to \infty$, where where $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$ is bounded by a constant.*

The authors state that a bound depending only on a and b can be effectively computed, but they do not provide such a bound.

We computed the diameter and girth of graphs for $a = 2$ and $b \in \{2, 3\}$ and the first 20 primes greater than 3, hoping that these graphs will provide new record holders. Although our hopes were not realised, the obtained results may be of independent interest. For example, in both families, neither girth nor diameter are monotonic in p.

	$a=2$	$b=2$	$a=2$	$b=3$
p	diam	girth	diam	girth
5	6	5	6	5
7	8	6	7	6
11	9	9	8	9
13	9	10	11	9
17	11	10	10	10
19	10	10	11	10
23	11	12	12	10
29	13	10	12	10
31	12	14	13	10
37	14	14	13	12

	$a=2$	$b=2$	$a=2$	$b=3$
p	diam	girth	diam	girth
37	14	14	13	12
41	14	10	13	14
43	13	14	13	10
47	14	14	15	12
53	14	14	14	14
59	14	14	14	14
61	15	16	15	14
67	14	14	14	14
71	15	14	15	10
73	16	14	16	14

7. Conclusion

Using an exhaustive computer search, we found the largest $(4, D)$-Cayley graphs for groups from the Small Groups Library in GAP for $D \leq 8$. As a result, we completely solved the degree/diameter problem for $(4, D)$-Cayley graphs with D equal to 5 and 6, and we found new record graphs for D equal to 7 and 8. Using the library of finite simple groups in GAP we also found a new record graph for D equal to 12. The results are summarized in the following table, where the underlined values mark our contribution.

$\Delta \backslash D$	2	3	4	5	6	7	8	9	10
4	**13**	**30**	**84**	**<u>216</u>**	**<u>513</u>**	<u>1200</u>	<u>3125</u>	7550	17608

The limitations of our approach are clear. Although the first two orders for which the complete list of groups is not available are 1024 and 2016, up to $M(4,7) - 1 = 4372$ there are 255 such orders, and up to $M(4,8) - 1 = 13120$ there are additional 1007 missing orders. For example, both the previous record 3080 and $M(4,8) - 1$ are missing. However, the relatively large distance between the records and the actual values $N_{Cay}(\Delta, D)$ from the Moore bound, together with our ability to solve some special cases, indicates that a dedicated search for (Δ, D)-Cayley graphs, based on techniques used by P. Potočnik. P. Spiga and G. Verret in [22] and R. J. Evans and P. Potočnik in [12] may also succeed for larger values of Δ and D.

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Declaration on Generative AI

During the preparation of this work, the author used Grammarly in order to: Grammar and spelling check. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the publication's content.

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